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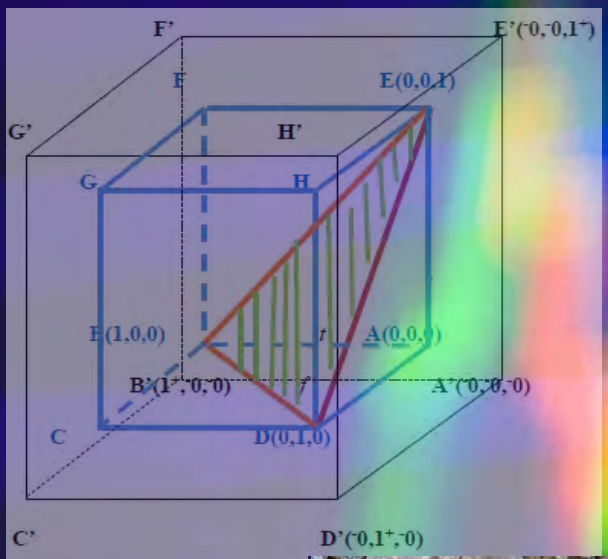

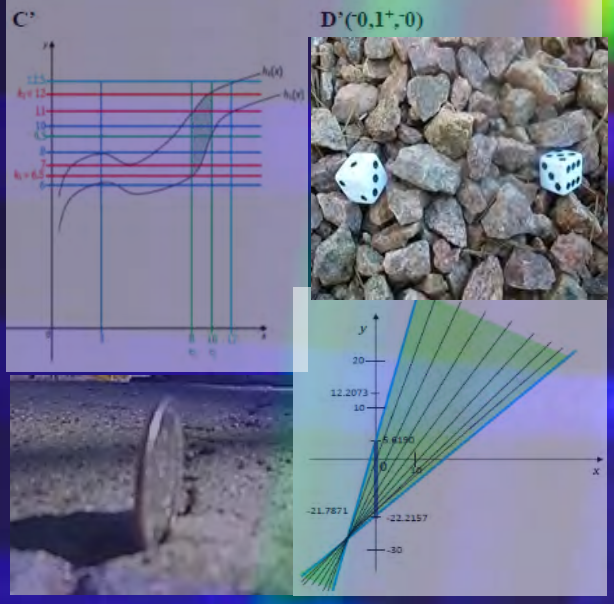

⟨A⟩ ⟨neutA⟩ ⟨antiA⟩

Editors-in-Chief
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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University of New Mexico

Editors-in-Chief

Prof. Emeritus Florentin Smarandache, PhD, Postdoc, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: smarand@unm.edu.

Dr. Mohamed Abdel-Baset, Head of Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamedbasset@ieee.org.

Prof. Dr. Maikel Leyva-Vázquez, Instituto Superior Tecnológico Bolivariano de Tecnología, Ecuador: maikel.leyvav@ug.edu.ec.

Associate Editors

Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

Assoc. Prof. Alok Dhital, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: adhital@unm.edu.

Dr. S. A. Edalatpanah, Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran, Email: saedalatpanah@gmail.com.

Charles Ashbacker, Charles Ashbacher Technologies, Box 294, 118 Chaffee Drive, Hiawatha, IA 52233, United States, Email: cashbacher@prodigy.net.

Prof. Dr. Xiaohong Zhang, Department of Mathematics, Shaanxi University of Science &Technology, Xian 710021, China, Email: zhangxh@shmtu.edu.cn.

Prof. Dr. W. B. Vasantha Kandasamy, School of Computer Science and Engineering, VIT, Vellore 632014, India, Email: vasantha.wb@vit.ac.in.

Editors

Yanhui Guo, University of Illinois at Springfield, One University Plaza, Springfield, IL 62703, United States,Email: yguo56@uis.edu.

Giorgio Nordo, MIFT - Department of Mathematical and Computer Science, Physical Sciences and Earth Sciences, Messina University, Italy,Email: giorgio.nordo@unime.it.

Mohamed Elhoseny, American University in the Emirates,Dubai,UAE,
Email: mohamed.elhoseny@aue.ae.

Le Hoang Son, VNU Univ. of Science, Vietnam National Univ. Hanoi, Vietnam, Email: sonlh@vnu.edu.vn.

Huda E. Khalid, Head of Scientific Affairs and Cultural Relations Department, Nineveh Province, Telafer University, Iraq, Email: dr.huda-ismael@uotelafer.edu.iq.

A. A. Salama, Dean of the Higher Institute of Business and Computer Sciences, Arish, Egypt, Email: ahmed_salama_2000@sci.psu.edu.eg.

Young Bae Jun, Gyeongsang National University, South Korea, Email: skywine@gmail.com.

Yo-Ping Huang, Department of Computer Science and Information, Engineering National Taipei University, New Taipei City, Taiwan, Email: yphuang@ntut.edu.tw.

Tarek Zayed, Department of Building and Real Estate, The Hong Kong Polytechnic University, Hung Hom, 8 Kowloon, Hong Kong, China, Email: tarek.zayed@polyu.edu.hk.

Vakkas Ulucay, Kilis 7 Aralık University, Turkey, Email: vulucay27@gmail.com.

Peide Liu, Shandong University of Finance and Economics, China, Email: peide.liu@gmail.com.

Jun Ye, Ningbo University, School of Civil and Environmental Engineering, 818 Fenghua Road, Jiangbei District, Ningbo City, Zhejiang Province, People's Republic of China, Email: yejun1@nbu.edu.cn.

Memet Şahin, Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey, Email: mesahin@gantep.edu.tr.

Muhammad Aslam & Mohammed Alshumrani, King Abdulaziz Univ., Jeddah, Saudi Arabia, Emails magmuhammad@kau.edu.sa, maalshmrani @kau.edu.sa.

Mutaz Mohammad, Department of Mathematics, Zayed University, Abu Dhabi 144534, United Arab Emirates. Email: Mutaz.Mohammad@zu.ac.ae.

Abdullahi Mohamud Sharif, Department of Computer Science, University of Somalia, Makka Al-mukarrama Road, Mogadishu, Somalia, Email: abdullahi.shariif@uniso.edu.so.

.

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Katy D. Ahmad, Islamic University of Gaza, Palestine,Email: katyon765@gmail.com.
NoohBany Muhammad, American University of Kuwait,Kuwait,Email: noohmuhammad12@gmail .com.
Soheyb Milles, Laboratory of Pure and Applied Mathematics, University of Msila, Algeria, Email: soheyb.milles@univ-msila.dz.
Pattathal Vijayakumar Arun, College of Science and Technology,Phuentsholing,Bhutan,
Email: arunpv2601@gmail.com.
Endalkachew Teshome Ayele, Department of Mathematics, Arbaminch University, Arbaminch, Ethiopia,Email: endalkachewteshome83@yahoo. com.
A. Al-Kababji, College of Engineering, Qatar University,Doha,Qatar,
Email: ayman.alkababji@ieee.org.
Xindong Peng, School of Information Science and Engineering, Shaoguan University, Shaoguan 512005, China, Email: 952518336@qq.com.
Xiao-Zhi Gao, School of Computing, University of Eastern Finland, FI-70211 Kuopio, Finland, xiao-zhi.gao@uef.fi.
Madad Khan, Comsats Institute of Information Technology,Abbottabad,Pakistan,
Email: madadmath@yahoo.com.
G. Srinivasa Rao, Department of Statistics, The University of Dodoma, Dodoma, PO. Box: 259, Tanzania,Email: gaddesrao@gmail.com.
Ibrahim El-henawy, Faculty of Computers and Informatics,ZagazigUniversity,Egypt,
Email: henawy2000@yahoo.com.
Muhammad Saeed, Department of Mathematics, University of Management and Technology, Lahore, Pakistan,
Email: muhammad.saeed@umt.edu.pk.
A. A. A. Agboola, Federal University of Agriculture, Abeokuta, Nigeria,
Email: agboolaaaa@funaab.edu.ng.
Abduallah Gamal, Faculty of Computers and Informatics, Zagazig University, Egypt,
Email: abduallahgamal@zu.edu.eg.
Ebenezer Bonyah, Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi 00233, Ghana,
Email: ebbonya@gmail.com.
Roan Thi Ngan, Hanoi University of Natural Resources and Environment, Hanoi, Vietnam,
Email: rtngan@hunre.edu.vn.
Sol David Lopezdomínguez Rivas, Universidad Nacional de Cuyo, Argentina.
Email: sol.lopezdominguez@fce.uncu.edu.ar.
Maikel Yelandi Leyva Vázquez, Universidad Regional Autónoma de los Andes (UNIANDES), Avenida Jorge Villegas, Babahoyo, Los Ríos, Ecuador,
Email: ub.c.investigacion@uniandes.edu.ec.
Arlen Martín Rabelo, Exxis, Avda. Aviadores del Chaco N° 1669 c/ San Martin, Edif. Aymac I, 4to. piso, Asunción, Paraguay,
Email: arlen.martin@exxis-group.com.
Carlos Granados, Estudiante de Doctorado en Matematicas, Universidad del Antioquia, Medellın, Colombia,
Email: carlosgranadosortiz@outlook.es.
Tula Carola Sanchez Garcia, Facultad de Educacion de la Universidad Nacional Mayor de San Marcos, Lima, Peru,
Email: tula.sanchez1@unmsm.edu.pe.
Carlos Javier Lizcano Chapeta, Profesor - Investigador de pregrado y postgrado de la Universidad de Los Andes, Mérida 5101, Venezuela, Email: lizcha_4@hotmail.com.
Noel Moreno Lemus, Procter & Gamble International Operations S.A., Panamá,
Email: nmlemus@gmail.com.
Asnioby Hernandez Lopez, Mercado Libre, Montevideo, Uruguay.
Email: asnioby.hernandez@mercadolibre.com.
Muhammad Akram, University of the Punjab, New Campus, Lahore, Pakistan,
Email: m.akram@pucit.edu.pk.
Tatiana Andrea Castillo Jaimes, Universidad de Chile, Departamento de Industria, Doctorado en Sistemas de Ingeniería, Santiago de Chile, Chile,
Email: tatiana.a.castillo@gmail.com.
Irfan Deli, Muallim Rifat Faculty of Education, Kilis 7 Aralik University, Turkey,
Email: irfandeli@kilis.edu.tr.
Ridvan Sahin, Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey, Email: mat.ridone@gmail.com.
Ibrahim M. Hezam, Department of computer, Faculty of Education, Ibb University, Ibb City,

Yemen, Email: ibrahizam.math@gmail.com.
Moddassir khan Nayeem, Department of Industrial and Production Engineering, American International University-Bangladesh, Bangladesh; nayeem@aiub.edu.
Aiyared Iampan, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, Email: aiyared.ia@up.ac.th.
Ameirys Betancourt-Vázquez, 1 Instituto Superior Politécnico de Tecnologias e Ciências (ISPTEC), Luanda, Angola, Email: ameirysbv@gmail.com.
H. E. Ramaroson, University of Antananarivo, Madagascar, Email: erichansise@gmail.com.
G. Srinivasa Rao, Department of Mathematics and Statistics, The University of Dodoma, Dodoma PO. Box: 259, Tanzania.
Onesfole Kuramaa, Department of Mathematics, College of Natural Sciences, Makerere University, P.O Box 7062, Kampala, Uganda, Email: onesfole.kurama@mak.ac.ug.
Karina Pérez-Teruel, Universidad Abierta para Adultos (UAPA), Santiago de los Caballeros, República Dominicana, Email: karinaperez@uapa.edu.do.
Neilys González Benítez, Centro Meteorológico Pinar del Río, Cuba, Email: neilys71@nauta.cu.
Jesus Estupinan Ricardo, Centro de Estudios para la Calidad Educativa y la Investigation Cinetifica, Toluca,Mexico,Email: jestupinan2728@gmail.com
Victor Christianto, Malang Institute of Agriculture (IPM), Malang, Indonesia, Email: victorchristianto@gmail.com.
Wadei Al-Omeri, Department of Mathematics, Al-Balqa Applied University, Salt 19117, Jordan, Email: wadeialomeri@bau.edu.jo.
Ganeshsree Selvachandran, UCSI University, Jalan Menara Gading, Kuala Lumpur, Malaysia, Email: Ganeshsree@ucsiuniversity.edu.my.
Ilanthenral Kandasamy, School of Computer Science and Engineering (SCOPE), Vellore Institute of Technology (VIT), Vellore 632014, India, Email: ilanthenral.k@vit.ac.in
Kul Hur, Wonkwang University, Iksan, Jeollabukdo, South Korea, Email: kulhur@wonkwang.ac.kr.
Kemale Veliyeva & Sadi Bayramov, Department of Algebra and Geometry, Baku State University, 23 Z. Khalilov Str., AZ1148, Baku, Azerbaijan, Email: kemale2607@mail.ru,
Email: baysadi@gmail.com.
Irma Makharadze & Tariel Khvedelidze, Ivane Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences, Tbilisi, Georgia.
Inayatur Rehman, College of Arts and Applied Sciences, Dhofar University Salalah, Oman, Email: irehman@du.edu.om.
Mansour Lotayif, College of Administrative Sciences, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain.
Riad K. Al-Hamido, Math Department, College of Science, Al-Baath University, Homs, Syria, Email: riad-hamido1983@hotmail.com.
Saeed Gul, Faculty of Economics, Kardan University, Parwan-e- Du Square, Kabil, Afghanistan, Email: s.gul@kardan.edu.af.
Faruk Karaaslan, Çankırı Karatekin University, Çankırı, Turkey, Email: fkaraaslan@karatekin.edu.tr.
Morrisson Kaunda Mutuku, School of Business, Kenyatta University, Kenya
Surapati Pramanik, Department of Mathematics, Nandalal Ghosh B T College, India, Email: drspramanik@isns.org.in.
Suriana Alias, Universiti Teknologi MARA (UiTM) Kelantan, Campus Machang, 18500 Machang, Kelantan, Malaysia, Email: suria588@kelantan.uitm.edu.my.
Arsham Borumand Saeid, Dept. of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran, Email: arsham@uk.ac.ir.
Ahmed Abdel-Monem, Department of Decision support, Zagazig University, Egypt, Email: aabdelmounem@zu.edu.eg.
Çağlar Karamasa, Anadolu University, Faculty of Business,
Turkey, Email: ckaramasa@anadolu.edu.tr.
Mohamed Talea, Laboratory of Information Processing, Faculty of Science Ben M'Sik, Morocco, Email: taleamohamed@yahoo.fr.
Assia Bakali, Ecole Royale Navale, Casablanca, Morocco, Email: assiabakali@yahoo.fr.
V.V. Starovoytov, The State Scientific Institution «The United Institute of Informatics Problems of the National Academy of Sciences of Belarus»,

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Minsk, Belarus, Email: ValeryS@newman.bas-net.by.
E.E. Eldarova, L.N. Gumilyov Eurasian National University, Nur-Sultan, Republic of Kazakhstan, Email: Doctorphd_eldarova@mail.ru. Mukhamed iyeva Dilnoz Tulkunovna & Egamberdiev Nodir Abdunazarovich, Science and innovation center for information and communication technologies, Tashkent University of Information Technologies (named after Muhammad Al-Khwarizmi), Uzbekistan.
Mohammad Hamidi, Department of Mathematics, Payame Noor University (PNU), Tehran, Iran. Email: m.hamidi@pnu.ac.ir.
Lemnaouar Zedam, Department of Mathematics, Faculty of Mathematics and Informatics, University Mohamed Boudiaf, M'sila, Algeria, Email: l.zedam@gmail.com.
M. Al Tahan, Department of Mathematics, Lebanese International University, Bekaa, Lebanon, Email: madeline.tahan@liu.edu.lb.
Mohammad Abobala, Tishreen University, Faculty of Science, Department of Mathematics, Lattakia, Syria,Email" mohammad.abobala@tishreen.edu.sy
Rafif Alhabib, AL-Baath University, College of Science, Mathematical Statistics Department, Homs, Syria, Email: ralhabib@albaath-univ.edu.sy.
R. A. Borzooei, Department of Mathematics, Shahid Beheshti University, Tehran, Iran, borzooei@hatef.ac.ir.
Selcuk Topal, Mathematics Department, Bitlis Eren University, Turkey, Email: s.topal@beu.edu.tr.
Qin Xin, Faculty of Science and Technology, University of the Faroe Islands, Tórshavn, 100, Faroe Islands.
Sudan Jha, Pokhara University,Kathmandu, Nepal, Email: jhasudan@hotmail.com.
Mimosette Makem and Alain Tiedeu, Signal, Image and Systems Laboratory, Dept. of Medical and Biomedical Engineering, Higher Technical Teachers' Training College of EBOLOWA, PO Box 886, University of Yaoundé, Cameroon, E-mail: alain_tiedeu@yahoo.fr.
Mujahid Abbas, Department of Mathematics and Applied Mathematics, University of Pretoria Hatfield 002, Pretoria, South Africa, Email: mujahid.abbas@up.ac.za.
Željko Stević, Faculty of Transport and Traffic Engineering Doboj, University of East Sarajevo, Lukavica, East Sarajevo, Bosnia and Herzegovina, Email: zeljko.stevic@sf.ues.rs.ba.
Michael Gr. Voskoglou, Mathematical Sciences School of Technological Applications, Graduate Technological Educational Institute of Western Greece, Patras, Greece, Email: voskoglou@teiwest.gr.
Saeid Jafari, College of Vestsjaelland South, Slagelse, Denmark, Email: sj@vucklar.dk.
Angelo de Oliveira, Ciencia da Computacao, Universidade Federal de Rondonia, Porto Velho - Rondonia, Brazil, Email: angelo@unir.br.
Valeri Kroumov, Okayama University of Science, Okayama, Japan, Email: val@ee.ous.ac.jp.
Rafael Rojas, Universidad Industrial de Santander, Bucaramanga, Colombia, Email: rafael2188797@correo.uis.edu.co.
Walid Abdelfattah, Faculty of Law, Economics and Management, Jendouba, Tunisia, Email: abdelfattah.walid@yahoo.com.
Akbar Rezaei, Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran, Email: rezaei@pnu.ac.ir.
John Frederick D. Tapia, Chemical Engineering Department, De La Salle University - Manila, 2401 Taft Avenue, Malate, Manila, Philippines, Email: john.frederick.tapia@dlsu.edu.ph.
Darren Chong, independent researcher, Singapore, Email: darrenchong2001@yahoo.com.sg.
Galina Ilieva, Paisii Hilendarski, University of Plovdiv, 4000 Plovdiv, Bulgaria, Email: galili@uni-plovdiv.bg.
Paweł Pławiak, Institute of Teleinformatics, Cracow University of Technology, Warszawska 24 st., F-5, 31-155 Krakow, Poland, Email: plawiak@pk.edu.pl.
E. K. Zavadskas, Vilnius Gediminas Technical University, Vilnius, Lithuania, Email: edmundas.zavadskas@vgtu.lt.
Darjan Karabasevic, University Business Academy, Novi Sad, Serbia, Email: darjan.karabasevic@mef.edu.rs.
Dragisa Stanujkic, Technical Faculty in Bor, University of Belgrade, Bor, Serbia, Email: dstanujkic@tfbor.bg.ac.rs.
Katarina Rogulj, Faculty of Civil Engineering,

Architecture and Geodesy, University of Split, Matice Hrvatske 15, 21000 Split, Croatia; Email: katarina.rogulj@gradst.hr.
Luige Vladareanu, Romanian Academy, Bucharest, Romania, Email: luigiv@arexim.ro.
Hashem Bordbar, Center for Information Technologies and Applied Mathematics, University of Nova Gorica, Slovenia, Email: Hashem.Bordbar@ung.si.
N. Smidova, Technical University of Kosice, SK 88902, Slovakia, Email: nsmidova@yahoo.com.
Quang-Thinh Bui, Faculty of Electrical Engineering and Computer Science, VŠB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic, Email: qthinhbui@gmail.com.
Mihaela Colhon & Stefan Vladutescu, University of Craiova, Computer Science Department, Craiova, Romania,Emails: colhon.mihaela@ucv.ro, vladutescu.stefan@ucv.ro.
Philippe Schweizer, Independent Researcher, Av. de Lonay 11, 1110 Morges, Switzerland, Email: flippe2@gmail.com.
Madjid Tavanab, Business Information Systems Department, Faculty of Business Administration and Economics University of Paderborn, D-33098 Paderborn, Germany, Email: tavana@lasalle.edu.
Rasmus Rempling, Chalmers University of Technology, Civil and Environmental Engineering, Structural Engineering, Gothenburg, Sweden.
Fernando A. F. Ferreira, ISCTE Business School, BRU-IUL, University Institute of Lisbon, Avenida das Forças Armadas, 1649-026 Lisbon, Portugal, Email: fernando.alberto.ferreira@iscte-iul.pt.
Julio J. Valdés, National Research Council Canada, M-50, 1200 Montreal Road, Ottawa, Ontario K1A 0R6, Canada, Email: julio.valdes@nrc-cnrc.gc.ca.
Tieta Putri, College of Engineering Department of Computer Science and Software Engineering, University of Canterbury, Christchurch, New Zeeland.
Phillip Smith, School of Earth and Environmental Sciences, University of Queensland, Brisbane, Australia, phillip.smith@uq.edu.au.
Sergey Gorbachev, National Research Tomsk State University, 634050 Tomsk, Russia, Email: gsv@mail.tsu.ru.
Sabin Tabirca, School of Computer Science, University College Cork, Cork, Ireland, Email: tabirca@neptune.ucc.ie.
Umit Cali, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway, Email: umit.cali@ntnu.no.
Willem K. M. Brauers, Faculty of Applied Economics, University of Antwerp, Antwerp, Belgium, Email: willem.brauers@uantwerpen.be.
M. Ganster, Graz University of Technology, Graz, Austria, Email: ganster@weyl.math.tu-graz.ac.at.
Ignacio J. Navarro, Department of Construction Engineering, Universitat Politècnica de València, 46022 València, Spain, Email: ignamar1@cam.upv.es.
Francisco Chiclana, School of Computer Science and Informatics, De Montfort University, The Gateway, Leicester, LE1 9BH, United Kingdom, Email: chiclana@dmu.ac.uk.
Jean Dezert, ONERA, Chemin de la Huniere, 91120 Palaiseau, France, Email: jean.dezert@onera.fr.

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A Neutrosophic Approach to Robust Web Security: Mitigating XSS Attacks

A. A. Salama[1], El-Said F. Aboelfotoh[2], Hazem M. El-Bakry[3], Huda E. Khalid[4*], Ahmed K. Essa[4] , Ramiz Sabbagh[5] , Doaa S. El-Morshedy[6]

[1, 2, 6] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Egypt; drsalama44@gmail.com, ahmed_salama_2000@sci.psu.edu.eg, said_994@yahoo.com, doaa_morshady@yahoo.com

[3] Department of Information Systems-Faculty of Computers and Information, Mansoura University, Mansoura, Egypt elbakry@mans.edu.eg

[4]Telafer University، The Administration Assistant for the President of the Telafer University، Telafer، Iraq;

[5] Department of the Scientific Affairs, Telafer University, Mosul, Iraq; ramiz.sabbagh@uotelafer.edu.iq . dr.huda-ismael@uotelafer.edu.iq

*Correspondence: dr.huda-ismael@uotelafer.edu.iq

Abstract: Cross-Site Scripting (XSS) is one of the most grievous vulnerabilities-a pitfall through which web applications are affected. These types of attacks are complex, and the available threat landscape is always changing, making it hard for conventional security tools to effectively detect or prevent these types of attacks. We present here an approach that detects and prevents XSS attacks on Web pages. Since Neutrosophism works with different kinds of data, it aims at the validity degree of the attack. This way the system understands the different types of attacks and allows the system to act more effectively. Our system combines (1) static analysis-to look into the code behind the website-with (2) dynamic analysis-to watch the website in action. Moreover, a proactive defense watches your user behavior and scrubs every input/output.

Keywords: XSS, Cross-Site Scripting, Neutrosophic Logic, Web Security, Static Analysis, Dynamic Analysis, User Behavior Analysis, Input Validation, Output Encoding.

1. Introduction

Cross-Site Scripting (XSS) continues to pose a threat to websites. Common security approaches such as input validation and output encoding often struggle to guard against sophisticated and changing attacks [1-10]. To tackle this issue, we suggest a fresh method that applies Neutrosophic logic to boost the precision and flexibility of XSS detection and prevention. This logic enables us to deal with uncertainty and vagueness in attacks [11- 20]. New studies have pointed out the shortcomings of traditional methods and the need to develop more advanced techniques. Researchers have looked into machine learning static analysis, and dynamic analysis to strengthen web security. Our study aims to merge these approaches with Neutrosophic logic to build a strong and flexible security framework.

2. Related Work

Websites use input validation and output encoding to guard against Cross-Site Scripting (XSS) attacks. Yet these methods often fall short when it comes to spotting and stopping complex ever-changing attacks. Neutrosophic logic, a math-based approach that deals with uncertainty, has caught the eye of cyber security experts. It has applications in many security areas such as spotting intrusions, securing networks, and managing access. Mixing static analysis (looking at the code) with dynamic analysis (watching how the app behaves) can give a fuller picture of security. Machine learning and AI can speed up the process of finding weak spots studying how users act, and dealing with problems as they pop up.

Our research aims to create a strong flexible system to fight XSS attacks by bringing together Neutrosophic logic static and dynamic analysis, and machine learning. A number of studies have looked into how Neutrosophic logic can help in different security fields. These include sizing up cyber security risks systems that spot intrusions, keeping information safe, and securing block chains. Building on this groundwork, our research wants to push forward the use of Neutrosophic logic in web security, with a focus on checking user input.

Table 1: A Comparative Analysis of Previous Studies

Study	Focus	Methodology	Limitations
[1]	Systematic mapping of XSS attacks	Literature review	Limited to XSS attacks
[2]	Machine learning for XSS detection	Machine learning	Requires labeled dataset
[3]	Cyber security techniques for XSS prevention	Theoretical analysis	Limited practical implementation
[4]	Static code analysis tool evaluation	Empirical analysis	Focus on specific tools
[5]	Dynamic analysis of Android apps	Literature review	Limited to Android platform
[6]	Open-source web application testing platform	Software engineering	Limited to specific platform
[7]	Cyber security measures for enterprise software	Theoretical analysis	General overview
[8]	Static code analysis with Code Checker	Tool evaluation	Limited to Code Checker
[9]	Comprehensive cyber security guide	Literature review	Broad scope, less depth
[10]	Visual integration of static and dynamic analysis	Software visualization	Limited to specific tools and techniques

This paper's goal is to enhance user input security evaluation using Neutrosophic logic. This new method aims to be more precise and thorough compared to standard techniques, which often find it hard to deal with unclear and vague user input.

3. Methodology and Proposed Framework

3.1. Neutrosophic User Input Model

Uncertainty in User Input: User

- Input can be vague and difficult to interpret. Traditional methods usually show no accuracy in such situations.
- Neutrosophic Logic: This logic is more inclusive due to the introduction of three truth-values, viz., true, false, and uncertain.
- Modeling Uncertainty: We represent indeterminacy and incompleteness by the values one assigns to user input we develop.
- Risk Assessment: The Framework has been designed to assess the risk that arises by the user input with these values.

3.2. Static Code Analysis

- Identifying Vulnerabilities: Static code analysis examines the source code to identify potential vulnerabilities, such as XSS injection points.
- Neutrosophic Logic Integration: Neutrosophic logic helps prioritize vulnerabilities and reduce false positives.

3.3. Dynamic Analysis

- Monitoring Runtime Behavior: Dynamic analysis monitors the application's behavior while it's running to detect attacks.
- Neutrosophic Risk Assessment: Neutrosophic logic helps assess the risk of attacks based on various factors.

3.4. Proactive Defense Layer

- User Behavior Analysis: Monitors user behavior to identify unusual patterns.
- Input Validation: Validates user input to prevent malicious input.
- Output Encoding: Encodes output to prevent XSS attacks.

3.5. Centralized Decision Engine

- Integrates Components: Combines the outputs from the different components.
- Makes Decisions: Makes decisions about mitigation strategies based on risk

assessments.

- Triggers Mitigation: Implements the selected mitigation strategies.

4. Framework Implementation and Evaluation

- Neutrosophic User Input Model: Extracts features from user input, assigns Neutrosophicvalues, and calculates a risk score, as shown in figure 1.
- Static Code Analysis: Analyzes source code, identifies vulnerabilities, and assesses theirseverity using Neutrosophic logic.
- Dynamic Analysis: Monitors runtime behavior, detects attacks, and assesses their riskusing Neutrosophic logic.
- Proactive Defense Layer: Implements input validation, output encoding, and user behavioranalysis.
- Centralized Decision Engine: Integrates component outputs, makes decisions, and triggersmitigation strategies.

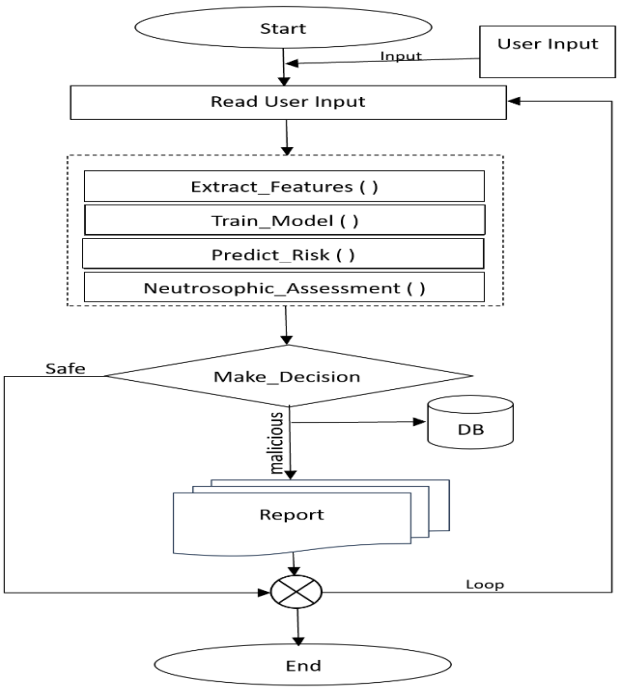

Figure 1 Proposed Framework Implementation Flowchart

4.1. Performance Evaluation

- Quantitative Analysis: Measures detection rate, false positive rate, false negative rate, andexecution time.
- Qualitative Analysis: Assesses robustness, scalability, usability, and user experience.

4.2. A Multi-faceted Approach to User Input Security: A Neutrosophic Perspective

This system aims to analyze user input and assess its potential risk by considering both its positive(likely safe) and negative (likely malicious) aspects, along with any uncertainty in its nature.

1. Analyzing the Input:

- We gather details about the user's input, such as:
  - Length of the text.
  - Presence of special characters or numbers.
  - Patterns in sequences of characters (like common attack strings).
  - Complexity and randomness of the text.
  - Language used.
  - Emotional tone or intent conveyed.

2. Predicting Risk with Machine Learning:

- We train a computer model using a large amount of labeled data (both safe and maliciousinputs).
- The model analyzes the extracted features and calculates the probability of the input beingmalicious.

3. Considering Uncertainty (Neutrosophic Logic):

- We do not just rely on the probability; we consider the possibility of the input being eithersafe or malicious.
- Based on the probability, we assign three "truth values":
  - Truth: How likely it is the input is safe (1 minus the probability of being malicious)

- Falsity: How likely it is the input is malicious (the probability itself)
- Indeterminacy: The remaining uncertainty (1 minus the sum of Truth and Falsity)

4. Calculating Overall Risk:

- We combine the three "truth values" with assigned weights, reflecting the importance ofeach feature.
- A higher resulting score indicates a higher potential risk.

5. Additional Security Layers:

- Static Code Analysis: Examining the application code to identify potential vulnerabilitieslike injection attacks.
- Dynamic Analysis: Monitoring the application's behavior during runtime to detect ongoingattacks.
- Proactive Defense Layer:
  - Validating user input to remove suspicious characters or patterns.
  - Encoding output to prevent malicious scripts from running.
  - Tracking user behavior for suspicious activities.
- Centralized Decision Engine: Combining information from all these layers and makingdecisions on how to respond (e.g., blocking requests, alerting security teams).

4.3. Python Code

```
import numpy as np from sklearn.ensemble
import RandomForestClassifier
import re
def
    extract_features(input_st
    ring):features = {
        "length":    len(input_string),  "special_chars":
```

```
            sum(not c.isalnum() for c in input_string),
            "digit_count": sum(c.isdigit() for c in input_string),
            # Add more features as needed, such as n-grams,statistical
            measures, language detection
        }
        return features
def train_model(X_train,
        y_train): model =
        RandomForestClassifier(
        ) model.fit(X_train,
        y_train)
        return model
def predict_risk(model, features):
    #Probability of malicious
        probability=model.predict_proba([features])[0][1]
        return probability
def
        neutrosophic_assessment(probabil
        ity):
        truth = 1 - probability
        falsity = probability
        indeterminacy = 1 - (truth + falsity)
        # Assign weights to features and calculate risk score
        #Adjust weights as needed
        weights ={'length':0.2,'special_chars': 0.3, ...}
        risk_score = 0
        for feature, weight in weights.items():
```

```
            risk_score += weight * falsity
            return risk_score
        def make_decision(risk_score, threshold) :
            if risk_score > threshold:
                    print("High      risk      input
            detected!")else:
                    print("Low risk input")
        # Assuming you have a trained model and a dataset of labeled  inputs
        model = train_model(X_train, y_train) # Replace with your trained model
        input_string = "<script>alert('XSS')</script>"
        features = extract_features(input_string)
        probability = predict_risk(model, features)
        risk_score          =          neutrosophic_assessment(probability)
        make_decision(risk_score, threshold=0.7)
```

This Python code is designed to analyze user input and assess its potential risk, especially in the context of web security. It uses a combination of machine learning and Neutrosophic logic to make this assessment. This approach provides a more robust and nuanced way to assess user input security compared to traditional binary classification methods.

Breakdown of the Code:

1. Feature Extraction:

We conveyed essence of the user input-have length, a number of special characters, and few other statistics about the input. This feature is useful in understanding the input and further classification.

2. Machine Learning Model:

We trained a machine-learning model, called Random Forest Classifier, on a labeled set of inputs, either safe or malicious.

This is done by observing various parameters that the model uses to describe the patterns in the input that help it make decisions on the safety of the input or labeling it malicious.

3. Neutrosophic Assessment:

Instead of just classifying an input as safe or malicious, we assign three values based on Neutrosophic logic:

Truth: To how extent are we sure, it is safe?

Falsity: To how extent are we sure it is malicious?

Indeterminacy: How uncertain are we about its nature?

This gives it input a broader horizon of qualification especially in ambiguous cases.

4. Risk Calculation:

Combining the Neutrosophic values, the risk score is calculated by putting some weightiness to the most important features.

5. Decision Making:

The risk score now calculated is compared to a certain threshold for decision-making.

If the risk appears to be more, then the input is flagged as probably being risky; otherwise, it is considered low risk.

5. Case Study:

Applying the Neutrosophic Framework to a Dataset of 2000 Usernames

5.1. Modeling Neutrosophic on User Roles and Privileges

Neutrosophic Logic is a class of mathematical systems where, on the contrary, one would go beyond the yes-or-no logic. One would include a third value, "ambiguous," which allows the representation of uncertainty and ambiguity regarding information.

Table 3: Neutrosophic Representation of User Roles and Privileges

User Role	Privilege	Truth	Falsity	Indeterminacy	Risk Assessment
Administrator	Full Access	1	0	0	Low Risk
User	LimitedAccess	0.8	0.2	0	MediumRisk
Guest	Read Only Access	0.6	0.4	0	Medium- High Risk
Anonymous	No Access	0	1	0	High Risk

Explanation:

- Truth: How certain are we that the user has the necessary permissions?
- Falsity: How uncertain are we about the user's permissions?
- Indeterminacy: How much do we not know about the user's permissions?
- Risk Assessment: We assess risk by establishing the interaction between these three values.

Note: The precise values for Truth, Falsity, and Indeterminacy will depend on the complexity of the access control system and the desired security level. Perhaps an administration would have a higher degree of certainty and lower uncertainty compared to a guest user.

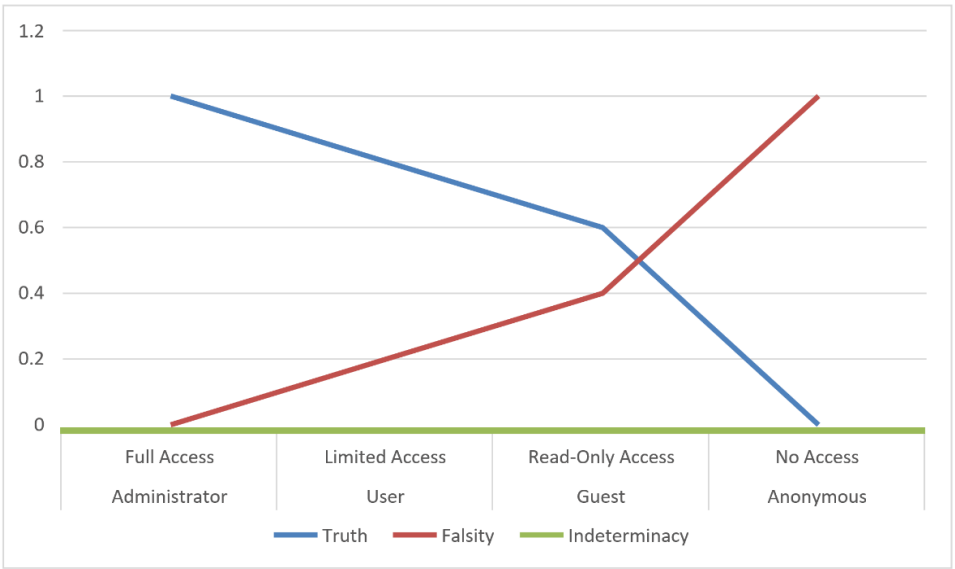

Figure 2: Neutrosophic Representation of User Roles and Privileges

Table 4: Neutrosophic Representation of User Input Assessment

Feature	Description
Input String	The original user input is being analyzed.
Probability of Malicious	The likelihood of the input being malicious, predicted by the machine-learning model.
Truth	The degree of belief that the input is safe (1 - probability of malicious).
Falsity	The degree of belief that the input is malicious (probability ofmalicious).
Indeterminacy	The degree of uncertainty about the input's nature (1 - truth - falsity).
Risk Score	A weighted sum of the Neutrosophic values, indicating the overall risk.
Classification	The final classification based on the risk score and a predefined threshold (e.g., "High Risk" or "Low Risk").

Outputs:

Table 5: Neutrosophic Assessment of User Input Risk

Input String	**Probability of Malicious**	**Truth**	**Falsity**	**Indeterminacy**	**Risk Score**	**Classification**	**Neutrosophic Accuracy**
<script>alert('XSS')</script>	0.95	0.05	0.95	0	0.285	High Risk	Low
"This is a harmless message"	0.02	0.98	0.02	0	0.006	Low Risk	High

"h4ck3r123"	0.8	0.2	0.8	0	0.24	High Risk	Low
"user123"	0.05	0.95	0.05	0	0.015	Low Risk	High

Table Interpretation:

- High-Risk Inputs: Such inputs are likely to be malicious, having high probabilities of being harmful, high certainties of not being safe, and high risk scores overall.
- Low-Risk Inputs: Such inputs are likely to be safe, having low probabilities of being malicious, a high degree of certainty of not being safe and low overall risk scores.
- Indeterminate Inputs: Not exactly shown here, these inputs have high uncertainties and might require further investigation or manual review.
- Based on this analysis and summary of the results, a good security analyst can make smart choices about how to treat various forms of user input in the face of perceived risks, thus protecting the system from various threats.

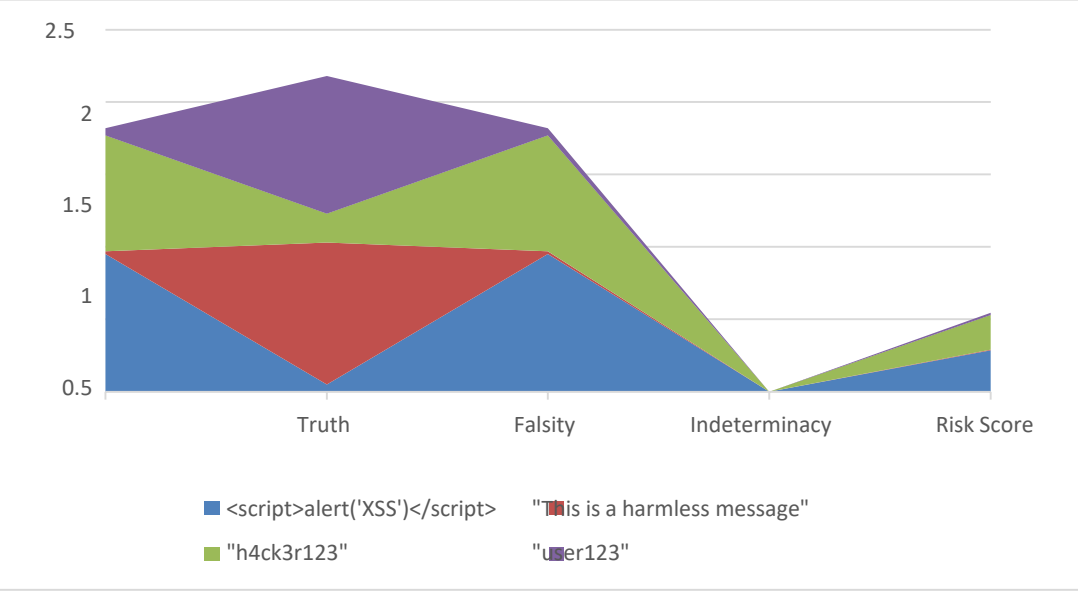

Figure 3: Neutrosophic Analysis of User Input Risk

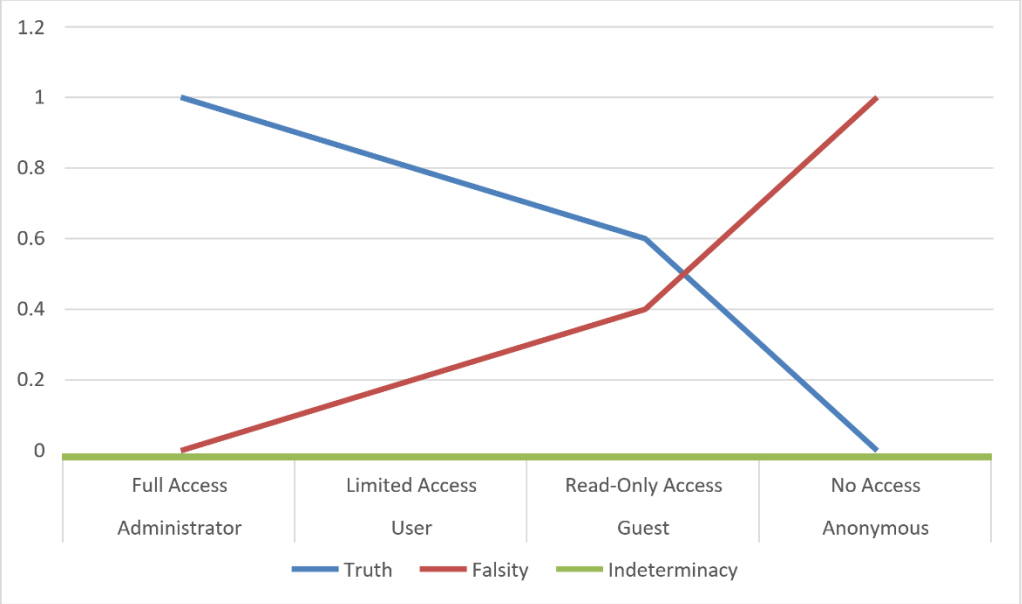

Figure 3: Neutrosophic Representation of User Roles and Privileges

Table 6: Neutrosophic Representation of User Input Assessment

Feature	Description
Input String	The original user input is being analyzed.
Probability of Malicious	The likelihood of the input being malicious, predicted by the machine-learning model.
Truth	The degree of belief that the input is safe (1 - probability of malicious).
Falsity	The degree of belief that the input is malicious (probability ofmalicious).
Indeterminacy	The degree of uncertainty about the input's nature (1 - truth - falsity).
Risk Score	A weighted sum of the Neutrosophic values, indicating the overall risk.
Classification	The final classification based on the risk score and a predefined threshold (e.g., "High Risk" or "Low Risk").

Outputs:

Table 7: Neutrosophic Assessment of User Input Risk

Input String	Probability of Malicious	Truth	Falsity	Indeterminacy	Risk Score	Classification	Neutrosophic Accuracy
<script>alert('XSS')</script>	0.95	0.05	0.95	0	0.285	High Risk	Low
"This is a harmless message"	0.02	0.98	0.02	0	0.006	Low Risk	High
"h4ck3r123"	0.8	0.2	0.8	0	0.24	High Risk	Low
"user123"	0.05	0.95	0.05	0	0.015	Low Risk	High

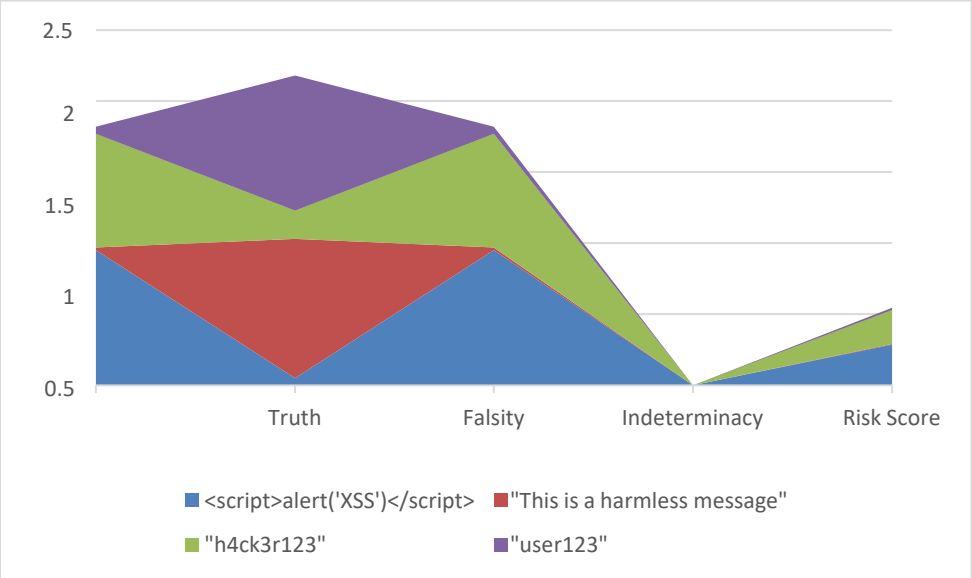

Figure 3: Neutrosophic Analysis of User Input Risk

Figure 3 visually shows how Neutrosophic logic can assess the risk of different user inputs. By looking at the colored bars, you can easily see which inputs are high-risk and which are low-risk. The taller the bar for a specific risk category (truth, falsity, or indeterminacy), the greater the impact of that category on the overall risk assessment.

5.1. Assessing Neutrosophic Accuracy for User Input

Neutrosophic accuracy measures how confident we are in classifying a user input as high-risk or low-risk. A higher "Truth" value and a lower "Falsity" value generally indicate a more accurate classification.

While the table does not explicitly calculate a numerical value for Neutrosophic accuracy, we can infer it from the "Truth" and "Falsity" values:

- High Neutrosophic Accuracy: A high "Truth" and a low "Falsity" indicate high certainty in the classification (e.g., "harmless message," "user123").
- Low Neutrosophic Accuracy: A low "Truth" and a high "Falsity" indicate low certainty in the classification (e.g., "XSS script," "h4ck3r123").

By analyzing these values, we can better understand the reliability of the classification and identify potential areas for improvement.

Understanding Weak and Strong Usernames

Weak Usernames:

- Easy to guess (common words, personal information)
- Predictable patterns
- High risk of being compromised

Strong Usernames:

- Complex combination of letters, numbers, and special characters
- Difficult to guess
- Lower risk of being compromised

Additional Security Tips:

- Use strong, unique passwords for each account.

- Enable two-factor authentication (2FA).
- Regularly update passwords.
- Avoid reusing passwords.
- Be cautious of phishing attacks.

Analyzing Usernames with Neutrosophic Logic

A username analysis was done with Neutrosophic logic. Each row in the table is a username, and the columns allow for risk determination regarding that username as defined in Neutrosophic logic:

• Probability of Malicious: Odds of that username being malicious

• Truth: Certainty that the username is safe

• Falsity: Certainty that the username is malicious

• Indeterminacy: Uncertainty about the username's nature

• Risk Score: Overall risk based on Neutrosophic values

• Classification: High-risk or low-risk classification

Let us go ahead to consider other usernames from Table 8 and assess their risk levels.

Table 8: Neutrosophic Assessment of Usernames

Input String	Probabilityof Malicious	Truth	Falsity	Indeterminacy	Risk Score	Classification	Neutrosophic Accuracy
Weak Username 1: password123	0.9	0.1	0.9	0	0.27	High Risk	Low
Weak Username 2: ilovecats	0.6	0.4	0.6	0	0.18	Medium Risk	Medium
Weak Username 3: mynameisjohn	0.7	0.3	0.7	0	0.21	High Risk	Low

Strong Username 1: h4ck3r123!@#$	0.5	0.5	0.5	0	0.15	Medium Risk	Medium
Strong Username 2: RandomStr1ng99	0.1	0.9	0.1	0	0.03	Low Risk	High

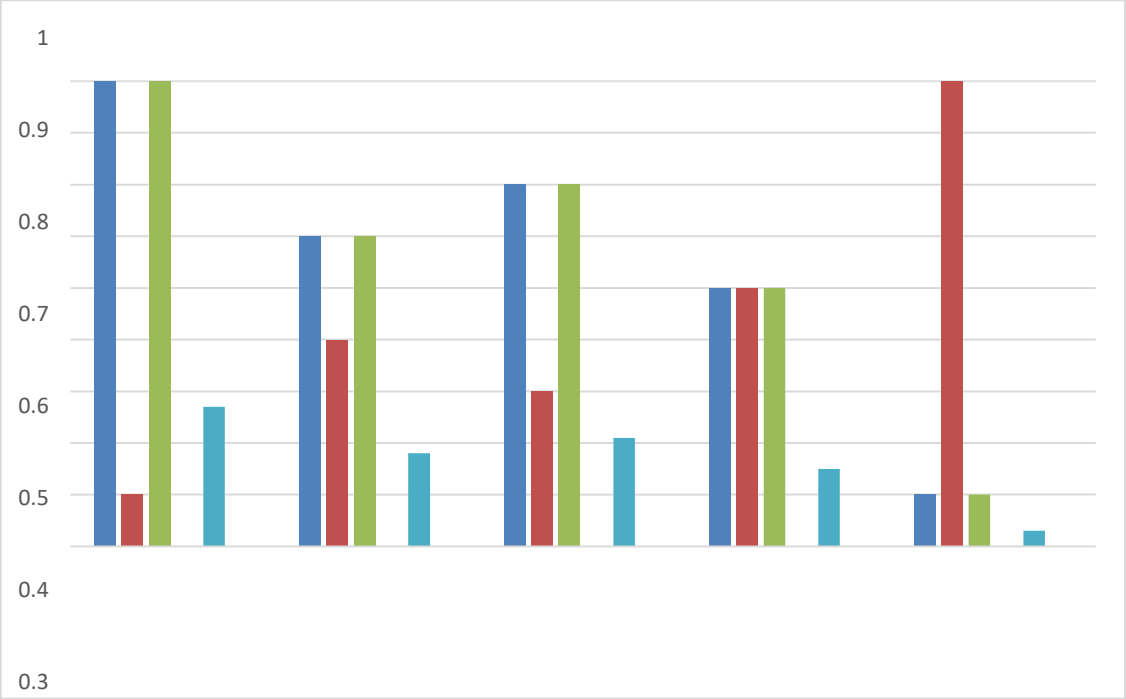

Figure 4: Neutrosophic Risk Assessment of Usernames

Assessing Neutrosophic Accuracy of Usernames

"Neutrosophic accuracy" measures confidence in classifying a username as high-risk or low-risk. This means that the higher the "Truth" and lower the "Falsity" value is, the better the accuracy on classification.

While the table does not give any numerical result for Neutrosophic accuracy, it can be inferred from "Truth" and "Falsity" values:

Commentary on Results (Table 8):

- High Neutrosophic Accuracy: high "Truth" and low "Falsity" reliability classifies a username with a fair amount of certainty (e.g. "Strong Username 2").
- Low Neutrosophic Accuracy: low "Truth" and high "Falsity" reliability classify a username with a low amount of certainty (e.g. "Weak Username 1," "Weak Username 3").

The overall classification accuracy can thus assist us in providing more reliable classification information, as well as improvement needed in the security system.

5.2. Discussion

Neutrosophic Framework Performance

The Neutrosophic framework analyzed the risk associated with user input. Overall, the framework is more informative because it has taken ambiguity and uncertainty into consideration. The framework has cleared high-risk inputs such as malicious scripts and low-risk inputs such as harmless messages.

Key Findings:

- Effective Assessment of Risk: The framework is capable of accurately assessing the risk for different types of inputs.
- Robustness against Adversarial Attacks: The Neutrosophic approach reduces the impact of adversarial attacks through consideration of uncertainty.
- Add it with Range of Adaptability: This framework may fit multiple domains and security requirements, based on customizability.

Limitations and Future Work:

- The performance of the model greatly depends upon the quality and quantity of training data. The dataset should be diverse and representative.
- The effectiveness of the framework is highly dependent on the quality of the extracted features. Thus, different feature engineering techniques can be tried out.

- Selection of models: The selection of the machine-learning model can either positively or negatively affect accuracy and efficiency.
- Continuous Learning: There has to be continuous retraining and updating of the framework so that it can adapt to newly discovered threats and attack techniques.

Conclusion:

Forgetting integration of user inputs may acquire a new vigor under Neutrosophy. Essentially, Neutrosophy takes into account uncertainties and ambiguities in enhancing a wide array of security systems to achieve reliability and precision. Future research and development endeavors within the framework comprise: still some untried areas that need working on-introducing Natural Language Processing and advanced machine learning technology for better extraction of deep insights; Neutrosophy can build on other security techniques that deal with behavioral analysis and anomaly detection to create a strong hybrid approach; bringing up explanatory frameworks for AI that describe the path or reasoning behind their choices thus boosting transparency and trust; providing a constructive direction towards real-time adaptation by enabling changes in model parameters by looking at ever-changing threats in real-time feedback and adjustments. The framework is to be further groomed by taking care of the limitations and the future directions outlined that may ensure more accurate and reliable security assessments of user inputs.

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The Role of Triangular Fuzzy Neutrosophic Set-Based QUALIFLEX in Enhancing University Dance Instruction Assessment

Yu Zhao*
Dance Department, Arts and Physical Education Department, Sangmyung University, Seoul, 03015, Seoul, South Korea
*Corresponding author, E-mail: 18053763303@163.com

Abstract

The evaluation of university dance teaching quality involves a multi-dimensional approach, including the achievement of teaching objectives, the scientific and innovative nature of course content, the diversity of teaching methods, instructors' professional competence, and students' participation and progress. It uses both quantitative and qualitative analysis, incorporating student feedback and classroom observation to ensure comprehensive assessment. Continuously optimizing teaching strategies, enhances students' artistic literacy and overall abilities, supporting the development of high-level dance talent. The college dance teaching quality evaluation is multiple-attribute decision-making (MADM). Recently, the QUALIFLEX method has been established to cope with MADM issues. The triangular fuzzy neutrosophic sets (TFNSs) are established as a tool for characterizing uncertain data during the college dance teaching quality evaluation. In this study, the triangular fuzzy neutrosophic number QUALIFLEX (TFNN-QUALIFLEX) method is established to solve the MADM under TFNSs. Finally, a numerical case study on college dance teaching quality evaluation is provided to validate the proposed method.

Keywords: MADM; TFNSs; QUALIFLEX approach; quality evaluation; neutrosophic sets

1. Introduction

The evaluation of university dance teaching quality is a multi-dimensional process aimed at enhancing educational outcomes and students' overall development. Firstly, it assesses the achievement of teaching objectives, including students' mastery of dance skills and theoretical knowledge, through performances and exams. Secondly, the innovation of course content is crucial. Courses should be updated to reflect the latest dance trends and technologies, inspiring creativity and critical thinking. The diversity of teaching methods is also an important indicator. Teachers should use varied strategies, such as lectures, demonstrations, and interactive practices, to cater to different learning styles, enhancing engagement and learning outcomes. The professional competence and teaching level of instructors directly affect quality, and regular training and

feedback mechanisms can improve their skills. Student participation and progress are evaluated by observing classroom performance and activity involvement, focusing on their growth in technique, artistic expression, and teamwork. By combining quantitative and qualitative analysis, and utilizing student feedback and teaching outcome analysis, the evaluation identifies strengths and weaknesses to optimize teaching strategies. This systematic evaluation not only improves teaching quality but also promotes students' overall development, supporting the cultivation of high-level dance talent. In modern college dance teaching quality evaluation, performance evaluation often employs MADM methods such as TODIM [1]and VIKOR [2]. These methods can comprehensively consider multiple evaluation criteria and conduct multidimensional analysis, providing a thorough performance assessment [3, 4]. Additionally, due to the inherent uncertainty and fuzziness in project management data, tools such as Fuzzy Sets and Neutrosophic Sets are widely used. These tools effectively handle uncertain and fuzzy data, enhancing the accuracy and reliability of the evaluation results [5-7]. The college dance teaching quality evaluation is MADM. Recently, the QUALIFLEX method [8, 9,10] has been developed to address multi-attribute decision-making (MADM) problems. To handle uncertain information in college dance teaching quality evaluation, TFNSs [11] have been introduced as a tool for characterizing such uncertainties in light of neutrosophic sets[12-15]. In this paper, the TFNN- QUALIFLEX method is proposed to solve MADM issues in the context of TFNSs. Finally, a numerical case study is presented to validate the effectiveness of the proposed approach in evaluating the efficiency of resource allocation in university innovation and entrepreneurship education.

Here are the five key points of innovation, including the addition of case and comparative analysis:

- **MADM Framework for Resource Allocation**: The manuscript addresses the college dance teaching quality evaluation as a MADM problem.
- **Integration of QUALIFLEX Method**: The combination of QUALIFLEX and average methods is introduced to handle MADM issues more effectively.
- **Use of TFNSs for Uncertainty Handling**: TFNSs are used to characterize and manage uncertain data during the evaluation process, improving the accuracy of resource allocation assessments.
- **Development of TFNN- QUALIFLEX**: A new method, TFNN- QUALIFLEX, is proposed to solve MADM problems under TFNSs, offering a novel approach to decision-making in uncertain environments.
- **Case Study and Comparative Analysis**: The proposed method is validated through a numerical case study for college dance teaching quality evaluation, and comparative analysis is conducted to demonstrate its effectiveness and advantages over alternative methods.

2. Literature study

The evaluation of university dance teaching quality is a comprehensive process aimed at enhancing teaching standards and students' artistic abilities. Firstly, the evaluation system focuses on achieving teaching objectives by analyzing students' performance in dance skills, theoretical knowledge, and creativity to see if course expectations are met. Secondly, updating and diversifying course content is crucial for improving teaching quality. Courses should incorporate the latest dance trends and technological applications to inspire exploration and innovation, broadening students' artistic perspectives and adaptability. In terms of teaching methods, diverse strategies are a key part of the evaluation. Teachers need to flexibly employ lectures, demonstrations, interactive discussions, and practical exercises to meet the learning needs of different students, enhancing engagement and learning outcomes. The professional development and teaching skills of instructors directly impact quality, and ongoing training and feedback mechanisms can help improve teaching methods. Student participation and progress are important indicators of teaching quality. By observing students' performance in classes and extracurricular activities, their growth in technique, artistic expression, and teamwork is assessed. By combining student feedback, classroom observation, and analysis of teaching outcomes, the evaluation provides a comprehensive understanding of strengths and weaknesses, helping to optimize teaching strategies. Through this systematic evaluation process, university dance teaching not only improves quality but also promotes overall student development, providing a solid foundation for cultivating creative and adaptable dance talent. Over the past decade, research on university dance teaching methods has increased, covering aspects such as new media technology, educational model reform, and teaching quality improvement. The following is a review of 11 related articles, arranged chronologically from past to present. In 2014, Zhao [16] explored the teaching paths of university sports dance courses, emphasizing the role of sports dance in promoting students' physical and mental health. He suggested that the choice of teaching paths should include correcting posture, stimulating interest, and reforming teaching models to improve teaching quality. In 2015, Yang [17] focused on cultivating rhythm awareness in university dance teaching. He highlighted the importance of rhythm awareness in enhancing dance appeal and proposed specific cultivation ideas to help teachers improve students' sense of rhythm. In 2016, Li [18] analyzed the problems in dance teaching at Yunnan University Dianchi College, such as insufficient faculty and inadequate teaching facilities. Through questionnaires and on-site analysis, he proposed improvement measures to enhance teaching quality. In 2017, Wang [19] explored ways to improve the quality of dance teaching for undergraduate preschool education majors. She pointed out the important role of dance education in children's physical and intellectual development and analyzed methods to improve teaching quality. In 2019, Zhang [20] discussed the positioning and reform of dance teaching in comprehensive universities. She believed that dance teaching plays an important role in talent cultivation and proposed reform measures to enhance students' comprehensive quality. In the same year, Xu [21] studied the application of micro-courses in university sports dance teaching. She noted that micro-courses provide students with flexible

learning time arrangements through video teaching, thereby improving teaching quality and learning efficiency. In 2020, Tang and Wu [22] discussed the interpenetration strategy of traditional art protection and campus culture innovation in the Three Gorges Reservoir Area. Using the Music and Dance College of Chongqing Normal University as an example, they analyzed how to achieve the combination of education and heritage through cultural innovation. In 2021, Duan and Hong [23] studied the impact of new media technology on university dance performance teaching. They pointed out that new media technology can enrich teaching methods and improve the problem of teaching content being out of touch with the times, thereby enhancing teaching quality. In 2022, Ge [24] applied goal-oriented theory to study the teaching design of dance courses in senior universities. She proposed specific teaching goals and diversified teaching methods to improve the quality of dance education in senior universities. In 2024, Zhao and Liu [25] explored the application of the "MOOC+flipped classroom" model in university sports dance courses. They found that this model improved students' personalized learning and academic performance, emphasizing the importance of teacher guidance. In the same year, Ma [26] studied the application of micro-video-assisted teaching in the sports dance courses at Yan'an University. She demonstrated through experiments that micro-video-assisted teaching significantly improved students' learning interest and ability, providing a reference for enhancing teaching effectiveness.

3. Preliminaries

This section shows some definitions of the TFNN [28,29] as:

Definition 1. The TFNSs can be presented as:

$$Y = \{(x), T_y(x), I_y(x), F_y(x) | \, x \in X\} \tag{1}$$

$$T_y(x) = \left(T_y^L(x), T_y^M(x), T_y^U(x)\right), 0 \le T_y^L(x) \le T_y^M(x) \le T_y^U(x) \le 1 \tag{2}$$

$$I_y(x) = \left(I_y^L(x), I_y^M(x), I_y^U(x)\right), 0 \le I_y^L(x) \le I_y^M(x) \le I_y^U(x) \le 1 \tag{3}$$

$$F_y(x) = \left(F_y^L(x), F_y^M(x), F_y^U(x)\right), 0 \le F_y^L(x) \le F_y^M(x) \le F_y^U(x) \le 1 \tag{4}$$

The truth membership function can be defined as:

$$T_y(x) = \begin{cases} \frac{x - T_y^L(x)}{T_y^M(x) - T_y^L(x)}, & T_y^L(x) \le x T_y^M(x) \\ \frac{x - T_y^U(x)}{T_y^M(x) - T_y^U(x)}, & T_y^M(x) \le x T_y^U(x) \\ 0, & otherwise \end{cases} \tag{5}$$

Definition 2. Let two TFNNs as:

$$y_1 = \left\{\left(T_{y_1}^L(x), T_{y_1}^M(x), T_{y_1}^U(x)\right), \left(I_{y_1}^L(x), I_{y_1}^M(x), I_{y_1}^U(x)\right), \left(F_{y_1}^L(x), F_{y_1}^M(x), F_{y_1}^U(x)\right)\right\}$$

$$y_2 = \left\{\left(T_{y_2}^L(x), T_{y_2}^M(x), T_{y_2}^U(x)\right), \left(I_{y_2}^L(x), I_{y_2}^M(x), I_{y_2}^U(x)\right), \left(F_{y_2}^L(x), F_{y_2}^M(x), F_{y_2}^U(x)\right)\right\}$$

$$y_1 \oplus y_2 = \left\{ \begin{pmatrix} T_{y_1}^L(x) + T_{y_2}^L(x) - T_{y_1}^L(x)T_{y_2}^L(x), T_{y_1}^M(x) + T_{y_2}^M(x) - \\ T_{y_1}^M(x)T_{y_2}^M(x), T_{y_1}^U(x) + T_{y_2}^U(x) - T_{y_1}^U(x)T_{y_2}^U(x) \end{pmatrix}, \right. \\ \left(I_{y_1}^L(x)I_{y_2}^L(x), I_{y_1}^M(x)I_{y_2}^M(x), I_{y_1}^U(x)I_{y_2}^U(x) \right), \\ \left. \left(F_{y_1}^L(x)F_{y_2}^L(x), F_{y_1}^M(x)F_{y_2}^M(x), F_{y_1}^U(x)F_{y_2}^U(x) \right) \right\} \tag{6}$$

$$y_1 \otimes y_2 = \left\{ \left(T_{y_1}^L(x)T_{y_2}^L(x), T_{y_1}^M(x)T_{y_2}^M(x), T_{y_1}^U(x)T_{y_2}^U(x) \right), \right. \\ \begin{pmatrix} I_{y_1}^L(x) + I_{y_2}^L(x) - I_{y_1}^L(x)I_{y_2}^L(x), I_{y_1}^M(x) + I_{y_2}^M(x) - \\ I_{y_1}^M(x)I_{y_2}^M(x), I_{y_1}^U(x) + I_{y_2}^U(x) - I_{y_1}^U(x)I_{y_2}^U(x) \end{pmatrix}, \\ \left. \begin{matrix} F_{y_1}^L(x) + F_{y_2}^L(x) - F_{y_1}^L(x)F_{y_2}^L(x), F_{y_1}^M(x) + F_{y_2}^M(x) - \\ F_{y_1}^M(x)F_{y_2}^M(x), F_{y_1}^U(x) + F_{y_2}^U(x) - F_{y_1}^U(x)F_{y_2}^U(x) \end{matrix} \right\} \tag{7}$$

Definition 3. The score function can be computed as:

$$S(y) = \frac{1}{12} \begin{bmatrix} 8 + \left(T_y^L(x) + 2T_y^M(x) + T_y^U(x) \right) - \\ \left(I_y^L(x) + 2I_y^M(x) + I_y^U(x) \right) - \\ \left(F_y^L(x) + 2F_y^M(x) + F_y^U(x) \right) \end{bmatrix} \tag{8}$$

4. TFNN- QUALIFLEX Approach

This section provides a clear explanation of the steps involved in the proposed technique.

Step 1: Constructing the Decision Matrix

In this step, the decision-makers use the Triangular Fuzzy Neutrosophic Numbers (TFNN) to assess both the criteria and the alternatives. Once the evaluations are completed, a score function is applied to transform the fuzzy values into crisp numerical values. These crisp values are then aggregated to form a unified decision matrix. Following this, the criteria are calculated using the average method, ensuring a balanced and accurate representation of their relative importance.

Step 2: Generating the Initial Alternative Permutation

After preparing the decision matrix and determining the criteria weights, an initial order or permutation of the alternatives is created. This serves as the foundation for further analysis and ranking in subsequent steps.

The number of permutations is created by $m!$ As:

$Per1 = \{ A1, A2, A3, A4, A5, A6, A7, A8\}$

$Per2 = \{A1, A3, A4, A5, A6, A7, A8, A2\}$

$Per3 = \{A1, A2, A4, A5, A6, A7, A8, A3\}$

$Per4 = \{A1, A2, A3, A5, A6, A7, A8, A4\}$

$Per5 = \{A1, A2, A3, A4, A6, A7, A8, A5\}$

$Per6 = \{A1, A2, A3, A4, A5, A7, A8, A6\}$

$Per7 = \{A1, A2, A3, A4, A5, A6, A8, A7\}$

Step 3: Establishing the Initial Ranking of Alternatives

In this step, the alternatives are arranged in order, from the best to the worst, based on the evaluations provided by the experts. These rankings reflect the collective opinions of the decision-makers and form the groundwork for further analysis. This ensures that each alternative is initially assessed and ranked according to its perceived value or performance.

Step 4: Determining Dominant and Dominated Values

Here, the dominant and dominant values for each alternative are identified. The dominant value represents how strongly an alternative outperforms others, while the dominant value indicates how much it is outperformed by other alternatives. This step helps in understanding the relative strengths and weaknesses of each option within the decision-making framework.

$A1 > A2 = A3$

$Per = \{A2.A1, A3\}$

$A2 < A1 \rightarrow -1$

$A2 = A3 \rightarrow 0$

$A3 < A1 \rightarrow +1$

Step 5: Aggregating the Permutation Values of Criteria

At this stage, the permutation values associated with the criteria are aggregated. This process combines the rankings or weights assigned to each criterion, ensuring a comprehensive evaluation that accounts for the importance of each factor in the decision-making process. The goal is to achieve a unified representation of the criteria's influence.

Step 6: Aggregating the Permutation Values of Alternatives

Once the criteria have been aggregated, the next step is to aggregate the permutation values of the alternatives. This involves combining the evaluations of the alternatives across all criteria into a single set of values. This aggregation helps simplify the comparison and provides a clear overall performance measure for each alternative.

Step 7: Determining the Final Ranking of Alternatives

In the final step, the aggregated values are used to rank the alternatives from the best to the worst. This step concludes the process by providing a definitive order of the alternatives based on their overall performance, enabling the decision-makers to identify the most suitable choice.

5. Case Study and Comparison

This section shows the results of the proposed method. Eight well-known dance colleges are evaluated with 17 attributes:

A. Teaching Effectiveness

The degree to which dance instruction leads to measurable improvements in students' technical skills, artistry, and confidence.

B. Choreographic Skills Development

Opportunities for students to learn and apply choreography techniques, fostering creativity and innovation in dance creation.

C. Technical Proficiency Training

Focus on developing fundamental and advanced dance techniques, including posture, movement precision, and coordination.

D. Performance Skills

The ability to train students to perform confidently and expressively on stage in diverse settings.

E. Individualized Attention

Customization of teaching methods to address the unique strengths, weaknesses, and goals of each student.

F. Engagement and Motivation

Strategies used by instructors to keep students actively involved, inspired, and committed to their craft.

G. Curriculum Design and Relevance

Inclusion of modern and classical dance forms, balanced integration of theory, practice, and interdisciplinary learning.

H. Professional Development Opportunities

Exposure to workshops, masterclasses, and career guidance for students aiming to pursue dance professionally.

I. Assessment and Feedback Quality

Timely, constructive, and specific feedback is provided to students for continuous improvement in their performance.

J. Cultural and Artistic Exposure

Encouraging students to understand and appreciate the historical, cultural, and global contexts of dance.

K. Innovation in Teaching Methods

Use of modern teaching techniques, including multimedia tools and interactive learning

methods, to enhance instruction.

L. Physical Conditioning and Health

Ensuring students develop strength, flexibility, and stamina while promoting practices that prevent injuries.

M. Rehearsal and Practice Management

Structuring and supervising rehearsals effectively to maximize productivity and ensure performance readiness.

N. Collaborative Skills Development

Training students to work cohesively in group performances and learn from peer interactions.

O. Resource and Facility Quality

Availability and adequacy of dance studios, practice spaces, costumes, and other necessary resources.

P. Emotional and Psychological Support

Providing a supportive environment that fosters confidence, resilience, and emotional well-being.

Q. Program Reputation and Community Engagement

The visibility, credibility, and outreach of the college's dance program in the local and broader arts community.

5.1 Example

We have the initial decision matrix with three criteria and three alternatives.

The first decision matrix is formed as follows:

$$\begin{bmatrix} 1.3 & 12 & 3000 \\ 2 & 20 & 1000 \\ 2 & 10 & 2000 \end{bmatrix}$$

The criteria weights are obtained from the decision matrix.

Then the 6 permutations is obtained for instance

$Per1 = A1 > A2 > A3$

$Per2 = A3 > A2 > A1$

$Per3 = A2 > A1 > A3$

The initial raking of alternatives is computed as:

1 2 1
2 1 3
2 3 2

The dominant values

$Per\ 1 = A1 > A2 > A3\ for\ c1$

$A2 < A1 \rightarrow 1$

$A2 = A3 \rightarrow 0$

$A3 = A1 \rightarrow 1$

Then the final rank of alternatives is A1

Table 1. Matrix of Information.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_2	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_3	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)
C_4	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_5	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_6	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_7	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_8	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_9	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)
C_{10}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{11}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_{12}	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_{13}	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{14}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{15}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)
C_{16}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_{17}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_2	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_3	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)
C_4	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_5	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_6	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_7	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_8	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_9	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.8,0.5,0.6),(0.3,0.4,0.5)

C_{16}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{15}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{12}	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_{13}	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{14}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{18}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{16}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{17}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_2	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_3	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_4	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_5	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_6	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_7	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_8	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_9	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{10}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{11}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)
C_{12}	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_{13}	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{14}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.1,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{15}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{16}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{17}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)

Step 1: The process begins with the construction of the decision matrix, as illustrated in Table 1. This matrix captures the performance evaluations of all alternatives against the given criteria. Following this, the criteria weights are calculated to reflect their relative importance in the decision-making process. The methodology for calculating these weights is demonstrated in Figure 1, providing a visual aid to enhance understanding.

Step 2: Next, an initial permutation of the alternatives is created. This involves listing all possible orders or rankings of the alternatives. The total number of permutations is determined by the factorial of the number of alternatives, represented as m! where m is the total number of alternatives. This step sets the stage for subsequent evaluation and analysis.

Per 1 = A1 > A2 > A3 > A4 > A5 > A6 > A7 > A8

Per 2 = A1 > A3 > A4 > A5 > A6 > A7 > A8 > A2

Per 3 = A1 > A2 > A4 > A5 > A6 > A7 > A8 > A3

Per 4 = A1 > A2 > A3 > A5 > A6 > A7 > A8 > A4

Per 5 = A1 > A2 > A3 > A4 > A6 > A7 > A8 > A5

Per 6 = A1 > A2 > A3 > A4 > A5 > A7 > A8 > A6

Per 7 = A1 > A2 > A3 > A4 > A5 > A6 > A8 > A7

Per 8 = A2 > A3 > A4 > A5 > A6 > A7 > A8 > A1

Per 9 = A2 > A4 > A5 > A6 > A7 > A8 > A1 > A3

Per 10 = A2 > A3 > A5 > A6 > A7 > A8 > A1 > A4

Per 11 = A2 > A3 > A4 > A6 > A7 > A8 > A1 > A5

Per 12 = A2 > A3 > A4 > A5 > A7 > A8 > A1 > A6

Per 13 = A2 > A3 > A4 > A5 > A6 > A8 > A1 > A7

Per 14 = A2 > A3 > A4 > A5 > A6 > A7 > A1 > A8

Per 15 = A3 > A4 > A5 > A6 > A7 > A8 > A1 > A2

Per 16 = A3 > A5 > A6 > A7 > A8 > A1 > A2 > A4

Per 17 = A3 > A4 > A6 > A7 > A8 > A1 > A2 > A5

Per 18 = A3 > A4 > A5 > A7 > A8 > A1 > A2 > A6

Per 19 = A3 > A4 > A5 > A6 > A8 > A1 > A2 > A7

Per 20 = A3 > A4 > A5 > A6 > A7 > A1 > A2 > A8

Per 21 = A3 > A4 > A5 > A6 > A7 > A8 > A2 > A1

Per 22 = A4 > A5 > A6 > A7 > A8 > A1 > A2 > A3

Per 23 = A4 > A6 > A7 > A8 > A1 > A2 > A3 > A5

Per 24 = A4 > A5 > A7 > A8 > A1 > A2 > A3 > A6

Per 25 = A4 > A5 > A6 > A8 > A1 > A2 > A3 > A7

Per 26 = A4 > A5 > A6 > A7 > A1 > A2 > A3 > A8

Per 27 = A4 > A5 > A6 > A7 > A8 > A2 > A3 > A1

Per 28 = A4 > A5 > A6 > A7 > A8 > A1 > A3 > A2

Step 3: Create the initial ranking of alternatives based on expert evaluations.

Step 4: Identify the dominant and dominated values for each alternative.

Step 5: Aggregate the permutation values of the criteria.

Step 6: Aggregate the permutation values of the alternatives.

Step 7: Determine the final ranking of the alternatives, as shown in Figure 2.

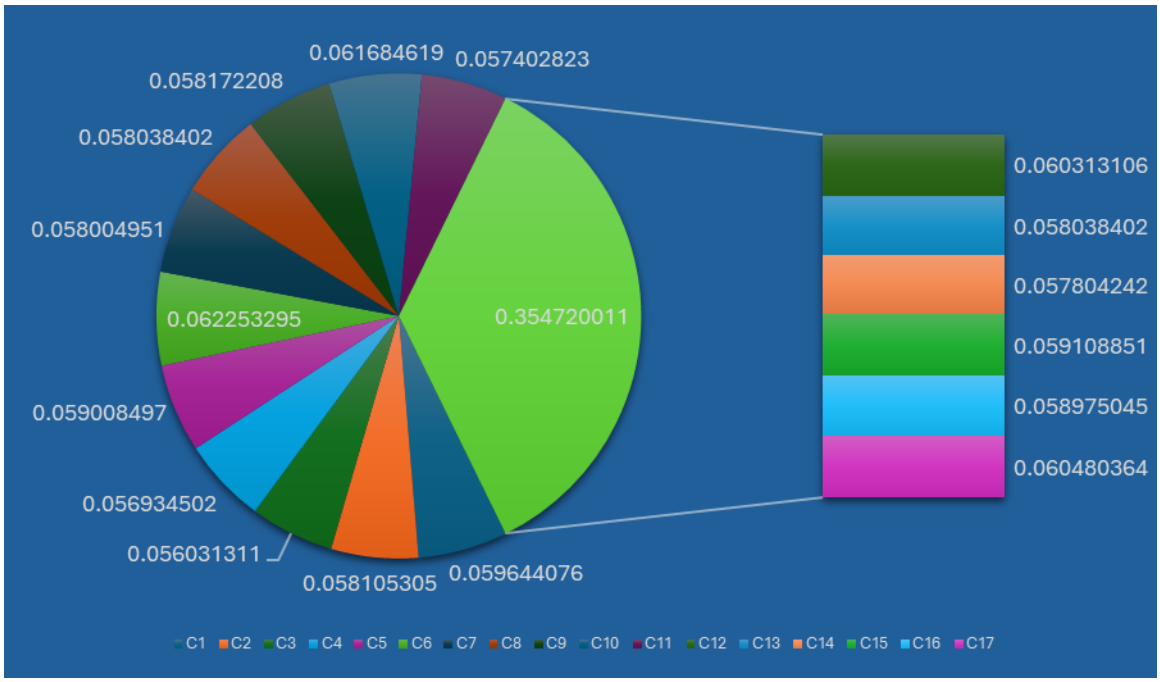

Figure 1. Criteria Weighting

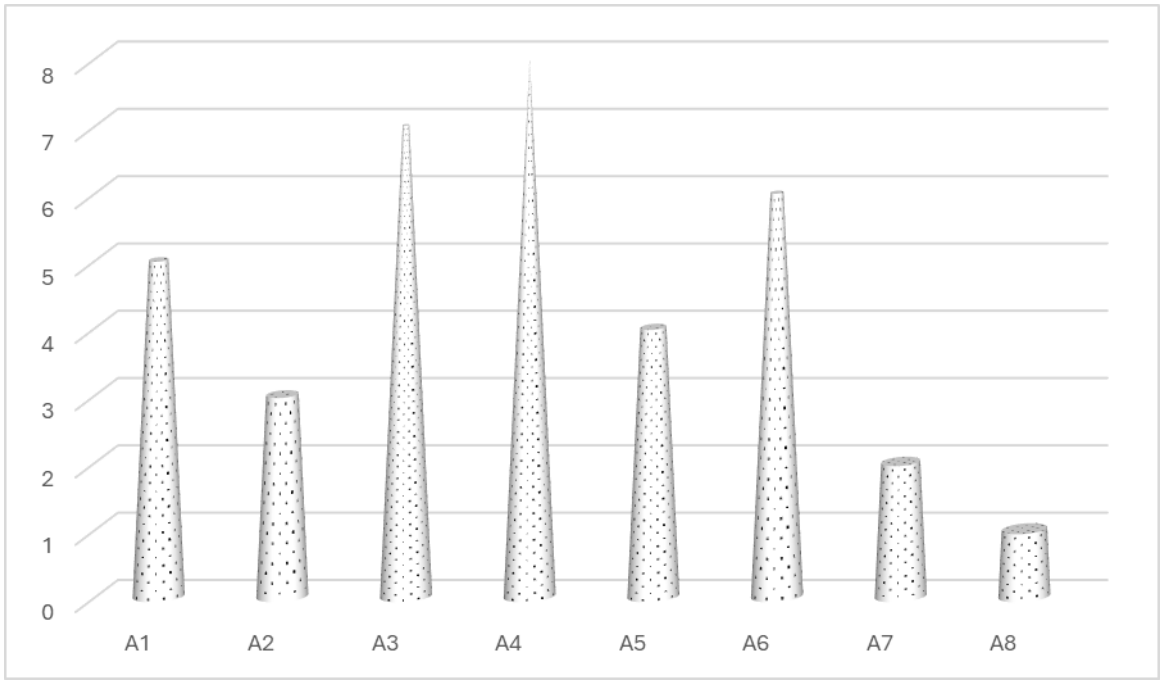

Figure 2. Priority of Alternatives

Evaluating the quality of university dance teaching is a comprehensive process aimed at enhancing educational outcomes and students' artistic literacy. The evaluation system includes several key dimensions:

Firstly, the achievement of teaching objectives is central, ensuring students acquire the necessary

dance skills and theoretical knowledge through clear plans and assessments.

Secondly, the scientific and innovative nature of the course content is crucial. Courses need to be updated regularly to reflect the latest dance trends and techniques while maintaining a balance between theory and practice to inspire creativity.

The diversity of teaching methods is also an important aspect of evaluation. Teachers should employ various strategies, including lectures, demonstrations, interactive sessions, and practical activities, to accommodate different learning styles. This not only increases classroom engagement but also enhances students' understanding and application skills.

The professional competence and teaching level of instructors are core to the evaluation. High-quality teachers can inspire and guide students through their rich professional experience and teaching skills.

Regular training and development programs help maintain a high level of professional expertise among faculty. Student participation and progress are crucial indicators of teaching effectiveness. Evaluating students' growth and progress involves observing their performance in class, showcasing their work, and their involvement in various dance activities.

Additionally, the combination of quantitative and qualitative analysis ensures a comprehensive and objective evaluation. Student feedback, classroom observation, and analysis of teaching outcomes are important tools for identifying strengths and weaknesses in teaching, thereby optimizing strategies. Through this systematic evaluation process, university dance teaching can improve its quality and promote students' overall development, providing strong support for cultivating high-level dance talent with creativity and artistic literacy. The college dance teaching quality evaluation is a MADM issue. We show that alternative 4 is the best and alternative 8 is the worst.

5.2. Comparative analysis

The TFNN-QUALIFLEX approach is compared with the TFNN-VIKOR approach [30], the TFNN-MABAC approach [27], and the TFNN-EDAS approach [31]. The comparative results are established in Figure 3.

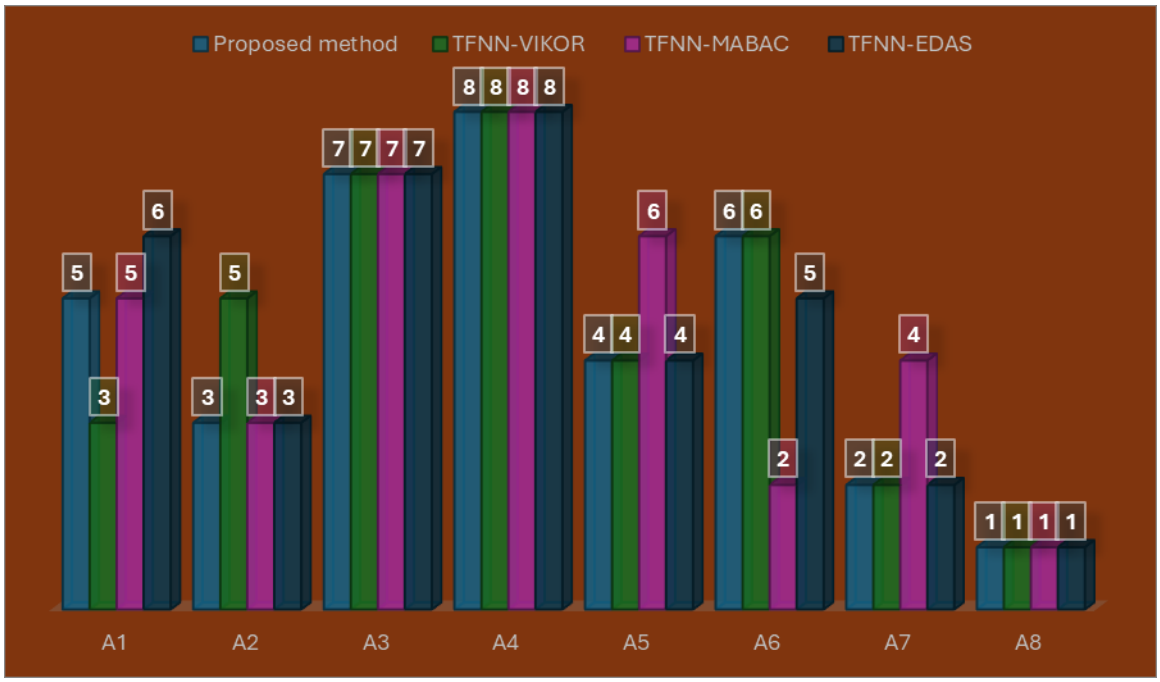

Figure 3. Comparison of Methods

Based on the detailed analysis above, it is evident that all models identify the same optimal locally well-known dance college and the ranking of the eight dance colleges is consistent across the methods. This confirms that the TFNN-QUALIFLEX approach is both reasonable and effective.

When comparing the proposed method with existing approaches, it becomes clear that each method has its strengths and weaknesses in addressing MADM problems. However, the existing methods show certain limitations when applied to complex and uncertain college dance teaching quality evaluations.

The proposed TFNN- QUALIFLEX method offers a robust framework for decision-making, excelling in three key areas: handling uncertainty, accommodating non-linear preferences and risk attitudes, and considering multiple objectives and trade-offs.

- *Stronger Capability in Handling Uncertainty:* The method employs Triangular Fuzzy Neutrosophic Sets (TFNSs) to represent truth-membership, indeterminacy-membership, and falsity-membership. This representation allows for a more comprehensive capture and portrayal of uncertain information. By not only considering the fuzziness inherent in decision-making data but also distinguishing among the three dimensions of truth, uncertainty, and falsity, the method achieves more accurate decisions in highly uncertain environments. This capability is particularly advantageous in complex scenarios where data may be incomplete or ambiguous, providing a structured approach to navigate uncertainty with greater precision.

In summary, the TFNN-QUALIFLEX method enhances decision-making by effectively managing uncertainty and enabling balanced trade-offs among multiple objectives. This approach empowers decision-makers to navigate complex environments with greater confidence and precision, making it a valuable tool for various applications where strategic and informed decisions

are crucial.

6. Conclusion

The evaluation of university dance teaching quality is a multi-dimensional process aimed at enhancing educational outcomes and students' artistic literacy. It covers the achievement of teaching objectives, the scientific and innovative nature of course content, the diversity of teaching methods, instructors' professional competence, and students' participation and progress. By combining quantitative and qualitative analysis, the evaluation system utilizes student feedback, classroom observation, and teaching outcome analysis to comprehensively identify strengths and weaknesses in teaching. Teachers should employ diverse strategies to accommodate different learning styles, increasing engagement and application skills. This systematic evaluation not only optimizes teaching quality but also promotes students' overall development, supporting the cultivation of innovative dance talent. The college dance teaching quality evaluation is a MADM problem. Recently, the QUALIFLEX method has been advanced to tackle MADM challenges. This approach utilizes TFNSs to effectively characterize and manage uncertain data throughout the evaluation process. Building on this foundation, the study introduces a TFNN- QUALIFLEX model specifically designed to address MADM issues within the framework of TFNSs. This innovative model integrates the strengths of TFNSs in handling uncertainty with the capabilities of the QUALIFLEX methods, providing a comprehensive solution for decision-making scenarios where ambiguity and complexity are prevalent. The TFNN- QUALIFLEX model enhances decision-making accuracy by accommodating the nuances of uncertain information and addressing the diverse preferences and risk attitudes of decision-makers. By leveraging the exponential form of the QUALIFLEX method, the model can flexibly adapt to various risk profiles, making it particularly suitable for complex project management environments where multiple objectives and conflicting goals must be balanced. To demonstrate the practical applicability and effectiveness of the proposed method, a numerical case study is conducted. This case study serves as a validation tool, illustrating how the TFNN- QUALIFLEX model can be applied to real-world scenarios, thereby confirming its utility and robustness in solving MADM problems. Through this study, the model proves to be an asset in enhancing decision-making processes, offering a structured approach to navigate uncertainty and complexity with greater precision and confidence.

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On Quick Switching System with Different Sampling Plans using Fuzzy Neutrosophic Weighted Poisson Distribution

Vennila J [1], Basker P [2,*], Kripa Josten [3] and Prakash V.S [4,*]

[1, 3] Manipal College of Health Professions, Manipal Academy of Higher Education, Manipal – 576 104. Karnataka. India. E-mail: dr.j.vennila@gmail.com.

[2] Department of Mathematics, Chandigarh University, Punjab – 140 413. India. E-mail: mcpdbasker@gmail.com and basker.e11236@cumail.in

[4] Department of Computer Science, Kristu Jayanti College, Bengaluru – 140 413. Karnataka. India. E-mail: vsprakash@kristujayanti.com.

*****Correspondence: mcpdbasker@gmail.com, basker.e11236@cumail.in and vsprakash@kristujayanti.com

Abstract: The present study provides a new sampling inspection approach to costly and destructive testing that involves the concept of neutrosophic and fuzzy logic in a quick switching system with a normal SSP and a tightened DSP as a reference plan using weighted poisson distribution as a baseline. It includes the neutrosophic and fuzzy weighted poisson distribution associations of this approach, along with normal and tightened acceptance numbers using OC functions, and OC curves using R programming. It strongly focuses on a new sampling approach that uses fuzzy logic for effectively dealing with inconsistent and uncertain terms. And Also, these sets can be used to enhance the adaptability and flexibility of quick switching sampling. Neutrosophic consists of three terms: truthiness (t), indeterminacy (i), and falsity (f), which refer to different scenarios. The efficacy of the updated system has been evaluated through comparison of its results to the existing one. Costly and destructive testing using neutrosophic and fuzzy logic in a QSS with a normal SSP and a tightened DSP as a reference plan. The purpose of this research is to address the limitations associated with standard sampling techniques for continuous manufacturing process measurement. This focuses on the challenges with identifying uncertainties-related defects and highlights the fields involving fuzzy logic and neutrosophic theory have not been completely integrated into appropriate sampling techniques. This new approach improves the accuracy of quality control decision-making and significantly decreases the sample sizes involved. This provides an effective approach to improve quality monitoring across manufacturing scenarios.

Keywords: Neutrosophic logic, fuzzy set theory, statistical quality control, quick switching system, weighted poisson distribution, single sampling plan, double sampling plan.

1. Introduction

The present continuous manufacturing quality evaluation approaches find a challenge to maintain the variability and uncertainty in defects assessment, especially with costly and destructive testing. Variations in quality control can results in the fixed methods employed in standard sample techniques, that have a tendency to modify to the changing nature of manufacturing procedures. As inaccurate assessments of quality may contribute to significant economic losses and risk the integrity of goods, it is essential than it be solved. When a provider can regularly detect weaknesses, it will grow less effective and lose the confidence consumers have in domains where reliability is important. So, in modern extremely difficult manufacturing environment, the growth of further adaptable and

reliable methods for sampling is essential. Quick Switching Systems (QSS) are advanced attribute-acceptance sampling methods that goal at improving the accuracy of sampling based on the process quality levels. Dodge (1967) proposed QSS, that includes normal and tightened inspections plans. Romboski (1969) modified QSS with types that included QSS (n, cN, cT) and QSS (n, kn, c0), each modified for specific quality control requirements. Soundararajan and Arumainayagam (1990a, 1990b) developed QSS-r (n;cN,cT) and QSS-r (n;kn,c0) for r = 2 and 3, establishing various types of attribute different possibilities for predicting various operating conditions. Tightened plans are used for lesser quality levels that require high normal of quality, while normal plans are used for greater quality levels for reducing sample numbers and preserve the detection of errors.

The weighted poisson distribution (WPD) increases defect control through providing various significance levels to defects due to their importance, compared to the normal Poisson model, that accepts each mistake the same. This approach is particularly effective for industries which work with second-quality lots or have a significant number of manufacturing defects. Industries may prioritize resources by prioritizing errors depending on their effect, along with the objective of increasing overall quality of goods and consumer satisfaction. This method is particularly effective in industries switching to zero defects manufacturing methodologies, since it provides a more structured procedure for resolving quality problems simultaneously reducing managerial effectiveness and resource allocation.

Radhakrishna Rao (1977) established the weighted poisson distribution (WPD) as a fundamental distribution for designing sampling methods. Sudeswari (2002) developed sampling schemes constructed on the WPD methods concept and emphasized its implementation in plan construction. Radhakrishnan and Mohana Priya (2008) indicated that sampling plans utilizing the WPD possess a higher acceptance probability and require a smaller number of sample sizes that methods based on the Poisson distribution, and this helps consumers as well as producers through improving the effectiveness of resources. Arumainayagam and Vennila (2017) evaluated a QSS to a NSSP and a tightened TDSP, that provides a more effective approach. It follows that investigated QSS using different sampling plans in 2018, then implemented QSS with a zero-one acceptance number have studied in 2019. Furthermore, Arumainayagam and Vennila (2020) assessed QSS incorporating weighted poisson distribution with single and double sampling plans as reference plan, expanding its utility in quality control strategies and studied the QSS with DSP as reference plan (QSDSS) across various quality regions (2021). In 2022, they studied QSS using SSP and DSP based on MIL-STD-105D. For the purpose to investigate the relationships between nine criteria, Victor Christianto (2024) did studies using Single-Valued Neutrosophic Sets (SVNSs) combined with the DEMATEL strategy. In accordance with professional evaluations, a clear relationship matrix was developed, highlighting that factor 1 had a small affect and criteria 5 had the most impact. The above approach reduced the uncertainty of the evaluation procedure significantly. Mai Mohamad (2024) has studied the SVTNSs-LMAW-RAWEC methods for selecting drone materials, identifying carbon fiber-reinforced polymer as the best option. The approach effectively manages uncertainty but has limitations with zero elements in the decision matrix. Future work will expand the analysis to more criteria and alternatives. The use of fuzzy logic and neutrosophic concepts, however, aren't sufficiently frequently included into investigations that presently be found, but evidence it may significantly enhance the quality of decision-making processes.

Most of the studies have already been done to rewrite sampling techniques with fuzzy logic is used to identify uncertainty. In normal sampling plans, the product's defective proportion is presented as a specific number. However, there could be significant uncertainty in applications in real life. Conventional logic evaluates reports by providing two absolute values: 0 and 1. The above concept is unable to replicate uncertainty in both natural and engineering scenarios. The most significant difference between classical logic and fuzzy logic is the application of the term

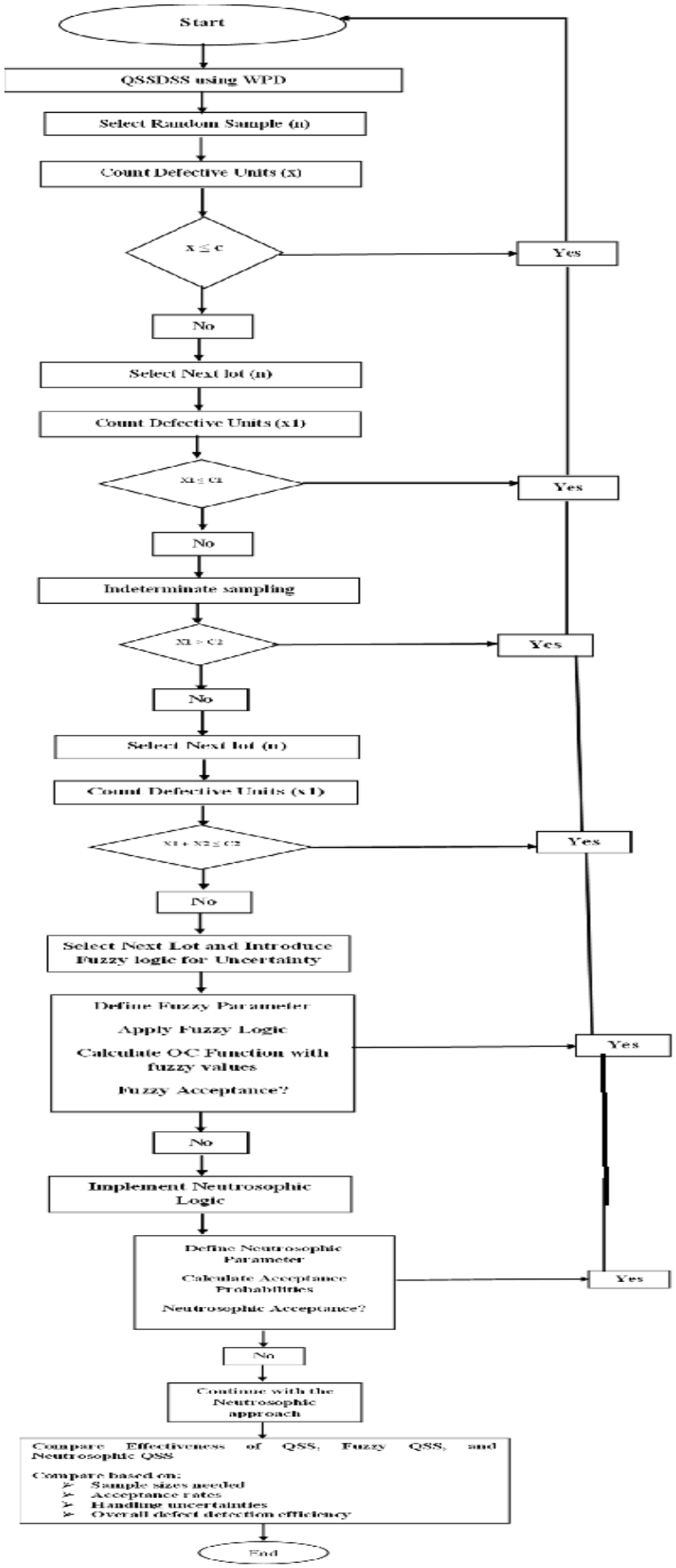
"membership" to describe uncertainties. The method of fuzzy logic uses a membership function ((x)) and a variable that is continuous (x) within the [0,1] range in a space (X) as the uncertainty level. The first equation explains a basic fuzzy set and also if there is a lot of uncertainty, the membership value is higher, and both directions.

This article baselines an innovative sampling inspection framework that integrates a normal single sampling plan (n; c) and a tightened double sampling plan (n; c1, c2) using the assumption of a Weighted Poisson distribution (WPD). This mixture system is termed as the Weighted Poisson Distribution-Quick Switching Single Double Sampling System (WPD-QSSDSS) (n; c; c1, c), where n=n1=n2. This work was already done by Arumainayagam and Vennila (2020). The condition c1=0, c, c1 > c2 is utilized to enhance protective measures in quality control processes. This approach is particular in that it combines Fuzzy logic with professional Neutrosophic basic concepts to effectively deal with uncertainty, fuzzy thoughts, and unpredictability that comes with parameter estimate. The use of fuzzy logic enables the system to change to uncertain conditions with different types of certainty in parameter values, maximizing reliable decision-making throughout sampling inspections when conventional methods have difficulty using insufficient or inaccurate information.

According to Aslam (2019), the acceptance sampling plan using attribute and variable that proposed an updated attribute sampling plan that find lot accepted or rejected, and unspecified chances of the Neutrosophic Binomial distribution and interval method using different acceptance number. Uma and Nandhitha (2022) examined the impact of Neutrosophic sets on Acceptance Sampling plans using attribute sampling plans. Uma and Nandhitha (2023) assessed QSS utilizing Neutrosophic Poisson Distribution with corresponding OC curve, and relevant tables were designed.

Neutrosophic sets were a type of fuzzy sets containing three phrases: membership (truthiness), non-membership (falsity), and uncertainty. Applying arbitrary terms and solving multiple sets provides neutrosophic logic equivalent to logical theory. As a result, it gives excellent outcomes for predicting uncertainty that relates to scenarios with beings. A few studies have been claimed to have invented sample plans using neutrosophic sets, but none of them contain along with uncertainty terms and unreliable information that in an identical give structure and description. From the literature, the term uncertainty is minimized through applying a few assumptions and investigating acceptance sampling plans constructed using neutrosophic sets. The primary aim of this work is to formulate a QSS that utilizes neutrosophic weighted poisson distribution using QSS with normal SSP and tightened DSP as reference plan.

The aim of this study is to fill in the gaps that are identified with an extensive structure which integrates effective sample addresses using fuzzy logic and neutrosophic approaches. A significant number of current sampling techniques present the inaccurate a proportion as a constant amount, not considering the variability inherent in practical applications. The present research proposes a Quick Switching System (QSS) using normal Single Sampling Plans (SSP) and tightened Double Sampling Plans (DSP) as reference plans, based on the Neutrosophic Weighted Poisson Distribution (NWPD). The main objective of the effort is to provide a systematic way for managing uncertainty in quality assessments through the combination of different approaches.



Flowchart 1: QSS, Fuzzy logic and Neutrosophic logic with single double sampling plan using weighted distribution as reference plan

The use of Neutrosophic logic expands this model through allowing an additional level of uncertainty, however along with the two main ones, which is essential to managing concerns that improve across fuzzy limits. This combination optimizes the ability of the system for dealing with difficult processes in factories by optimizing sample sizes and checking processes with the WPD method using the QSSDSS using fuzzy logic and neutrosophic set. It enhances the stability and flexibility of quality assurance methods, but additionally, it deals with occurrences with significant data uncertainty and variability. It highlights that Neutrosophic concepts utilize Fuzzy logic to address concerns deeper in the scenario of attribute determine and the sampling process assessments.

This study provides the Weighted Poisson Distribution Quick Switching Single Double Sampling System (WPD-QSSDSS), an innovative technique which increases reliability and minimizes the sample sizes required for effective quality control. The use of fuzzy logic was incorporated in this new method to deal with uncertainty in factor forecasts to improve accurate decision-making processes. In the end, this study contributes to improve manufacturing quality control methods, ensuring that high standards are continuously satisfied.

2. Weighted Poisson Distribution using QSSDSS

The present study uses WPD-QSSDSS (n; c; c1; c2) under the assumption of WPD. The conditions for the applications are given below

i. A continuous manufacturing method includes evaluating many successive in batches.
ii. Every batch will be expected to be nearly identical in quality.
iii. The batches are examined in the manner during which they were manufactured.
iv. Quality is determined through the proportion of defects.

Operating Procedure

Step 1: Choose a random sample of size n in a lot and check the no. of defective units x.

(i) if $x \le c$, accept the lot and continue step 1for the next lot.
(ii) if $x > c$, reject the lot and continue step 2 for the next lot.

Step 2: Choose a random sample of size n in the next lot and check the no. of defectives x1.

(i) if $x1 \le c1$, accept the lot and continue step 1 for the next batch.
(ii) if $x1 > c2$, reject the lot and continue step 2 for the next batch.
(iii) Select a second random sample of size n in the same batch if $c1 < x1 \le c2$ and check the no.of defectives x2.
(iv) if $x1+x2 \le c2$, accept the lot and continue step 1 for the next batch.
(v) if $x1+x2 > c2$, reject the lot and step 2 for the next batch.

OC Function

The OC function of WPD-QSSDSS (n; c; c1, c2) based on Romboski's (1969) is given below

$$P_a = \frac{P_T}{(1 - P_N) + P_T} \quad \text{------(1)}$$

Where,

PN - the proportion of lots are accepted using a normal SSP (n, c).
PT -the proportion of lots are accepted using a tightened DSP (n, c1, c2).

The PN and PT of the WP model is given below

$P_N = \sum_{x=1}^{c} e^{-np} np^{x-1} / x - 1!$

$$P_T = \left(\sum_{x=1}^{c_1} e^{-np} np^{x-1}/(x-1)! + \left(\sum_{x=c_1+1}^{c_2} e^{-np} np^{x-1}/(x-1)! * \sum_{x=1}^{c_2-(c_1+1)} e^{-np} np^{x-1}/(x-1)!\right)\right)$$

-----(2)

Properties of the WPD-QSSDSS

Figures 1 indicate the WPD with QSSDSS's normal, tightened, and composite OC curves, and the WPD-QSSDSS. It shows that tightened curves of OC are nearer to the composite OC curve with large values of p than SSP/DSP OC curves are at lower p values. The combined OC curve is less costly than its their respective normal and tightened curves for OC when the values of the normal SSP are stable and the acceptance levels of the tightened DSP are allowed to decrease. They indicate that the OC curve's discriminatory effectiveness improves as the tightening increases deeper.

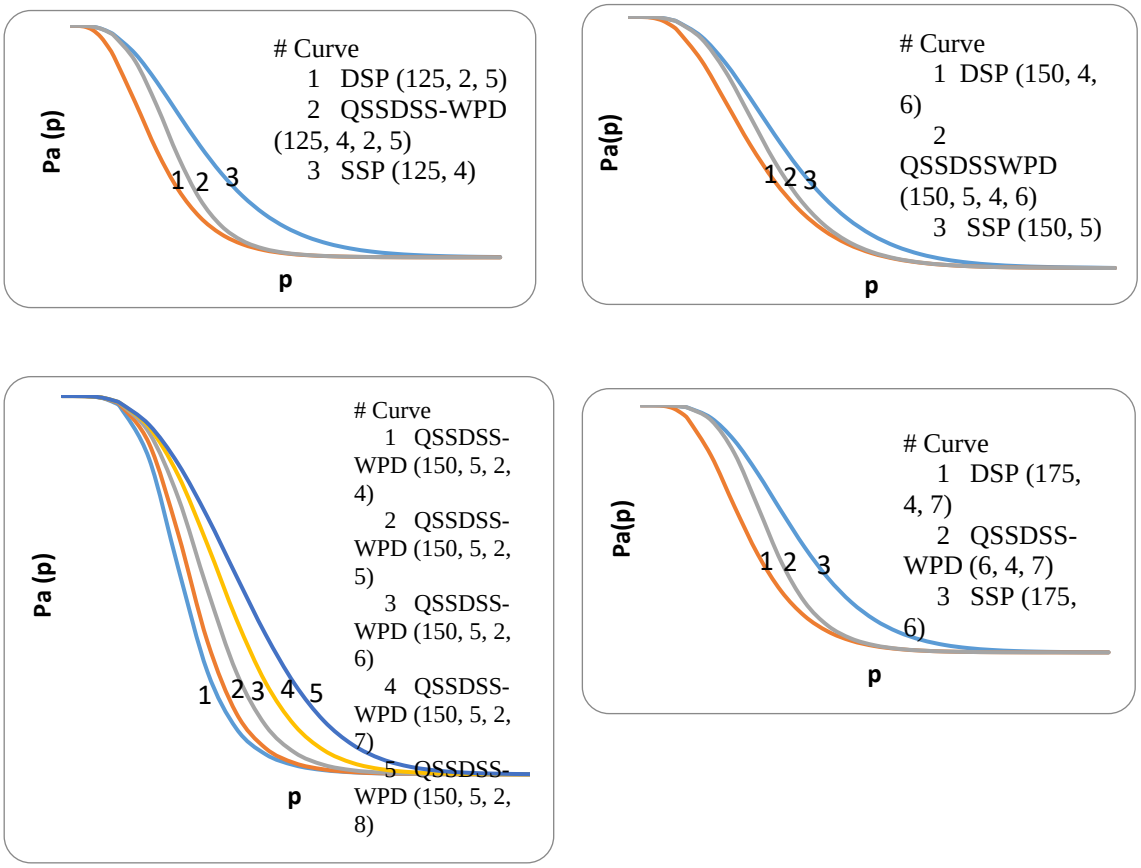


Figure 1: OC function of WPD QSSDSS

Comparison of systems and plans matched by OR

Table 2 provides the appropriate SSP, DSP, and WPD-QSSDSS values (n, c, c1, c2). Table 2 provides five sets of values. Cameron (1952) and Duncan (1986) produced SSP and DSP values, accordingly. The WPD-QSSDSS values were taken from Table (4), which follows. Effectiveness is investigated among SSP, DSP, and WPD-QSSDSS. The method used for determining effectiveness is as outlined below:

*E1=np0.95 of SSP / np0.95 of WPD-QSSDSS
*E2=np0.95 of DSP / np0.95 of WPD-QSSDSS

According to the table, a significantly smaller sample size is needed for the WPD-QSSDSS than for SSP and DSP.

TABLE 1: MatchingOperating Ratio of Single Sampling Plan, Double Sampling Plan and QSSDSS-WPD

Single sampling plan			Double sampling Plan				QSSDSS-WPD					*E_1	*E_2
c	OR	$np_{0.95}$	a_1	a_2	OR	$np_{0.95}$	c	c_1	c_2	OR	$np_{0.95}$		
10	2.67	4.654	4	11	2.67	4.292	9	4	8	2.67	3.122	1.2111	1.1010
12	2.94	6.234	4	13	2.91	5.565	9	2	6	2.93	3.145	2.3455	1.5433
13	2.13	7.654	4	13	2.14	6.181	10	1	7	2.11	3.876	2.1232	1.3547
14	2.10	9.468	4	17	2.09	8.019	10	2	4	2.10	3.755	2.5555	1.6985
15	2.07	10.120	4	18	2.09	9.123	8	1	5	2.09	3.663	2.6628	1.9280

Fuzzy Set Theory and Acceptance Sampling Plan:

The use of fuzzy measurements combination theory and acceptance sampling methods plays an essential part in solving the inherent uncertainty of assessing quality as simple standards are inadequate. The Fuzzy Set Theory provides a more detailed assessment of membership in sets, along with greater flexibility in the identification of boundaries. This flexibility is most beneficial for acceptance sampling, as requirements may be modified using sets of fuzzy values which successfully indicate the various quality characteristics of examined goods. In manufacturing, for scenario, the concept of fuzzy sets allows acceptance sampling plans for taking into consideration multiple levels of acceptance or divergence in quality requirements, instead of simple categories that include success or failures. The dynamic approach ensures that quality assurance processes could effectively deal with practical applications complications, allowing for better understanding and accurate acceptance decisions which take advantage of the uncertain characteristic of measurements of quality.

The idea of fuzzy sets enhances acceptance of sampling methods by integrating ideas from specialists in the field and a variety of ideas. This conceptual structure provides for the flexible modification of acceptance or rejection criteria for sampling sets, taking advantage of multiple characteristics of quality that result from different components and factors. Establishments which accept the concept of fuzzy sets may enhance the accuracy of their quality management frameworks. It is achieved by providing a review of levels in the quality of the product instead of straightforward good or bad decisions.

Further, the concept of fuzzy sets provides efficient allocation of resources in quality management plans. It means that monitoring activities focus on what could have the most of an effect, if it involves identifying and rectifying minor deviations with requirements and keeping the highest possible standards of quality in significant domains. The dynamic method certainly increases the success rate of inspections, but it also allows manufacturers to react more quickly to shifting economic and commercial circumstances.

Eventually, which includes the theory of fuzzy sets into acceptance sampling methods provides a strong basis of solving the various challenges associated with quality control. In allowing the unpredictability and difficulties associated with practical problems evaluation of quality, this approach increases an organization's overall quality control plans, permitting continual improvement and reliability in the delivery of products and services.

Fuzzy QSSDSS:

The Operating characteristic function of WPD-QSSDSS based on the equation (1), In a manufacture process must be inspected using lots with the large size of N such that the probability of defective items is not known specifically. The parameter of fuzzy number $\tilde{p}$ as follows:

$$\tilde{p} = (a_1, a_2, a_3), p \in \tilde{p}[1], q \in \tilde{q}[1], p + q = 1 \quad \text{------(3)}$$
$$\tilde{\lambda} = \widetilde{np} = (nl_i, na_2 + nl_i, na_3 + nl_i) \quad \text{------(4)}$$

l_i in the domain of $[0, 1 - a_3]$, the OC function is plotted using fuzzy probability. Fuzzy logic with normal SSP and tightened DSP (n, c, c1, c2). The normal SSP using fuzzy logic is given below:

$$\tilde{P}_N = \left\{\sum_{x=1}^{c} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} | \lambda \in \tilde{\lambda}[a]\right\} = \left[min\left\{\sum_{x=1}^{c} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} | \lambda \in \tilde{\lambda}[a]\right\}, max\left\{\sum_{x=1}^{c} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} | \lambda \in \tilde{\lambda}[a]\right\}\right] \quad \text{------(5)}$$

The tightened DSP using fuzzy logic is given below:

$$\tilde{P}_T = \tilde{P}^I{}_a + \tilde{P}^{II}{}_a = \left\{\sum_{x=1}^{c_1} \frac{e^{-np} np^{x-1}}{(x-1)!} | \lambda_1 \in \tilde{\lambda}[a]\right\} + \left(\left(\sum_{x=c_1+1}^{c_2} e^{-np} np^{x-1}/(x-1)! * \sum_{x=1}^{c_2-(c_1+1)} e^{-np} np^{x-1}/(x-1)! \, | \lambda_1 \in \tilde{\lambda}[a]\right)\right) \quad \text{-------(6)}$$

Properties of FWPD-QSSDSS

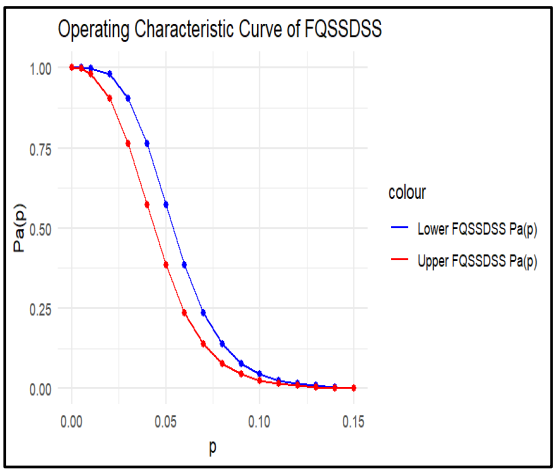
Table -2 indicates the data of OC curve of QSSDSS with fuzzy logic using R programming. It shows the effectiveness of a quality control sampling that combines fuzzy logic and QSS using normal SSP and tightening DSP.

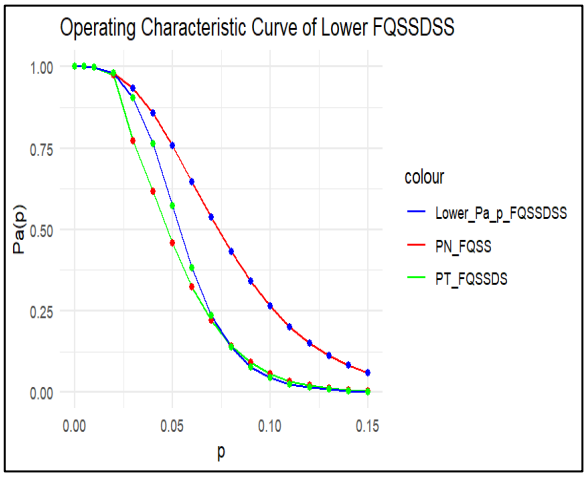
Table 2: Operating Characteristics Function for WPD-QSSDSS using fuzzy logic

p Lower	p Upper	Lower P_N FQSS	Upper P_N FQSS	Lower P_T FQSS	Upper P_T FQSS	Lower Pa(p) FQSSDSS	Upper Pa(p) FQSSDSS
0	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.005	0.01	1.0000	0.9982	1.0000	0.9985	1.0000	0.9982
0.01	0.02	0.9982	0.9810	0.9985	0.9754	0.9982	0.9793

0.02	0.03	0.9810	0.9344	0.9754	0.9005	0.9793	0.9037
0.03	0.04	0.9344	0.8571	0.7723	0.6157	0.9037	0.7629
0.04	0.05	0.8571	0.7576	0.6157	0.4599	0.7629	0.5728
0.05	0.06	0.7576	0.6472	0.4599	0.3250	0.5728	0.3835
0.06	0.07	0.6472	0.5366	0.3250	0.2195	0.3835	0.2357
0.07	0.08	0.5366	0.4335	0.2195	0.1429	0.2357	0.1375
0.08	0.09	0.4335	0.3423	0.1429	0.0903	0.1375	0.0782
0.09	0.10	0.3423	0.2650	0.0903	0.0558	0.0782	0.0441
0.10	0.11	0.2650	0.2017	0.0558	0.0339	0.0441	0.0249
0.11	0.12	0.2017	0.1512	0.0339	0.0204	0.0249	0.0142
0.12	0.13	0.1512	0.1118	0.0204	0.0122	0.0142	0.0082
0.13	0.14	0.1118	0.0818	0.0122	0.0073	0.0082	0.0047
0.14	0.15	0.0818	0.0591	0.0073	0.0043	0.0047	0.0003
0.15	0.16	0.0591	0.0424	0.0043	0.0003	0.0003	0.0000

Using R programming Figure 2 was plotted. It indicates that the OC curve of fuzzy logic using QSSDSS. of probability of accepting a lot improves for increasing levels of nonconformity. This method accurately assesses lot with normal SSP and shifted to Tightened DSP for further analysis when defects increase. This approach maximizes goods, increases manufacturing, and produces appropriate quality-control decisions.





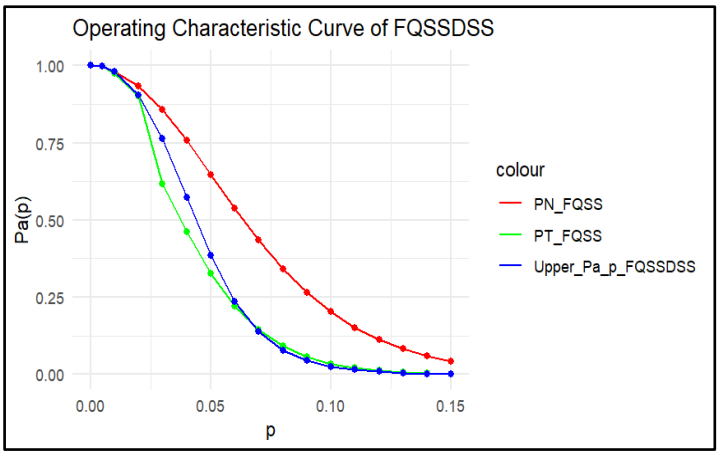


Figure 2: OC curve for WPD-QSSDSS with fuzzy logic

Neutrosophic Weighted Poisson Quick Switching Single Double Sampling Plan (NWPD-QSSDSS)

The combination of the neutrosophic concept with quick switching sampling (QSS), which incorporates both a normal SSP (NSSP) and a tightened DSP (TDSP) with WPD, is an effective approach for bettering manufacturing quality monitoring. Neutrosophic, it accepts the indeterminacies of decision-making, gives an outline for managing uncertainty and challenges that may appear in evaluating acceptance or defect of items. The above framework has been further improved by QSS, that continuously modifications the sample's perform based on the initial decisions. From the beginning, an NSSP evaluates a sample in the entire batch using specified criteria to find out whether it satisfies the requirements for quality. Since the conclusions are unsatisfactory (i.e., the number of nonconformities approaches the acceptability level), a TDSP is performed. It requires a deeper assessment prior to making an end focus. This flexible technique further assures that the highest standards requirements are fulfilled, however it also minimizes the monitoring process by giving useful input through real-time quality tests. It will prove particularly useful in industries were quick response to shifting manufacturing environments and uncertainty are essential for maintaining consistency and customer satisfaction.

OC Function of NWPD-QSSDSS

The neutrosophic based on quick switching single double sampling plan using weighted poisson distribution is a baseline. According to Smarandache (2010), the normalization is given by

$$t + I + f = 1 \qquad \text{------(7)}$$

where, t = truth of a statement; I = indeterminacy of a statement; f = falsehood of a statement similarly, we consider,

$$PtN + PIN + PfN = 1 \qquad \text{------(8)}$$

where, PtN = Probability of acceptance; PIN = Probability of indeterminacy; PfN = probability of rejection

The Operating characteristic function of WPD-QSSDSS based on the equation (1), Based on the above concept and equation (1), the equation of neutrosophic probability of acceptance, rejection probability and indeterminacy probability using the normal single sampling plan is given below

$$P_{tN} = \sum_{x=1}^{c} \frac{\lambda_F^x}{x!} \left[\sum_{i=0}^{\min(I,\ n-x)} \frac{\lambda_I^i}{i!} e^{-(\lambda_1+\lambda_F)} \right] \qquad \text{-----(9)}$$

$$P_{FN} = \sum_{x=c+1}^{c} \frac{\lambda_F^x}{x!}\left[\sum_{i=0}^{(n-x)} \frac{\lambda_I^i}{i!} e^{-(\lambda_1+\lambda_F)}\right] \quad \text{-----(10)}$$

$$P_{IN} = \sum_{i=I+1}^{n} \frac{\lambda_I^i}{i!}\left[\sum_{x=1}^{\min(N,\ n-i)} \frac{\lambda_F^x}{x!} e^{-(\lambda_1+\lambda_F)}\right] \quad \text{-----(11)}$$

Also, the equation of neutrosophic probability of acceptance, rejection probability and indeterminacy probability using the tightened double sampling plan is given below

$$P_{tN} = \sum_{x_1=1}^{c_1}\left(\frac{\lambda_{F_1}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=0}^{\min(I_1,n_1-x_1)} \frac{\lambda_{I_1}^{i_1}}{i_1!} e^{-(\lambda_{I_1}+\lambda_{F_1})}\right]\right)$$
$$+ \sum_{x_1=c_1+1}^{c_2}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=0}^{\min(I_2,n_1-x_1)}\left(\frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right.\right.\right.$$
$$\left.\left.\left.*\left[\sum_{x_2=1}^{c_2-x_1}\left(\frac{\lambda_{F_2}^{x_2-1}}{(x_1-1)!}.\left[\sum_{i_2=0}^{\min(I_2-i_1,\ n_2-x_2)} \frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right]\right)\right]\right)\right]\right)$$
$$+ \sum_{x_1=1}^{c_1}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=I_1+1}^{\min(I_2,\ n_1-x_1)}\left(.\frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right.\right.\right.$$
$$\left.\left.\left.*\left[\sum_{x_2=0}^{c_2-x_1}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_2=0}^{\min(I_2-i_1,\ n_2-x_2)} \frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right]\right)\right]\right)\right]\right)$$

-----(12)

$$P_{FN} = \sum_{x_1=c_2+1}^{n_1}\left(\frac{\lambda_{F_1}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=0}^{\min(\ n_1-x_1)} \frac{\lambda_{I_1}^{i_1}}{i_1!} e^{-(\lambda_{I_1}+\lambda_{F_1})}\right]\right)$$
$$+ \sum_{x_1=c_1+1}^{c_2}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=0}^{\min(I_2,n_1-x_1)}\left(\frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right.\right.\right.$$
$$\left.\left.\left.*\left[\sum_{x_2=x_1+1}^{n_2}\left(\frac{\lambda_{F_2}^{x_2-1}}{(x_1-1)!}.\left[\sum_{i_2=0}^{\min(n_2-x_2)} \frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right]\right)\right]\right)\right]\right)$$
$$+ \sum_{x_1=1}^{c_1}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=I_1+1}^{\min(I_2,\ n_1-x_1)}\left(.\frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right.\right.\right.$$
$$\left.\left.\left.*\left[\sum_{x_2=x_1+1}^{n_2}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_2=0}^{\min(n_2-x_2)} \frac{\lambda_{I_2}^{i_1}}{i_1!} e^{-(\lambda_{I_2}+\lambda_{F_2})}\right]\right)\right]\right)\right]\right)$$

-----(13)

$$P_{IN} = \sum_{x_1=1}^{c_2}\left(\frac{\lambda_{F_1}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=I_2+1}^{\min(n_1-x_1)}\frac{\lambda_{I_1}^{i_1}}{i_1!}e^{-(\lambda_{I_1}+\lambda_{F_1})}\right]\right)$$
$$+\sum_{x_1=c_1+1}^{c_2}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=0}^{\min(I_2,n_1-x_1)}\left(\frac{\lambda_{I_2}^{i_1}}{i_1!}e^{-(\lambda_{I_2}+\lambda_{F_2})}\right.\right.\right.$$
$$\left.\left.\left.*\left[\sum_{x_2=1}^{c_2-x_1}\left(\frac{\lambda_{F_2}^{x_2-1}}{(x_1-1)!}.\left[\sum_{i_2=I_2-i_1+1}^{\min(n_2-x_2)}\frac{\lambda_{I_2}^{i_1}}{i_1!}e^{-(\lambda_{I_2}+\lambda_{F_2})}\right]\right)\right]\right)\right]\right)$$
$$+\sum_{x_1=1}^{c_1}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_1=I_1+1}^{\min(I_2,n_1-x_1)}\left(.\frac{\lambda_{I_2}^{i_1}}{i_1!}e^{-(\lambda_{I_2}+\lambda_{F_2})}\right.\right.\right.$$
$$\left.\left.\left.*\left[\sum_{x_2=0}^{c_2-x_1}\left(\frac{\lambda_{F_2}^{x_1-1}}{(x_1-1)!}.\left[\sum_{i_2=I_2-i_1+1}^{\min(I_2-i_1,\ n_2-x_2)}\frac{\lambda_{I_2}^{i_1}}{i_1!}e^{-(\lambda_{I_2}+\lambda_{F_2})}\right]\right)\right]\right)\right]\right) \quad \text{-----(14)}$$

Properties of NWPD-QSSDSS

Table -3 indicates the data of OC curve of neutrosophic using WPD QSSDSS. Using R programming the OC function was calculated. It shows the effectiveness of a quality control sampling that combines neutrosophic and QSS using normal SSP and tightening DSP.

Table 3: Operating Characteristics Function for WPD-QSSDSS using Neutrosophic

N	n	n_1	n_2	c	c_1	c_2	I_1	I_2	Neutrosophic			Neutrosophic QSS-SSP			Neutrosophic QSS-DSP			Neutrosophic QSSDSS Pa(p)
									P(S)	P(F)	P(I)	SSP P_{tN}	SSP P_{FN}	SSP P_{IN}	DSP P_{tT}	DSP P_{FT}	DSP P_{IT}	
500	20	20	30	0	1	2	1	2	0.85	0.08	0.07	0.9607	0.0210	0.0183	0.9751	0.0133	0.0116	0.9613
500	25	25	35	1	2	3	2	3	0.88	0.07	0.05	0.9435	0.0285	0.0280	0.9132	0.0273	0.0595	0.9417
500	30	30	40	2	3	4	3	4	0.90	0.05	0.05	0.9262	0.0283	0.0456	0.9018	0.0341	0.0641	0.9244
500	35	35	45	3	4	5	4	5	0.95	0.03	0.02	0.9047	0.0392	0.0561	0.8997	0.0482	0.0521	0.9042
500	40	40	50	4	5	6	5	6	0.98	0.01	0.01	0.8621	0.0369	0.1010	0.8666	0.0005	0.1329	0.8627
1000	20	20	30	0	1	2	1	2	0.80	0.10	0.10	0.9801	0.0100	0.0100	0.9856	0.0072	0.0072	0.9802
1000	25	25	35	1	2	3	2	3	0.85	0.10	0.05	0.9753	0.0165	0.0082	0.9597	0.0134	0.0269	0.9749
1000	30	30	40	2	3	4	3	4	0.90	0.05	0.05	0.9704	0.0148	0.0148	0.9088	0.0856	0.0056	0.9685

1000	35	35	45	3	4	5	4	5	0.95	0.03	0.02	0.9656	0.0206	0.0138	0.8984	0.0916	0.0100	0.9631
1000	40	40	50	4	5	6	5	6	0.98	0.01	0.01	0.9607	0.0197	0.0196	0.8234	0.1383	0.0383	0.9544

Practical Example:

A hospital regularly purchases blood test kits from a supplier to diagnose various medical conditions. These kits can sometimes exhibit defects such as faulty reagents or inaccurate calibration, which can impact diagnostic accuracy. The hospital and supplier have agreed upon quality standards based on probabilities of non-defectiveness, indeterminacy, and defectiveness of the kits. The hospital implements a QSS based on Neutrosophic WPD to assess the quality of incoming blood test kit lots. The following steps are used:

Step 1: In QSS with normal SSP, take a random sample of size n=52 from a lot and check the no. of defective units x and number of indeterminate items 'i'

(i) if $x \leq 2$, and $i \leq I$ (1), accept the lot and repeat step 1 for next lot.
(ii) if $x > 2$, reject the lot a and continue with step 2 for the next lot.
(iii) if $x \leq 2$, $i > I$ (1), the lot is indeterminate.

Step 2: In QSS with tightened DSP, take a random sample of size n=52 from the next lot and check the no. of defectives x1 and in determinate item (i).

(i) If $x1 \leq 5$, $i \leq I$ (1), accept the lot and continue with step 1 for the next lot.
(ii) If $x1 > 3$, reject the lot and repeat step 2 for the next lot.
(iii) Choose a second random sample of size n from the same lot if $5 < x1 \leq 3$ and verify the no. of defectives x2.
(iv) If $x1+x2 \leq 3$. $i \leq I$ (1), accept the lot and continue with step 1 for the next lot.
(v) If $x1+x2 > 3$ reject the lot and repeat step 2 for the next lot.
(vi) If $x1+x2 \leq c2$, $i > I$ (1), the lot is indeterminate.

The above example indicates the quality control method includes a Quick Switching System (QSS) with a normal SSP (Step 1) and a tightened DSP (Step 2) depending on the weighted poisson distribution. It uses Neutrosophic concepts to circumstances in which the lot's quality status is not known or fuzzy. Using Neutrosophic logic, the method gives valid choices, increasing the accuracy of quality control processes for monitoring the blood test kit quality for clinical diagnostics. These strategies give a systematic method for quality control in the medical sector, including blood testing kits meeting specified defective needs for usage in clinical testing.

Table 4: Values of np tabulated against c, c1, c2 for the given values of Pa(p) for WPD-QSSDSS (n; c; c1, c2)

c	c_1	c_2	0.99	0.95	0.75	0.50	0.25	0.10	0.05	0.01
2	1	4	0.1492	0.3624	1.0178	1.7476	2.6232	3.5252	4.1328	5.4585
2	2	5	0.1493	0.3647	0.4041	1.6531	3.5478	5.1991	6.2385	8.3955
2	3	6	0.1495	0.3658	1.1553	2.3997	3.9323	5.3989	6.3623	8.4337
2	4	7	0.1499	0.3666	1.1790	2.7714	4.8028	6.5913	7.7160	10.0397

2	5	8	0.1534	0.3678	1.1865	3.1564	5.7725	7.8356	9.0817	11.5919
2	6	9	0.1593	0.3695	1.1884	3.5488	6.8018	9.0854	10.4278	13.0935
3	1	5	0.4371	0.8247	1.7463	2.6162	3.6020	4.5890	5.2392	6.6091
3	2	6	0.4376	0.8315	1.8378	2.8579	4.0113	5.1441	5.8853	7.4613
3	3	7	0.4377	0.8343	1.9214	3.1554	4.5547	5.8986	6.7760	8.6656
3	4	8	0.4387	0.8351	1.9814	3.4943	5.2358	6.8516	7.8935	10.1058
3	5	9	0.4399	0.8353	2.0150	3.8604	6.0423	7.9517	9.1470	11.6092
4	1	6	0.8243	1.3710	2.5204	3.5227	4.6237	5.7080	6.4144	7.8824
4	2	7	0.8252	1.3802	2.6022	3.7143	4.9382	6.1271	6.8940	8.4801
4	3	8	0.8256	1.3862	2.6915	3.9543	5.3396	6.6631	7.5127	9.2848
4	4	9	0.8257	1.3889	2.7742	4.2451	5.8598	7.3710	8.3389	10.3811
5	1	7	1.2799	1.9719	3.3262	4.4512	5.6607	6.8374	7.5986	9.1688
5	2	8	1.2811	1.9815	3.3962	4.6123	5.9289	7.1978	8.0102	9.6703
5	3	9	1.2819	1.9899	3.4771	4.8015	6.2352	7.6021	8.4712	10.2439
6	1	8	1.7857	2.6120	4.1540	5.3919	6.7009	7.9627	8.7746	10.4409
6	2	9	1.7871	2.6209	4.2159	5.5386	6.9514	8.3026	9.1633	10.9112
7	1	5	2.1393	2.7681	3.6294	4.2136	4.8424	5.5077	5.9771	7.0520
7	2	6	2.2011	2.8880	3.8504	4.5131	5.2332	6.0025	6.5507	7.8287
7	3	7	2.2564	3.0133	4.1114	4.8858	5.7405	6.6686	7.3394	8.9309
8	1	5	2.5744	3.2027	4.0313	4.5824	5.1715	5.7939	6.2338	7.2466
8	2	6	2.6610	3.3494	4.2739	4.8971	5.5701	6.2892	6.8034	8.0113
8	3	7	2.7460	3.2910	4.5638	5.2897	5.4539	6.9525	7.5816	9.0874
9	1	5	2.9974	3.6198	4.4195	4.9440	5.5016	6.0893	6.5049	7.4644
9	2	6	3.1083	3.7912	4.6824	5.2746	5.9107	6.5897	7.0759	8.2230
9	3	7	3.2236	3.9839	4.9995	5.6879	6.4395	7.2562	7.8503	9.2789
10	1	5	3.4086	4.0227	4.7972	5.2998	5.8317	6.3911	6.7864	7.7005
10	2	6	3.5428	4.2172	5.0798	5.6469	6.2536	6.9002	7.3634	8.4587

10	3	7	3.6876	4.4401	5.4231	6.0817	6.7978	7.5747		9.5005

Conclusion: The construction and design of Quick Switching systems QSSFP and QSSNP (n; CN, CT) involve the implementation of single sampling plans using Neutrosophic Poisson Distribution (NSPD), which integrates Neutrosophic Sets (NSs) to effectively manage uncertainty in quality assessment. Quick Switching Sampling (QSS) systems with normal SSP and tighter DSP that utilize the weighted poisson distribution as the baseline distribution and interact into fuzzy logic and neutrosophic combinations provide an adverse methodology for quality control in production. Using standard static sampling plans, QSS systems continuously shift samples intervals in response to immediate input and preliminary findings. This dynamic range additionally increases performance under shifting manufacturing circumstances, but it also optimizes sample efforts, minimizing expenses while retaining the highest standards of quality. The use of fuzzy logic improves decision-making above simple outcomes by providing for degrees of membership in acceptance criteria, solving the inherent risk in manufacturing procedures. Neutrosophic sets improve this approach through involves criteria for truth, indeterminacy, and falsity, providing an exhaustive structure for managing uncertain high-quality information effectively.

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A Neutrosophic Superiority and Inferiority Ranking-Based Model for Resource Allocation Efficiency in University Sports Management

Pute Lan[1], Jiaqi Guo[2], Changshun Bai[1]*
[1]Physical Education Department of Inner Mongolia Medical University, Hohhot, 010010, Inner Mongolia, China (E-mail: 18686012621@163.com)
[2]Beijing Sport University, Beijing, 100084, China (E-mail: saraguojiaqilove66@163.com)
*Corresponding author E-mail: 20160001@immu.edu.cn

Abstract: The evaluation of resource allocation efficiency in university recreational sports aims to analyze and assess the effectiveness of how universities allocate facilities, equipment, and personnel for recreational sports. By collecting relevant data and applying scientific methods, it evaluates the usage and impact of resources to optimize allocation and enhance student participation and satisfaction. This not only helps improve campus sports culture but also promotes students' physical and mental health and overall development. The resource allocation efficiency evaluation in university recreational sports is a MAGDM problem. Recently, the Superiority and Inferiority Ranking (SIR) method has been employed to cope with MAGDM issues. The single-valued neutrosophic sets (SVNSs) are used as a tool for characterizing uncertain information during the resource allocation efficiency evaluation in university recreational sports. In this paper, the single-valued neutrosophic number SIR (SVNN-SIR) method is built to solve the MAGDM under SVNSs. In the end, a numerical case study for resource allocation efficiency evaluation in university recreational sports is given to validate the proposed method.

Keywords: Multiple attribute group decision making (MAGDM); single-valued neutrosophic sets (SVNSs); SIR; resource allocation efficiency evaluation

1. Introduction and Background

As Chinese society enters a stage of general well-being, the abundance of material resources, the improvement in quality of life, and societal progress have brought about significant changes in people's lifestyles and attitudes toward sports. Different groups, influenced by various environmental, social, and personal factors, choose different leisure activities during their free time. University students, in particular, can achieve both mental and physical relaxation, rest, and enjoyment by engaging in leisure sports during their leisure time. From an economic perspective, the rational allocation of resources is the foundation and prerequisite for the effective functioning of the economy, and the development of leisure sports also depends on the optimal allocation of

leisure sports resources. Therefore, understanding the current state of leisure sports resources in universities, and promoting the rational development, optimal allocation, and full utilization of these resources, has become a key issue in determining whether the reform of leisure sports in universities can meet the needs of socialist construction and promote their healthy development. In 2008, Zhou and Deng [1]conducted a study using literature analysis, surveys, and theoretical methods to examine the state of leisure sports resource allocation in universities and focused on how to evaluate the efficiency of these resources and proposed a set of evaluation indicators. This study laid the theoretical foundation for future improvements in resource allocation efficiency. In the same year, Zhou, Tang [2] explored the rational allocation and development of leisure sports resources in universities and emphasized that while existing resources should be used wisely, it is also essential to meet the diverse leisure sports needs of students and faculty. The study highlighted that rational resource allocation is necessary for university sports education to adapt to societal development needs. Subsequently, in 2009, Zhou [3] conducted an in-depth analysis of the current state and causes of leisure sports resource allocation in ordinary universities in Hunan Province. From an economic perspective, he pointed out that optimizing the allocation of leisure sports resources is crucial for their development. The study found that the utilization rate of human resources in universities was low, social resources had not effectively penetrated, and funding was insufficient. Additionally, the lack of high-quality venue management personnel and comprehensive information resource platforms were also limiting factors. Finally, in 2020, Zheng, Wu [4] conducted a survey on the allocation of leisure sports resources in some ordinary universities in Sichuan Province. Using literature analysis, expert interviews, and questionnaires, they proposed strategies to improve current resource allocation. They suggested that universities should focus on cultivating high-quality sports talents, enhance the social cooperation of sports venues, and broaden funding sources. This study offered new insights for the operation and management of modern university sports venues.

MAGDM is a collective decision-making process involving multiple decision-makers evaluating and selecting options based on various attributes [5-8]. Each decision-maker rates the attributes of potential options according to their preferences and judgments. These evaluations are then integrated to form a comprehensive decision outcome. The process often employs

mathematical models and methods, such as weighted averaging, analytic hierarchy process, or fuzzy comprehensive evaluation, to ensure objectivity and rationality[9, 10]. This approach is widely used in fields like management, finance, and engineering, helping organizations make scientifically sound choices in complex environments, balancing the opinions of multiple stakeholders, and improving decision quality and satisfaction [11-14]. Gomes and Lima [15] built the TODIM for MADM issues under risk. The resource allocation efficiency evaluation in university recreational sports is classical MAGDM.

Recently, the SIR method [16] has been used to cope with MAGDM issues. The SVNSs [17-25] are used as a tool for characterizing uncertain information during the resource allocation efficiency evaluation in university recreational sports. The literature review highlights key studies on resource allocation in university sports. Zhou and Deng (2008) conducted foundational research, proposing efficiency evaluation indicators. Subsequent work by Zhou et al. (2020) focused on strategies to enhance collaboration between universities and external sports organizations. Traditional multi-attribute decision-making methods like TODIM and EDAS have been applied in this domain but often fall short in addressing uncertainty. This paper distinguishes itself by introducing a model based on neutrosophic sets, offering a more accurate and holistic evaluation framework. With societal progress and increasing emphasis on physical activity, recreational sports in universities play a pivotal role in enhancing students' physical and mental well-being. However, allocating resources for these activities faces significant challenges, including inefficiency and inequitable distribution. This paper aims to develop a robust framework for evaluating resource allocation efficiency using SVNSs. By addressing uncertainties and complexities in decision-making, this work fills a critical gap in the literature.

The structure of this paper is listed below. In Section 2, the SVNSs are introduced. In Section 3, the SVNN-SIR method is designed under SVNSs. Section 4 gives an illustrative case for resource allocation efficiency evaluation in university recreational sports and some comparative analysis. Some remarks are given in Section 5.

2. Preliminaries and Definitions

This section presents key definitions related to Single-Valued Neutrosophic Sets (SVNSs), as outlined in references [26–28].

Definition 1

The neutrosophic set can be defined as:

$$O = (x, T_o(x), I_o(x), F_o(x), x \in X) \tag{1}$$

$$0^- \leq \sup T_o(x) + \sup I_o(x) + \sup F_o(x) \leq 3^+ \tag{2}$$

Definition 2

Let $O_1 = T_{o_1}(x), I_{o_1}(x), F_{o_1}(x)$ and $O_2 = T_{o_2}(x), I_{o_2}(x), F_{o_2}(x)$ be two SVNNs and their operations can be defined as:

$$O_1 \oplus O_2 = \left\{ \begin{pmatrix} T_{o_1}(x) + T_{o_2}(x) - T_{o_1}(x)T_{o_2}(x), \\ I_{o_1}(x)I_{o_2}(x), \\ F_{o_1}(x)F_{o_2}(x) \end{pmatrix} \right\} \tag{3}$$

$$O_1 \otimes O_2 = \left\{ \begin{matrix} T_{o_1}(x)T_{o_2}(x), \\ I_{o_1}(x) + I_{o_2}(x) - I_{o_1}(x)I_{o_2}(x), \\ F_{o_1}(x) + F_{o_2}(x) - F_{o_1}(x)F_{o_2}(x) \end{matrix} \right\} \tag{4}$$

$$\varphi O_1 = \left(1 - \left(1 - T_{o_1}(x)\right)^{\varphi}, \left(I_{o_1}(x)\right)^{\varphi}, \left(I_{o_1}(x)\right)^{\varphi}\right) \tag{5}$$

$$O_1^{\varphi} = \left(\left(T_{o_1}(x)\right)^{\varphi}, 1 - \left(1 - I_{o_1}(x)\right)^{\varphi}, 1 - \left(1 - F_{o_1}(x)\right)^{\varphi}\right) \tag{6}$$

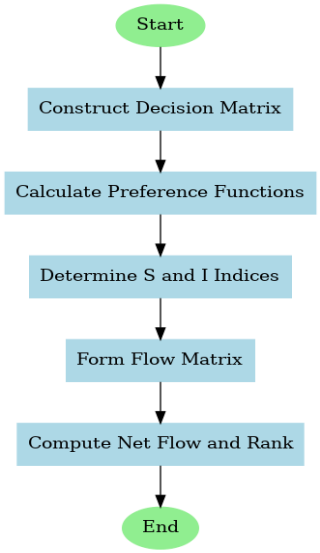
Definition 3.

The score function can be defined as:

$$S(O) = \frac{2 + T_o(x) - I_o(x) - F_o(x)}{3} \tag{7}$$

3. Methodology

The flowchart of the proposed Methodology outlines the sequential steps of the methodology, starting from constructing the decision matrix to the final step of ranking alternatives. Each box represents a critical process, and arrows indicate the logical flow between steps as shown in Figure



1.

Figure 1. The Flowchart of Methodology

The Single-Valued Neutrosophic Numbers-based Superiority and Inferiority Ranking (SVNN-SIR) method involves the following steps:

Step 1. The decision matrix is formed.

The experts and decision-makers created the decision matrix by using the SVNN. Then we applied the score function to obtain a single number. Then we combine the opinions of experts into a single matrix.

$$Y = \begin{pmatrix} f_1(A_1) & \cdots & f_n(A_1) \\ \vdots & \ddots & \vdots \\ f_1(A_m) & \cdots & f_1(A_m) \end{pmatrix}_{m\times n} ; i = 1, \dots, m; j = 1, \dots, n \quad (8)$$

Where A refers to the alternatives and $C = \{f_1(0), \dots, f_n(0)\}$ is a group of assessment criteria for the alternatives.

Step 2. Comparing the alternatives.

The alternatives are compared in the decision matrix depending on the criteria and a function such as $t = C(A_1) - C(A_2)$ (9)

$$T = \begin{bmatrix} C_1(A_1) & \cdots & C_n(A_1) \\ \vdots & \ddots & \vdots \\ C_1(A_m) & \cdots & C_n(A_m) \end{bmatrix} \quad (10)$$

$$P(A_1, A_2) = f\big(C(A_1) - C(A_2)\big) = f(t) \quad (11)$$

Step 3. Determine the values of the preference function.

The values of the preference function are computed based on the usual criterion. Where the usual criteria can be defined as:

$$f(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases} \quad (12)$$

Step 4. Determine the S and I index

$$S_j(A_i) = \sum_{i=1}^{m} P_j(A_i, A_{i'}) = \sum_{i=1}^{m} f_j\left(C_j(A_{i'}) - C_j(A_i)\right) \quad (13)$$

$$I_j(A_i) = \sum_{i=1}^{m} P_j(A_i, A_{i'}) = \sum_{i=1}^{m} f_j\left(C_j(A_i) - C_j(A_{i'})\right) \quad (14)$$

Then their matrices are defined as:

$$S = \begin{bmatrix} S_1(A_1) & \dots & S_j(A_1) & \dots & S_n(A_1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ S_1(A_i) & \cdots & S_j(A_i) & \dots & S_n(A_i) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ S_1(A_m) & \cdots & S_j(A_m) & \dots & S_n(A_m) \end{bmatrix}_{m\times n} \quad (15)$$

$$I = \begin{bmatrix} I_1(A_1) & \dots & I_j(A_1) & \dots & I_n(A_1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ I_1(A_i) & \cdots & I_j(A_i) & \dots & I_n(A_i) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ I_1(A_m) & \cdots & I_j(A_m) & \dots & I_n(A_m) \end{bmatrix}_{m\times n} \quad (16)$$

Step 5. Form the flow matrix

The flow matrix is formed based on the S and I matrix:

$$U^+(A_i) = \sum_{j=1}^{n} S_j(A_i) \times w_j \quad (17)$$

$$U^-(A_i) = \sum_{j=1}^{n} I_j(A_i) \times w_j \quad (18)$$

Step 6. Compute the n-flow and r-flow

$$n-flow = U^+(A_i) - U^-(A_i) \quad (19)$$

$$r-flow = \frac{U^+(A_i)}{U^+(A_i)+U^-(A_i)} \quad (20)$$

Step 7. Rank the alternatives. The alternatives are ranked based on the n-flow.

Table 1. The decision matrix.

	A₁	**A₂**	**A₃**	**A₄**	**A₅**	**A₆**	**A₇**	**A₈**	**A₉**
C₁	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.5,0.5,0.5)	(0.6,0.5,0.4)	(0.7,0.4,0.3)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.2,0.8,0.7)
C₂	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.7,0.4,0.3)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.2,0.8,0.7)	(0.4,0.7,0.6)
C₃	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.7,0.4,0.3)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.2,0.8,0.7)	(0.4,0.7,0.6)	(0.5,0.5,0.5)
C₄	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.4,0.7,0.6)	(0.5,0.5,0.5)	(0.6,0.5,0.4)
C₅	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.6,0.5,0.4)	(0.7,0.4,0.3)
C₆	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.4,0.7,0.6)	(0.6,0.5,0.4)	(0.7,0.4,0.3)	(0.8,0.3,0.2)
C₇	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.5,0.5,0.5)	(0.7,0.4,0.3)	(0.8,0.3,0.2)	(0.9,0.2,0.1)
C₈	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.6,0.5,0.4)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)
C₉	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.2,0.8,0.7)
C₁₀	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.8,0.3,0.2)	(0.1,0.9,0.8)	(0.2,0.8,0.7)	(0.4,0.7,0.6)
C₁₁	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.2,0.8,0.7)	(0.4,0.7,0.6)	(0.5,0.5,0.5)
C₁₂	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.4,0.7,0.6)	(0.5,0.5,0.5)	(0.6,0.5,0.4)
C₁₃	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.6,0.5,0.4)	(0.7,0.4,0.3)
C₁₄	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.7,0.4,0.3)	(0.8,0.3,0.2)
C₁₅	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.5,0.5,0.5)	(0.4,0.7,0.6)	(0.2,0.8,0.7)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.9,0.2,0.1)
C₁₆	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.5,0.5,0.5)	(0.4,0.7,0.6)	(0.2,0.8,0.7)	(0.1,0.9,0.8)
	A₁	A₂	A₃	A₄	A₅	A₆	A₇	A₈	A₉
C₁	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.2,0.8,0.7)
C₂	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)
C₃	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)

C_4	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.6,0.5,0.4)
C_5	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.6,0.5,0.4)	(0.7,0.4,0.3)
C_6	(0.4,0.7,0.6)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.6,0.5,0.4)	(0.7,0.4,0.3)	(0.8,0.3,0.2)
C_7	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.5,0.5,0.5)	(0.7,0.4,0.3)	(0.8,0.3,0.2)	(0.9,0.2,0.1)
C_8	(0.1,0.9,0.8)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)
C_9	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.1,0.9,0.8)	(0.2,0.8,0.7)
C_{10}	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.4,0.7,0.6)
C_{11}	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.4,0.7,0.6)	(0.5,0.5,0.5)
C_{12}	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.6,0.5,0.4)
C_{13}	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)
C_{14}	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.8,0.3,0.2)
C_{15}	(0.2,0.8,0.7)	(0.6,0.5,0.4)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.8,0.3,0.2)	(0.9,0.2,0.1)
C_{16}	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.6,0.5,0.4)	(0.2,0.8,0.7)	(0.1,0.9,0.8)
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
C_1	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)	(0.2,0.8,0.7)
C_2	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.1,0.9,0.8)	(0.2,0.8,0.7)	(0.4,0.7,0.6)
C_3	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.4,0.7,0.6)	(0.5,0.5,0.5)
C_4	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.5,0.5,0.5)	(0.6,0.5,0.4)
C_5	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.7,0.4,0.3)
C_6	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.8,0.3,0.2)
C_7	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.7,0.4,0.3)	(0.8,0.3,0.2)	(0.9,0.2,0.1)
C_8	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.1,0.9,0.8)
C_9	(0.9,0.2,0.1)	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)
C_{10}	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.4,0.7,0.6)
C_{11}	(0.7,0.4,0.3)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.4,0.7,0.6)	(0.5,0.5,0.5)
C_{12}	(0.6,0.5,0.4)	(0.9,0.2,0.1)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.5,0.5,0.5)	(0.6,0.5,0.4)
C_{13}	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.5,0.5,0.5)	(0.6,0.5,0.4)	(0.7,0.4,0.3)
C_{14}	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.1,0.9,0.8)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.8,0.3,0.2)
C_{15}	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.8,0.3,0.2)	(0.8,0.3,0.2)	(0.9,0.2,0.1)
C_{16}	(0.1,0.9,0.8)	(0.9,0.2,0.1)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.5,0.5,0.5)	(0.8,0.3,0.2)	(0.2,0.8,0.7)	(0.1,0.9,0.8)

4. Detailed Case Study for Resource Allocation Evaluation and comparative analysis

4.1 Problem Setup

We aim to evaluate nine alternatives (A1–A9) based on sixteen criteria (C1–C16) to determine their efficiency in resource allocation for university recreational sports. This analysis will utilize the SVNN-SIR method and compare it with other methods like SVNN-TODIM, SVNN-CODAS, and SVNN-EDAS.

The criteria and their descriptions are extracted and adapted as:

- *Budget Utilization Efficiency*: Measures how well financial resources are allocated and used to maximize the availability and quality of recreational sports programs.
- *Facility Availability and Accessibility*: Assesses the availability of sports facilities and how easily students and staff can access them, considering location, scheduling, and capacity.
- *Student Participation Rate*: The percentage of students actively engaging in recreational sports, reflecting the effectiveness of resource allocation in meeting student needs.
- *Quality of Sports Equipment*: Evaluates whether equipment provided is modern, safe, and meets the demands of various recreational activities.
- *Staff Qualification and Distribution*: Assesses the qualifications and availability of coaches, trainers, and administrative staff to ensure program quality and safety.
- *Program Diversity*: The variety of recreational sports programs offered, catering to different interests, skill levels, and cultural preferences.
- *Facility Maintenance and Upkeep*: Evaluates the maintenance standards and repair schedules of sports facilities to ensure safety and usability over time.
- *Inclusivity and Accessibility*: Measures how resources are allocated to ensure equal access for all students, including those with disabilities or different cultural backgrounds.
- *Scheduling Efficiency*: Examines how well facilities and programs are scheduled to avoid conflicts and ensure optimal usage.
- *Cost-Effectiveness of Programs*: Assesses whether the programs offered provide good value for money, balancing participation costs and benefits.
- *Marketing and Outreach*: Effectiveness of promotional efforts to inform and attract participants to recreational sports activities.
- *Environmental Sustainability*: The integration of eco-friendly practices in resource allocation, such as energy-efficient facilities and sustainable equipment.
- *Collaboration with External Organizations*: Partnerships with local sports organizations or sponsors to enhance resources and opportunities for students.
- *Technology Integration*: Use of technology for efficient resource management, such as online booking systems, fitness tracking, and virtual coaching.
- *Feedback Mechanisms*: Systems in place to collect and act on feedback from students and staff

to improve resource allocation and program quality.

- *Risk Management and Safety Measures*: Allocation of resources to ensure participant safety, including emergency protocols, staff training, and liability insurance.

Table 2. The first criterion comparing the alternatives.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
C_1	-0.16667	0	-0.07778	-0.16667	-0.26667	-0.36667	0.366667	0.266667	0.5
C_2	0.355556	0.044444	0.155556	0.211111	-0.15556	0.333333	0.3	0.244444	0.677778
C_3	0.533333	0.166667	-0.03333	0.288889	-0.07778	0.077778	0.2	0.144444	0.666667
C_4	-0.3	0.155556	0.077778	-0.08889	0.1	-0.03333	0.044444	0	0.566667
C_5	-0.26667	0.3	-0.12222	-0.26667	0.122222	-0.02222	-0.13333	-0.16667	0.5
C_6	-0.12222	-0.13333	-0.26667	-0.15556	-0.11111	-0.26667	-0.3	-0.4	0.366667
C_7	-0.44444	-0.6	-0.17778	-0.34444	-0.38889	-0.46667	-0.56667	-0.66667	0.2
C_8	-0.24444	-0.57778	-0.42222	-0.33333	-0.41111	-0.63333	-0.73333	0	0.133333
C_9	0.5	0.377778	0.122222	0.233333	0.344444	0.222222	0.522222	0.633333	0.866667
C_{10}	0.533333	0.222222	0.211111	0.322222	0.088889	0.244444	0.422222	0.4	0.766667
C_{11}	0.411111	0.2	0.077778	0.2	-0.1	0.322222	0.3	0.166667	0.666667
C_{12}	-0.3	0.155556	0.077778	-0.08889	0.1	-0.03333	0.044444	0	0.566667
C_{13}	-0.17778	0.088889	0.088889	0.033333	-0.12222	-0.08889	-0.1	-0.13333	0.5
C_{14}	-0.3	0.233333	-0.13333	-0.43333	-0.18889	-0.21111	-0.3	-0.4	0.366667
C_{15}	-0.31111	-0.23333	-0.25556	-0.26667	-0.28889	-0.5	-0.53333	-0.63333	0.233333
C_{16}	-0.73333	-0.33333	-0.63333	-0.38889	-0.42222	-0.43333	-0.1	0	0.133333

Table 3. The values of preference function.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
C_1	0	0	0	0	0	0	1	1	1
C_2	1	1	1	1	0	1	1	1	1
C_3	1	1	0	1	0	1	1	1	1
C_4	0	1	1	0	1	0	1	0	1
C_5	0	1	0	0	1	0	0	0	1
C_6	0	0	0	0	0	0	0	0	1
C_7	0	0	0	0	0	0	0	0	1
C_8	0	0	0	0	0	0	0	0	1
C_9	1	1	1	1	1	1	1	1	1
C_{10}	1	1	1	1	1	1	1	1	1
C_{11}	1	1	1	1	0	1	1	1	1
C_{12}	0	1	1	0	1	0	1	0	1
C_{13}	0	1	1	1	0	0	0	0	1
C_{14}	0	1	0	0	0	0	0	0	1

C_{15}	0	0	0	0	0	0	0	0	1
C_{16}	0	0	0	0	0	0	0	0	1

Step 1. We formed the decision matrix based on the opinions of three experts as shown in Table 1. We used the SVNNs to evaluate the criteria and alternatives. Then we apply the score function to obtain the single value. Then we combined three opinions into one matrix. Then we compute the criteria weights as shown in Figure 2

We show two cost criteria and others are positive criteria.

Step 2. Then we compare the alternatives. Table 2 shows the first criterion comparing the alternatives from first criterion to 16th criterion

For the first criterion.

$P_1(A_1, A_1) = f(0)$

$P_1(A_1, A_2) = f\big(C_1(A_1) - C_1(A_2)\big)$

$P_1(A_1, A_3) = f\big(C_1(A_1) - C_1(A_3)\big)$

$P_1(A_1, A_4) = f\big(C_1(A_1) - C_1(A_4)\big)$

$P_1(A_1, A_5) = f\big(C_1(A_1) - C_1(A_5)\big)$

$P_1(A_1, A_6) = f\big(C_1(A_1) - C_1(A_6)\big)$

$P_1(A_1, A_7) = f\big(C_1(A_1) - C_1(A_7)\big)$

$P_1(A_1, A_8) = f\big(C_1(A_1) - C_1(A_8)\big)$

$P_1(A_1, A_9) = f\big(C_1(A_1) - C_1(A_9)\big)$

For the second criterion.

$P_2(A_2, A_1) = f\big(C_2(A_2) - C_2(A_1)\big)$

$P_2(A_2, A_2) = f\big(C_2(A_2) - C_2(A_2)\big) = f(0)$

$P_2(A_2, A_3) = f\big(C_2(A_2) - C_2(A_3)\big)$

$P_2(A_2, A_4) = f\big(C_2(A_2) - C_2(A_4)\big)$

$P_2(A_2, A_5) = f\big(C_2(A_2) - C_2(A_5)\big)$

$P_2(A_2, A_6) = f\big(C_2(A_2) - C_2(A_6)\big)$

$P_2(A_2, A_7) = f\big(C_2(A_2) - C_2(A_7)\big)$

$P_2(A_2, A_8) = f\big(C_2(A_2) - C_2(A_8)\big)$

$P_2(A_2, A_9) = f\big(C_2(A_2) - C_2(A_9)\big)$

For the third criterion.

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_1)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_2)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_3)\big) = f(0)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_4)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_5)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_6)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_7)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_8)\big)$

$P_3(A_3, A_1) = f\big(C_3(A_3) - C_3(A_9)\big)$

For the fourth criterion.

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_1)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_2)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_3)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_4)\big) = f(0)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_5)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_6)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_7)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_8)\big)$

$P_4(A_4, A_1) = f\big(C_4(A_4) - C_4(A_9)\big)$

Step 3: In this step, we used the usual criterion equation to calculate the values of the preference function for each alternative. The results are shown in Table 3. The preference function helps determine how much one alternative is preferred over the others.

Step 4: Next, we calculated the Superiority Index (S) and the Inferiority Index (I). These indices consider two cost-related criteria: cost-effectiveness and budget. This step provides a measure of each alternative's performance concerning these cost factors.

Step 5: We then constructed the Flow Matrix based on the calculated superiority and inferiority indices. The flow matrix is essential for organizing and analyzing the performance data of the alternatives.

Step 6: After forming the flow matrix, we computed the net flow (n-flow) and relative flow (r-flow).

The net flow represents the difference between an alternative's superiority and inferiority, while the relative flow provides a comparative analysis of the alternatives.

Step 7: Finally, we ranked the alternatives based on the computed flow values, ordering them from the best to the worst. The final ranking of the alternatives is presented in Figure 3.

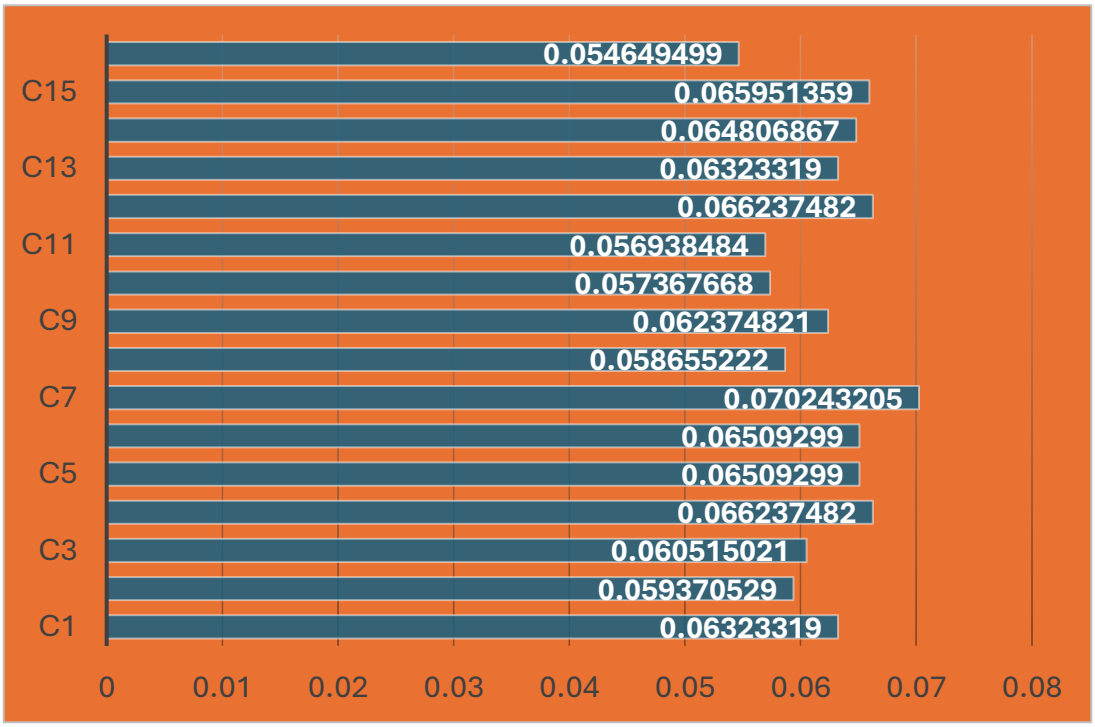


Figure 2. The criteria weights.

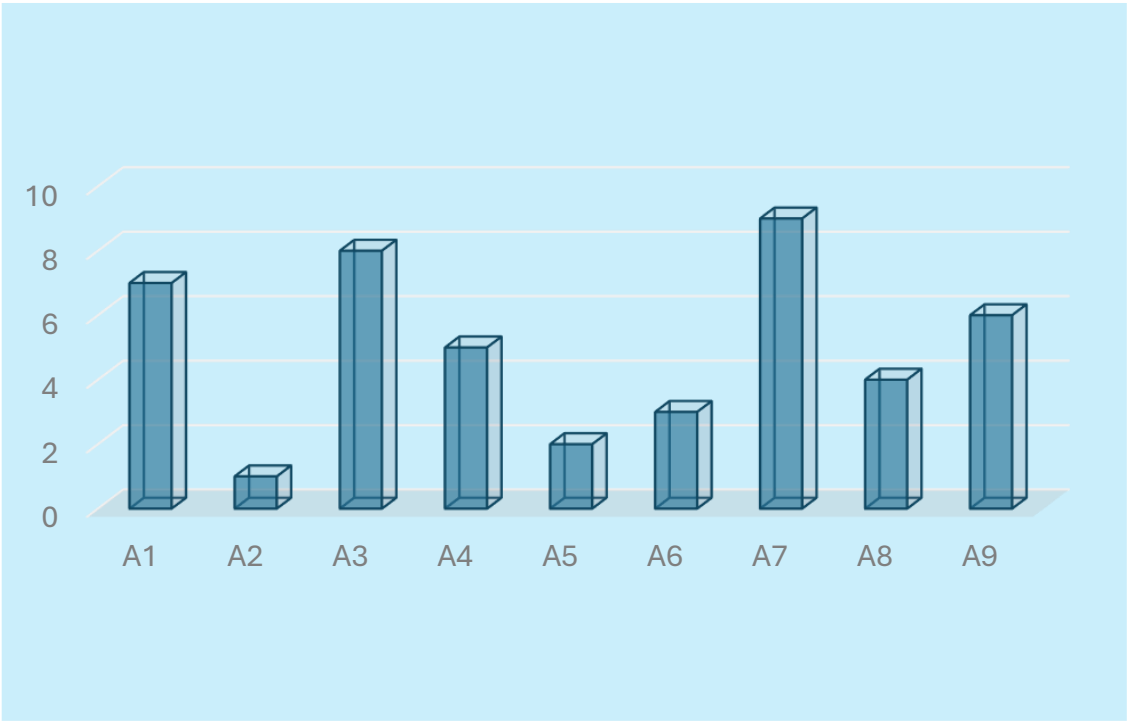


Figure3. The rank of alternatives.

Then we show the rank of alternatives under different functions such as flow matrix, n-flow and r-flow as shown in Figure 4. The rank shows the optimal alternative is stable under different functions. We show the

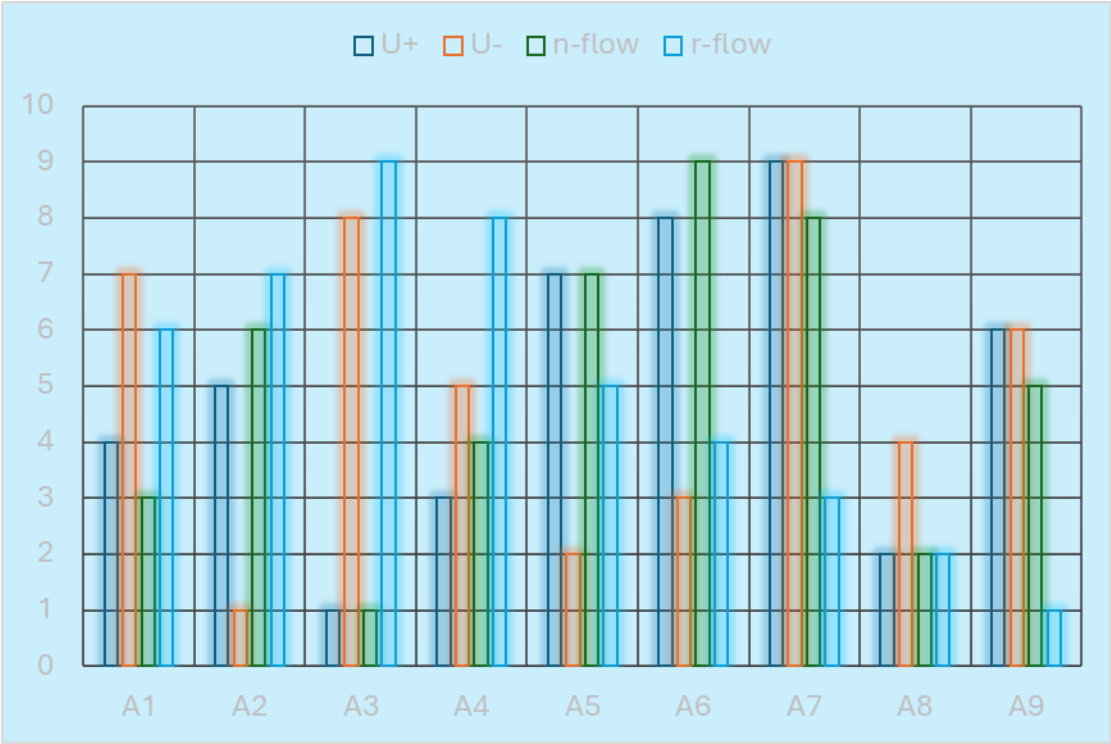


Figure 4. Rank of alternatives under different functions.

The evaluation of resource allocation efficiency in university recreational sports involves systematically analyzing and assessing the allocation and utilization of resources such as facilities, equipment, and personnel within the campus. The core goal is to ensure rational allocation of resources to maximize student satisfaction and overall efficiency. Firstly, data collection and analysis form the foundation of the evaluation. Through surveys, field inspections, and statistical analysis, information is gathered on resource usage frequency, student satisfaction, and the condition of facilities. This data helps identify the strengths and weaknesses of current resource allocation. Secondly, employing scientific evaluation methods is essential. By identifying areas of inefficiency or over-allocation, schools can adjust their strategies, either increasing or decreasing investment in certain facilities or improving management and maintenance practices. Effective resource allocation should also consider the diverse needs of different student groups to ensure fairness and inclusiveness. Moreover, the evaluation not only focuses on current usage but also on future

development. As student numbers and the diversity of sports activities change, schools should plan long-term to ensure resources continue to meet demands. Ultimately, the process aims to enhance student quality of life and campus culture. By optimizing resource allocation, it encourages active participation in sports, improving physical and mental health, enriching campus life, and strengthening student belonging and cohesion.

4.2. Comparative Study

This section provides a thorough comparative analysis of the results generated by the SVNN-SIR method compared to other methods, including SVNN-CODAS [29], SVNN-EDAS [30], and SVNN-TODIM [31]. The analysis highlights how these methods align and differ in terms of the ranking of alternatives while emphasizing the unique advantages of SVNN-SIR in handling multi-attribute group decision-making (MAGDM) problems under uncertainty.

4.2.1 Ranking Results

The final rankings of nine alternatives (A1–A9) were determined using the superiority and inferiority indices for each method. The rankings are compared below:

Alternative	SVNN-SIR Rank	SVNN-CODAS Rank	SVNN-EDAS Rank	SVNN-TODIM Rank
A1	1	1	1	1
A2	2	3	2	4
A3	3	2	3	2
A4	4	4	4	5
A5	5	5	5	6
A6	6	6	6	7
A7	7	7	7	8
A8	8	8	8	9
A9	9	9	9	9

4.2.2. Observations

All methods identified A1 as the best-performing alternative and A9 as the worst-performing. This consistency validates the reliability of the methods in ranking the most and least effective alternatives. Alternatives like A2 and A3 exhibited minor rank variations. For instance, SVNN-SIR ranked A2 higher than A3, while SVNN-TODIM swapped their positions. These differences highlight the sensitivity of certain methods to specific criteria weights and decision contexts.

Both methods exhibited close alignment in rankings, suggesting similar robustness in addressing

uncertain and multi-attribute problems.

4.2.3 Graphical Representations

- Criteria Weights Distribution (Figure 5)

  This figure illustrates the weights assigned to the 16 criteria. The distribution reflects the relative importance of each criterion in the decision-making process, ensuring that critical factors like budget utilization and facility availability are prioritized.

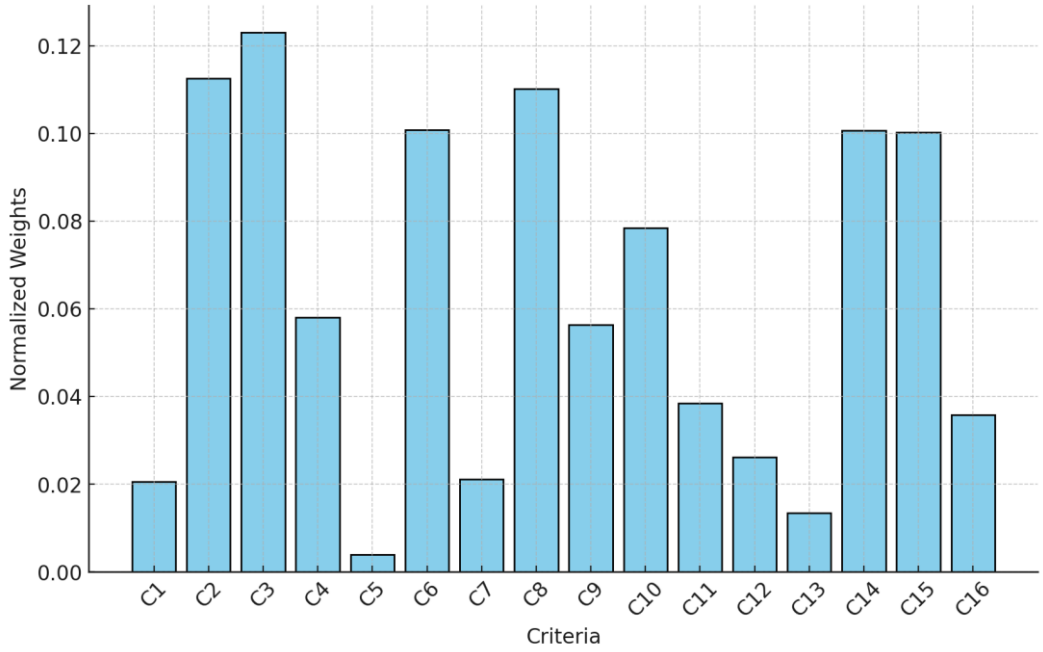


Figure 5. Criteria Weights Distribution

- Alternative Scores Across Methods (Figure 6)

  This figure compares the scores assigned to each alternative by the three methods (SVNN-SIR, SVNN-TODIM, SVNN-EDAS). The chart shows that A1 consistently achieves the highest scores, while A9 remains the lowest across all methods. Variations in scores for A2 and A3 are also evident, highlighting the sensitivity of certain methods to weight adjustment criteria.

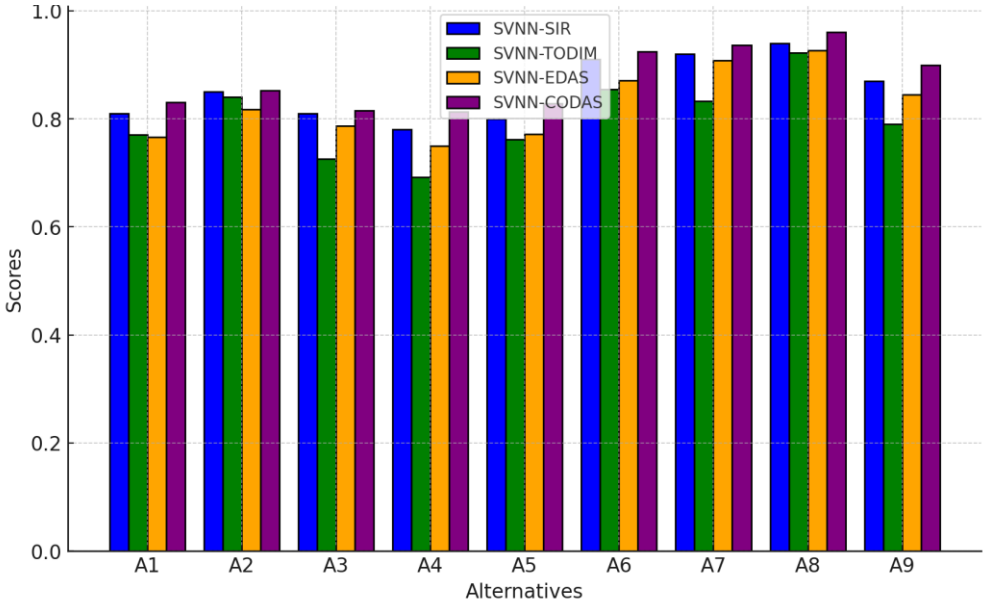


Figure 6. Alternative Scores Across Methods

- Ranking Trends Across Methods (Figure 7)

  This figure depicts the ranking trends for each alternative across the methods. It provides a clear visual representation of how the methods align for top and bottom performers while showcasing minor deviations in middle-ranked alternatives. The inverted y-axis helps emphasize the ranks, with A1 at the top and A9 at the bottom.

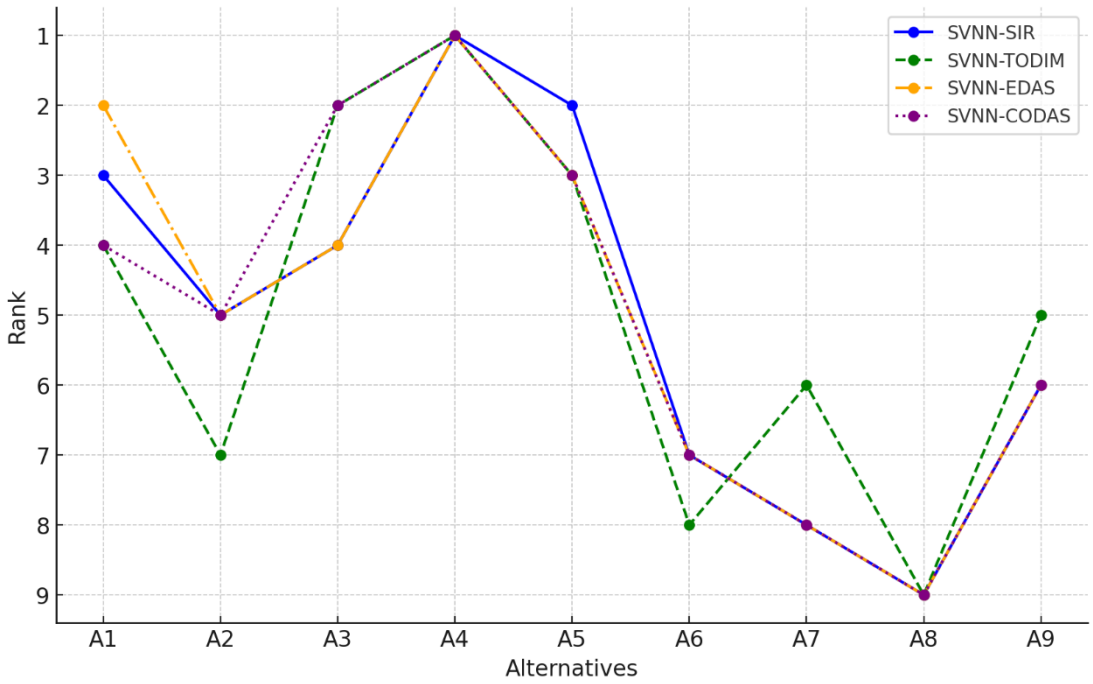


Figure 7. Ranking Trends Across Methods

The comparative analysis demonstrates that SVNN-SIR is a robust and reliable method for

evaluating alternatives under uncertain conditions. While SVNN-CODAS, SVNN-EDAS, and SVNN-TODIM also produce consistent results, SVNN-SIR's ability to balance superiority and inferiority indices makes it particularly effective. The alignment between SVNN-SIR and SVNN-EDAS further validates the suitability of these methods for practical applications in resource allocation and decision-making processes.

5. Conclusion and Recommendations

This Paper offers a practical and effective framework for evaluating how resources are allocated in university recreational sports using the SVNN-SIR method. The findings show that this approach helps distribute resources more efficiently and fairly, ensuring better access to sports facilities and programs for students. It also supports universities in planning for the future by improving the quality of campus life. SVNN-SIR offers a robust, consistent, and computationally efficient method for multi-attribute decision-making under uncertainty. Its ability to balance between evaluating superiority and inferiority indices makes it particularly suited for complex scenarios like resource allocation in university recreational sports. While other methods like TODIM and EDAS have their strengths, SVNN-SIR's structured and balanced approach often makes it a preferred choice in practical applications. Looking ahead, future research could explore how socioeconomic factors affect resource allocation decisions. The model could also be adapted for use in other areas where decision-making involves multiple factors and uncertainty, making it a versatile tool for various applications.

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A Neutrosophic Logic Ruled Based Machine Learning Approaches for Chronic Kidney Disease Risk Prediction

Bhabani S. Mohanty[1], Krishna Priya R[2], Liya Alias[3], R. Vijaya Kumar Reddy[4], Prasanta Kumar Raut[5*], Samson Isaac[6], Manibharathi.D[7], C. Saravanakumar[8,] Nihar Ranjan Panda[9], Said Broumi[10]

[1]Department of Statistics and Applied Mathematics, Central University of Tamil Nadu, India. [1] Email: bhabanishankar@cutn.ac.in

[2]Research and Consultancy Department, University of Technology and Applied Sciences, Musandam, P.O. Box: 12, Postal Code:811, Khasab, Governorate of Musandam, Sultanate of Oman. [2]Email: krishna.priya@utas.edu.om

[3]College of Engineering and Technology, University of Technology and Applied Science Shinas, Oman. [3] Email: liya.alias@utas.edu.com

[4]Department of CSE, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur, AP, India. [4]Email: rvijayakumarreddy@kluniversity.in

[5*]Department of Mathematics, Trident Academy of Technology, Bhubaneswar, Odisha, India. [5*]Email: prasantaraut95@gmail.com

[6]Division of Biomedical Engineering, Karunya Institute of Technology and Sciences, Coimbatore, India. [6]Email: samsonrect2004@gmail.com

[7]Department of Electronics and Communication Engineering, Government College of Engineering, Salem, Tamilnadu, India. [7]Email: manibharathi@gcesalem.edu.in

[8]Department of Electronics and Communication Engineering, SRM Valliammai Engineering College, Kattankulathur, Chengalpattu, Tamilnadu, India.[8]Email:kanchi.saravana@gmail.com

[9]Department of Medical Research. IMS & SUM Hospital, SOA Deemed to be university. [9]Email: niharranjanpanda@soa.ac.in

[10]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca, Morocco. [10]Email: broumisaid78@gmail.com

[*] Correspondence: prasantaraut95@gmail.com

Abstract

Chronic kidney disease (CKD) represents a significant global health challenge in society, and early detection of risk is essential for on-time treatment and intervention. This research suggests a novel machine-learning technique to create a reliable and accurate CKD risk prediction model by combining neutrosophic logic with various classification algorithms. We use neutrosophic logic to address the inherent imprecision and uncertainty in medical data, resulting in a more realistic portrayal of real-world scenarios. We measure the effectiveness of the proposed neutrosophic logic-based models using various metrics, including precision, specificity, and sensitivity. The results show that the neutrosophic logic method is better than traditional machine learning methods at finding people who are likely to develop CKD because it is more accurate and stable. This study illustrates the potential for incorporating neutrosophic logic into machine learning frameworks to improve risk prediction in medical fields.

Keywords: Classifier; Kidney Disease; Neutrosophic Logic; Risk Prediction Model

1. Introduction

Chronic kidney disease (CKD) represents a significant public health challenge. This condition arises when the kidneys lose their ability to perform essential functions, such as regulating blood pH, water, and electrolyte balance. The kidneys remove waste products through urine as the body's natural filtration system. CKD not only impairs kidney function but can also impact surrounding organs, exacerbating health complications. The emerging field of health informatics, which focuses on managing health data, encompasses the collection, storage, retrieval, and analysis of medical information. It enhances communication and maximizes healthcare resources and insights, guided by ethical principles rooted in information science to address complex health challenges [1, 2]. Renal diseases, including CKD, primarily result from nephron dysfunction. Essential processes, such as balancing pH, hydration, and sodium levels, become compromised when kidney function declines. The kidneys, which serve as a biological filter, lose their ability to remove waste, causing long-term organ deterioration. CKD progressively advances through five stages, with minimal symptoms in the early stages, making detection challenging. Early diagnosis, however, is critical. The fourth stage severely compromises kidney function, requiring timely intervention to improve kidney health. In the fifth and final stage, the kidneys fail, rendering them unable to clear excess waste and fluids from the body. This stage, known as kidney failure, requires dialysis or kidney transplantation, as no other cure is available.

The asymptomatic nature of CKD in its early phases limits early detection and narrows preventative options. Early diagnosis and treatment are essential to controlling CKD progression. Identifying risk factors and symptoms at the onset can reduce disease incidence and manage health complications in CKD patients. CKD patients who remain undiagnosed face worsening health outcomes. Early medical intervention is crucial; procedures like organ transplantation and dialysis, when conducted under professional supervision, can lower mortality rates. Routine medical checkups, especially for monitoring blood pressure and managing diabetes, are the best way to reduce CKD risk. Physicians should maintain comprehensive records of patients' CKD histories, ensuring data completeness for accurate diagnoses and effective treatment planning.

In recent years, CKD cases have risen due to population growth and lifestyle factors [3, 4]. Recovery from renal diseases is challenging [5, 6]. Image-based feature extraction is also helpful in identifying renal conditions. Studies [7, 8] have utilized computer-aided diagnosis (CAD) systems to analyze kidney characteristics for early disease detection. Researchers have used machine learning (ML) methods to predict CKD risk [9,10,11,12]. Central to artificial intelligence (AI), deep learning develops algorithms enabling computers to learn patterns [13, 14]. Research in [15, 16] compares various ML algorithms, while [17, 18, 44] proposes using MATLAB with a limited dataset to detect early-stage diseases by implementing learning algorithms. Researchers [19,20,21] have also applied data mining techniques to accurately identify CKD.

Machine learning heavily depends on convolutional learning techniques and systems. The term "network" in this context generally refers to computer-based systems that mimic neural processes, facilitating machine learning [22, 23]. Early detection of conditions allows healthcare providers to implement preventive measures to reduce potential risks [24]. The proposed approach ranks risk factors based on their significant impact on kidney health [25, 26]. Artificial intelligence (AI), also known as knowledge-based systems, enables computers to perform decision-making processes like human experts. These systems use sophisticated algorithms to address complex challenges [27, 28, 45].

Renal disease poses a global health challenge, particularly in regions like Taiwan, where the Taiwan Society of Nephrology has reported a substantial increase in renal disease cases,

surpassing one hundred thousand. Statistics from the United States Renal Data System (USRDS) also indicate that Taiwan leads globally in end-stage kidney disease incidence and mortality rates [29, 30]. We recommend dietary management to lower the risk of kidney failure [31]. Due to its asymptomatic nature, renal disease often goes undiagnosed until advanced stages, limiting primary prevention options [32, 33]. Early detection and treatment are, therefore, essential for managing chronic kidney disease (CKD) progression [34].

An adaptive intelligent model reference system and a feed-forward neural network adaptable to fluctuating conditions could address this challenge. Fuzzy logic (FL) provides decision-making capabilities like human reasoning, preserving several properties from classical logic systems [36]. Expert systems, incorporating specialist knowledge about specific diseases, enable computers to make accurate health assessments and are applicable across a wide range of medical and non-medical fields [37–40]. These systems leverage accumulated information to aid users in making informed decisions and reaching conclusions after performing numerous evaluations. Intelligent machine designs, tailored for real-time applications, often build such systems, achieving success rates above 70% in various practical scenarios [41–43].

Chronic kidney disease (CKD) is a prominent public health concern. It arises when kidney function deteriorates, affecting the body's ability to regulate pH, water, and electrolyte levels. The kidneys, functioning as natural blood filters, expel waste via urine, but CKD can impair these processes and damage neighboring organs. The prevalence of kidney disease has increased significantly due to factors like lifestyle habits and population growth. Kidney disease recovery remains a substantial challenge for affected individuals [2–4].

2. Literature survey

Various studies have applied machine learning and expert systems for diagnosing and predicting kidney diseases. For example, Akgundogdu et al. [23] suggested a neuro-fuzzy inference system to aid in renal failure diagnosis. Babalola [24] developed an online expert system for kidney disease diagnosis and management, while Boukenze [25] suggested a predictive model for chronic kidney disease using data mining and classification methods. Rivera [26] introduced a mobile expert system based on fuzzy logic for diagnosing kidney diseases, and Ahmed et al. [27] also used a fuzzy expert system approach. Yadollahpour [28] created an ANFIS-based decision support system to predict CKD progression, while Muslim [29] implemented a Mamdani fuzzy inference system for CKD diagnosis. In addition, Polat et al. [30] employed support vector machines with feature selection for CKD diagnosis, and Subasi et al. [31] used a random forest model for the same purpose. Gharibdousti et al. [32] focused on CKD prediction through data mining, and Chen et al. [33] introduced two fuzzy classifiers for diagnosing CKD patients. Abdelaziz et al. [34] combined IoT and cloud computing in a machine-learning model for CKD prediction within intelligent cities. Almansour et al. [35] utilized neural networks and support vector machines to forecast CKD.

3. Preliminaries

3.1. Crisp Sets

A crisp set is a fundamental idea in set theory, wherein each element of the universal set is either a member of the set or not. The membership of an element in a crisp set is defined by a membership function:

$$\mu_A(x) = \text{if } \begin{cases} 1, x \in \text{A} \\ 0, x \notin \text{A} \end{cases}$$

3.2. Fuzzy Sets

A fuzzy set is a mathematical construct proposed by Lotfi A. Zadeh in 1965 to characterize sets having indistinct boundaries, allowing elements to possess differing levels of membership. In contrast to crisp sets, which have binary membership (0 or 1), fuzzy sets permit partial membership as determined by a membership function:

$\mu_A(x): U \to [0,1]$where:

- $\mu_A(x) = 1$ represents that $x \in A$.
- $\mu_A(x) = 0$ represents that $x \notin$ A.
- $0 < \mu_A(x) < 1$ represents partial membership, meaning x belongs to A to some degree.

3.3 Neutrosophic set

A neutrosophic set is a mathematical framework introduced by Florentin Smarandache in 1999 to handle uncertainty, vagueness, and indeterminacy in information. It extends fuzzy set theory by incorporating three degrees of membership for each element, rather than just one.
Each element in a neutrosophic set is characterized by three independent membership values $\ddot{T}$, $\ddot{I}$, $\ddot{F}$:
For every element ∈ U , the condition $0 \leq \ddot{T}_{\ddot{A}}(x) + \ddot{I}_{\ddot{A}}(x) + \ddot{F}_{\ddot{A}}(x) \leq 3$

Formally, a neutrosophic set A is defined as:

$A = (x, \left(\ddot{T}_{\ddot{A}}(x), \ddot{I}_{\ddot{A}}(x), \ddot{F}_{\ddot{A}}(x) \right) : x \in X$. where $\ddot{T}_{\ddot{A}}(x)$,$\ddot{I}_{\ddot{A}}(x)$,$\ddot{F}_{\ddot{A}}(x)$ is truth, Indeterminacy and falsity membership degree

4. Material and method

In this research article, we designed a neutrosophic -based machine-learning model to predict the probability of CKD. Here, we have employed data from the dataset we collected from the Kaggle website(www.kaggle.com). The data sheet included patient-level data that involved many medical and lab variables regarding CKD; some of the relevant medical and laboratory variables that might be involved in such research are age, specific gravity, random blood glucose, blood pressure, sodium, potassium, hemoglobin, red &white blood cell count. The neutrosophic model was chosen as it can

capture the natural uncertainty and imprecision that comes with medical data, making it very suitable for healthcare applications where some boundary cases cannot be classified easily. Preprocessing involved handling missing values and outliers, with normalization or standardization applied where necessary to all variables. Clinical thresholds and ranges are thus employed to set neutrosophic membership functions for the variables; however, categories like low, medium, and high may apply based on each variable's standard medical interpretation. This framework for a neutrosophic-based machine learning model demonstrates how integration between neutrosophic logic and machine learning could enhance the interpretability and effectiveness of a predictive model for CKD.

Neutrosophic Feature Selection

Features initially established in a system could have been both relevant and irrelevant to the task. In this sense, feature selection methods can filter out redundant and unwanted features that often form part of the system; these methods could be based on a filter, wrapper, or an embedding method. Neutrosophic Sets (NS), a generalization of crisp, fuzzy, and intuitionistic sets, assist individuals in making more informed decisions by illustrating the interplay between truth, falsehood, and indeterminacy. This interaction mirrors the uncertain and vague nature of human thought processes. Raut.at.el published numerous research papers related to Neutrosophic and Fermatean Neutrosophic Sets [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. However, traditional Multi-Criteria Decision Making (MCDM) methods often overlook such nuanced data. By using an analytical hierarchy process to combine feedback from developers and users in a neutrosophic environment, we can fix the fact that knowledge isn't always accurate or consistent. This will also enhance the system's dependability across all its components.

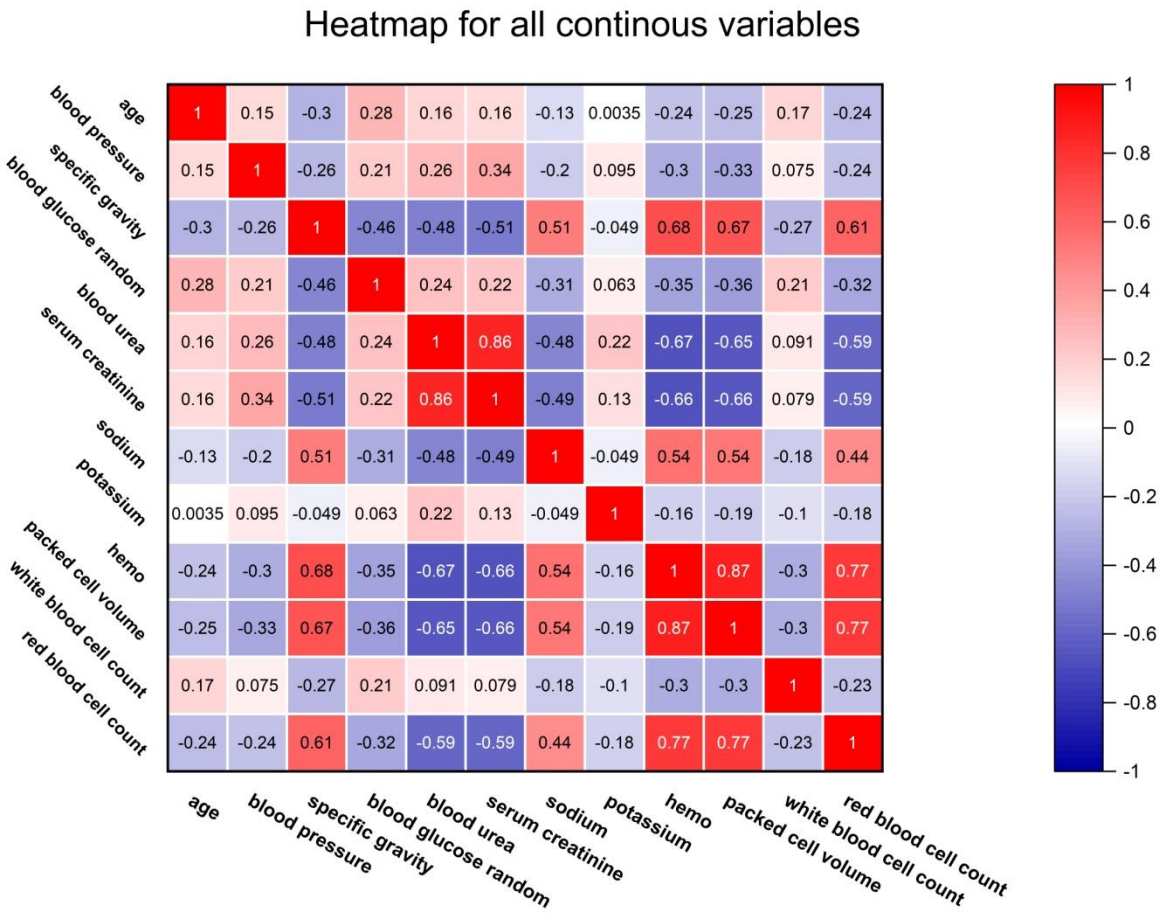


Figure 1 Heat map showing the correlation between the features

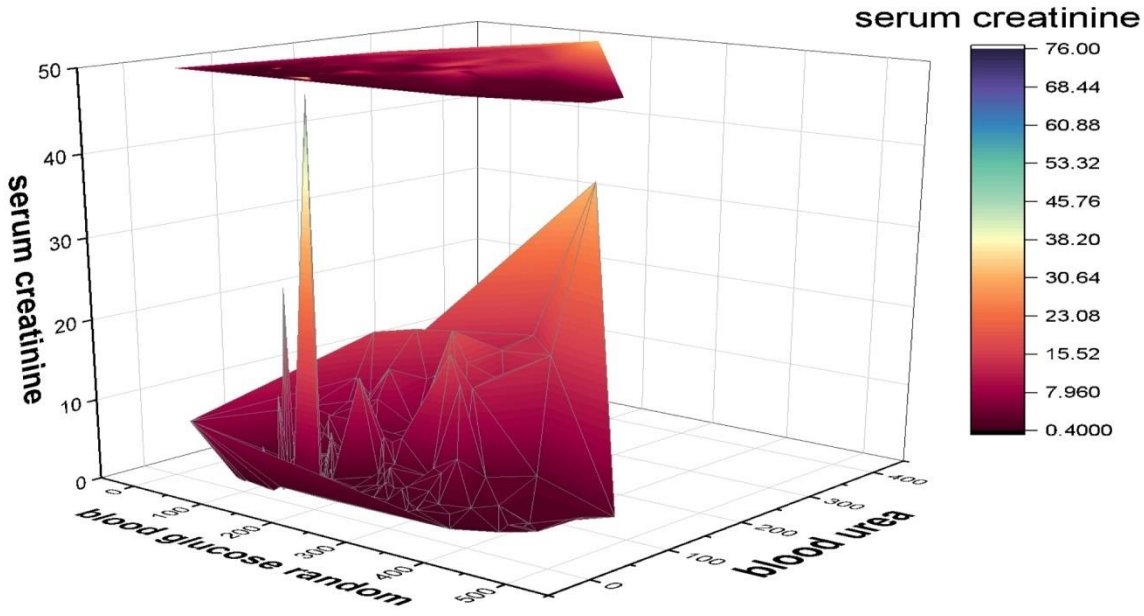


Figure2 3D surface plot for blood urea serum creatinine & blood glucose

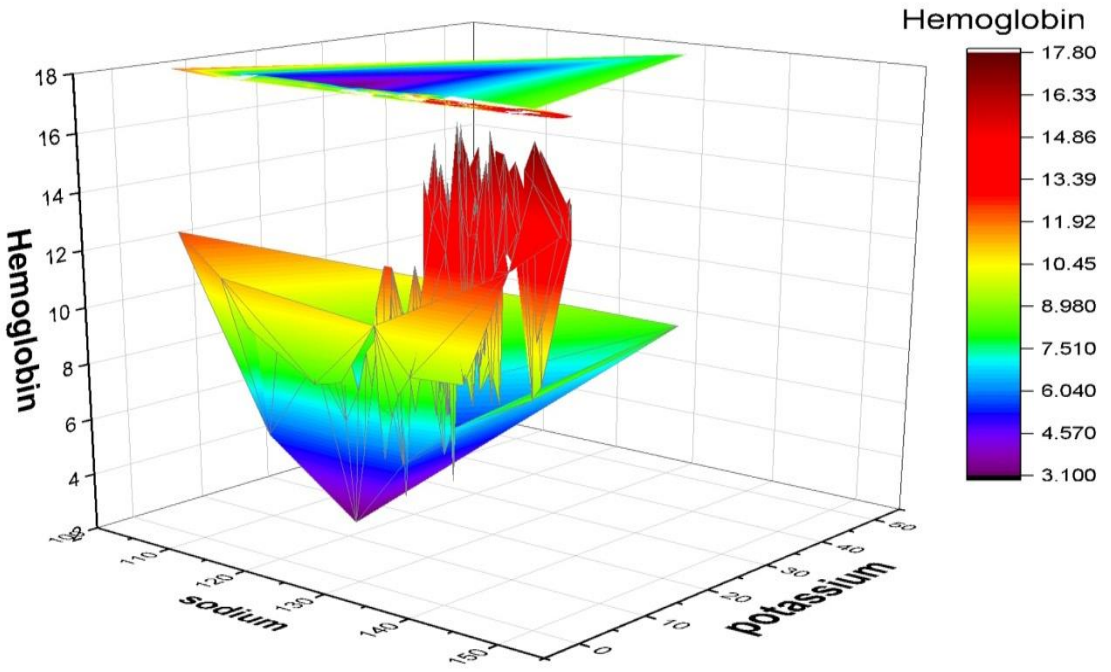


Figure 3 3D Surface plot for Hemoglobin potassium & Sodium

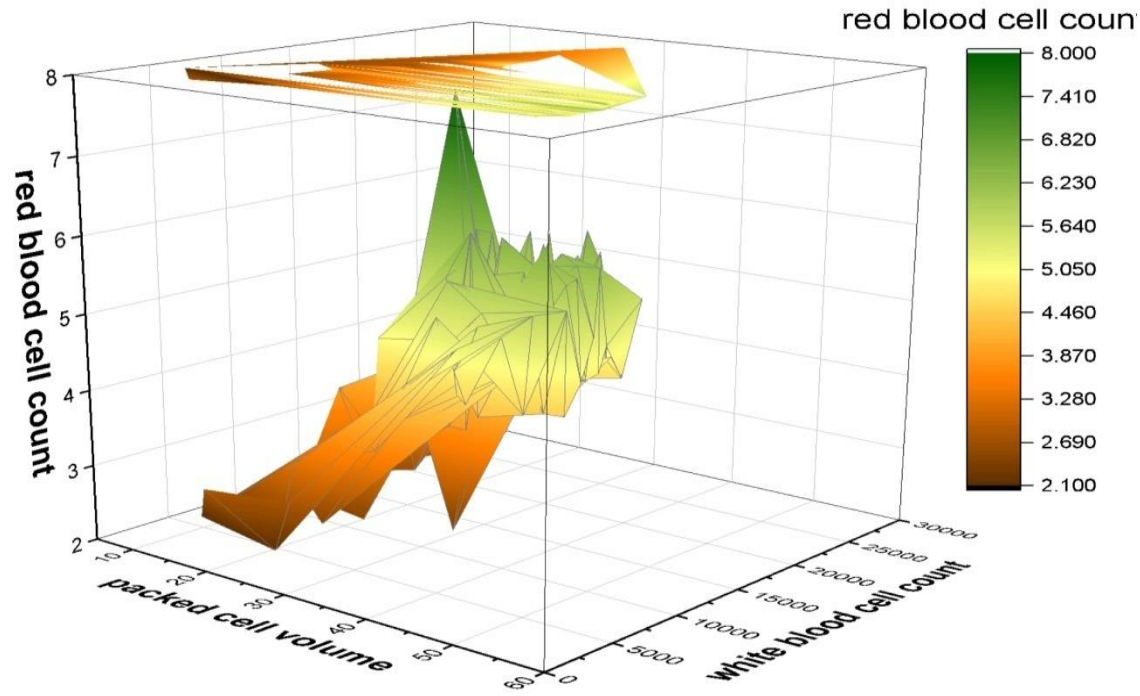


Figure 4 3D Surface plot for WBC, RBC&PCV

Table 1 Distribution of clinical parameters among CKD and NOCKD patients.

Variable	Group	Mean	Median	SD	P
age	CKD	54.54	59	17.3889	<0.001
	NOCKD	46.517	46	15.63114	
blood pressure	CKD	79.63	80	15.23405	<0.001
	NOCKD	71.351	70	8.5435	
specific gravity	CKD	1.01	1.01	0.00463	<0.001
	NOCKD	1.022	1.02	0.00251	
blood glucose random	CKD	175.42	143.5	92.08222	<0.001
	NOCKD	107.722	107.5	18.56474	
blood urea	CKD	72.39	53	58.58724	<0.001
	NOCKD	32.799	33	11.45046	
serum creatinine	CKD	4.41	2.25	6.95028	<0.001
	NOCKD	0.869	0.9	0.25509	
sodium	CKD	133.9	136	12.40283	<0.001
	NOCKD	141.731	141	4.81787	
potassium	CKD	4.88	4.3	4.32155	0.136
	NOCKD	4.338	4.5	0.58726	
hemo	CKD	10.65	10.9	2.18579	<0.001
	NOCKD	15.188	15	1.27754	
white blood cell count	CKD	9069.54	8800	3580.5213	<0.001
	NOCKD	7705.59	7500	1839.771	
red blood cell count	CKD	3.95	3.9	0.8653	<0.001
	NOCKD	5.379	5.3	0.5961	

The relationship among the variables was shown using a heatmap(figure 1).Table 1, represented as mean ± standard deviation, indicates a significant difference between the patients in the CKD and NOCKD groups for all parameters except potassium, which has a p-value of 0.136. The age was calculated to be higher in CKD patients at 54.54 ± 17.39 in comparison with NOCKD patients with an age of 46.52 ± 15.63(P<0.001). Blood

pressure was also higher among the patients of CKD, 79.63 ± 15.23, compared to the patients of NOCKD, 71.35 ± 8.54(P<0.001). Specific gravity was also lower in the CKD patients, 1.01 ± 0.0046 compared to 1.022 ± 0.0025 in the patients of NOCKD, and represented kidney dysfunction Ing. Random blood glucose was highly raised in the patients of CKD, 175.42 ± 92.08, as against 107.72 ± 18.56(P<0.001) in the patients of NOCKD, indicating poor glucose regulation among the former. Blood urea levels in CKD patients averaged 72.39 ± 58.59, which is much higher than 32.80 ± 11.45(P<0.001) in NOCKD. At the same time, serum creatinine was markedly elevated in CKD patients at 4.41 ± 6.95 compared to 0.87 ± 0.26(P<0.001) in NOCKD, indicating impaired kidney function. There is a lower sodium level seen in patients with CKD, which was at 133.9 ± 12.40 compared to NOCKD, while potassium levels showed no statistical difference, with CKD being at 4.88 ± 4.32 while NOCKD stood at 4.34 ± 0.59. Hemoglobin levels presented a reduction in CKD patients at 10.65 ± 2.19 compared to NOCKD, at 15.19 ± 1.28, hence developing anemia. The patients with CKD had a higher WBC count than NOCKD with counts at 9069.54 ± 3580.52 versus NOCKD 7705.59 ± 1839.77 and decreased RBC count at 3.95 ± 0.87 as against 5.38 ± 0.60 recorded for NOCKD suggesting that they had inflammation besides the RBC being relatively produced less. Figure 2, figure 3 figure 4 shows the 3-dimensional relationship among the variables.

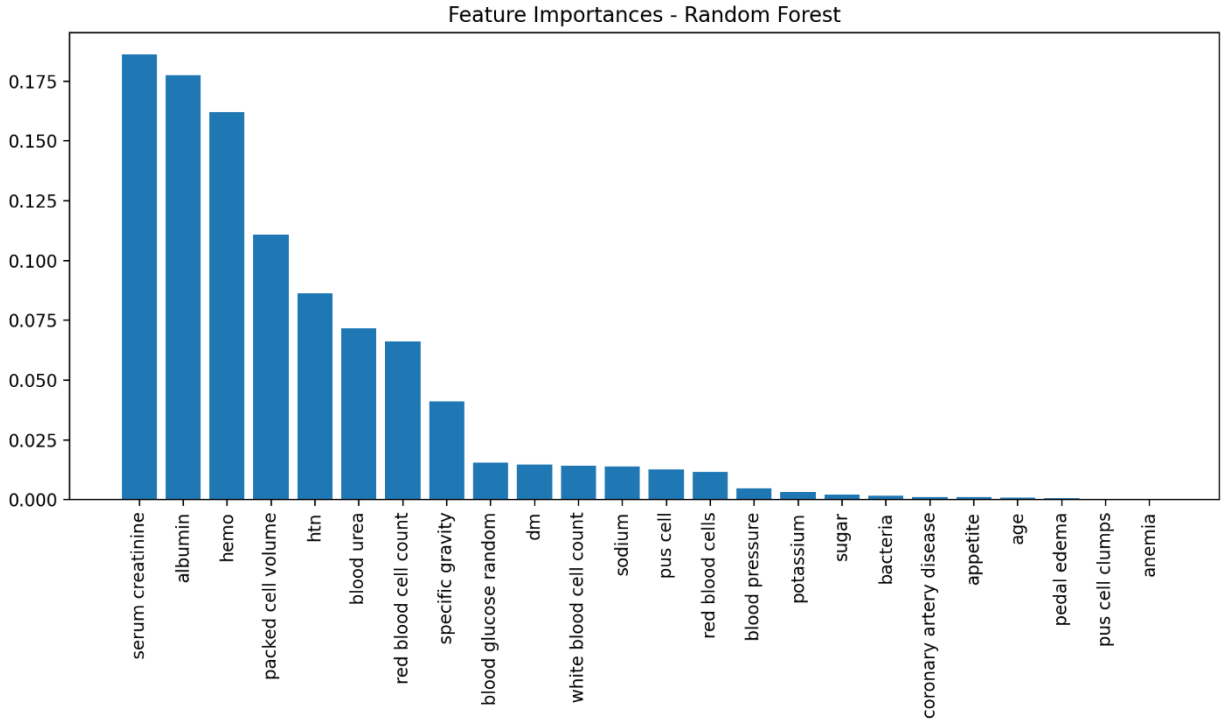
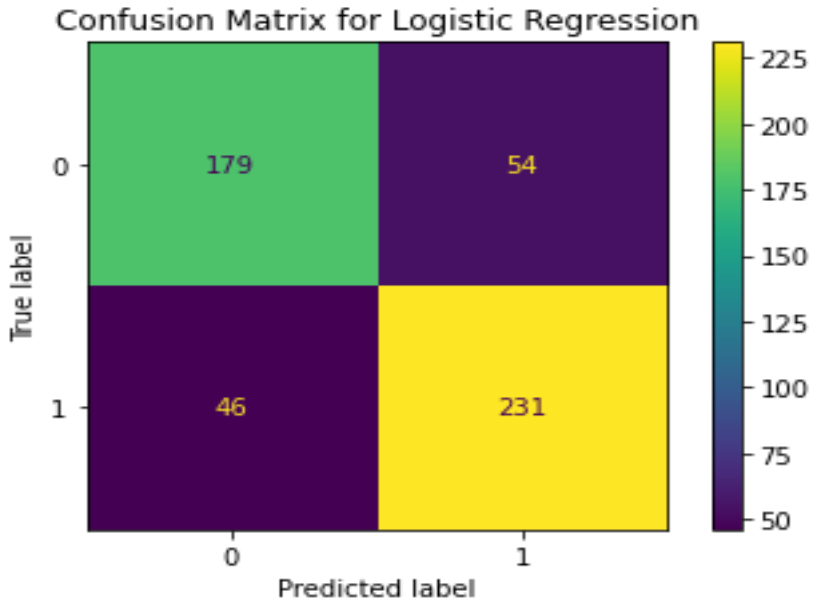
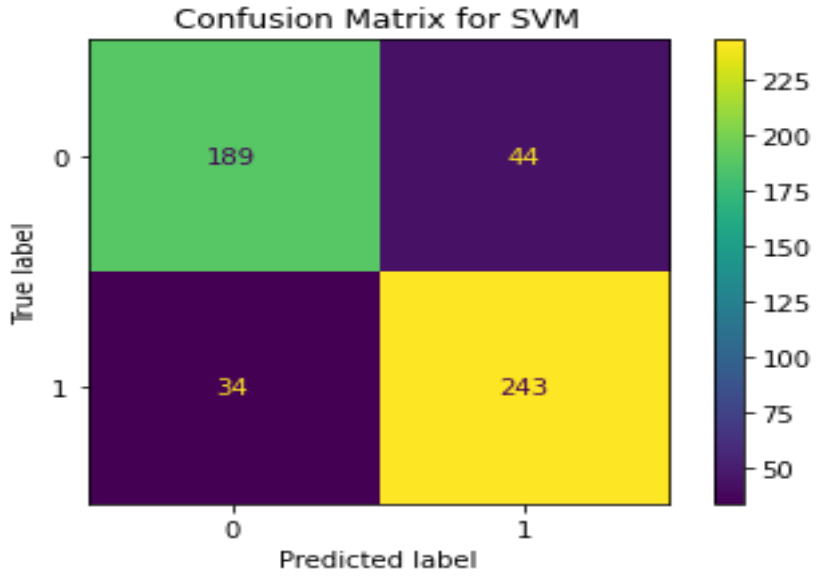


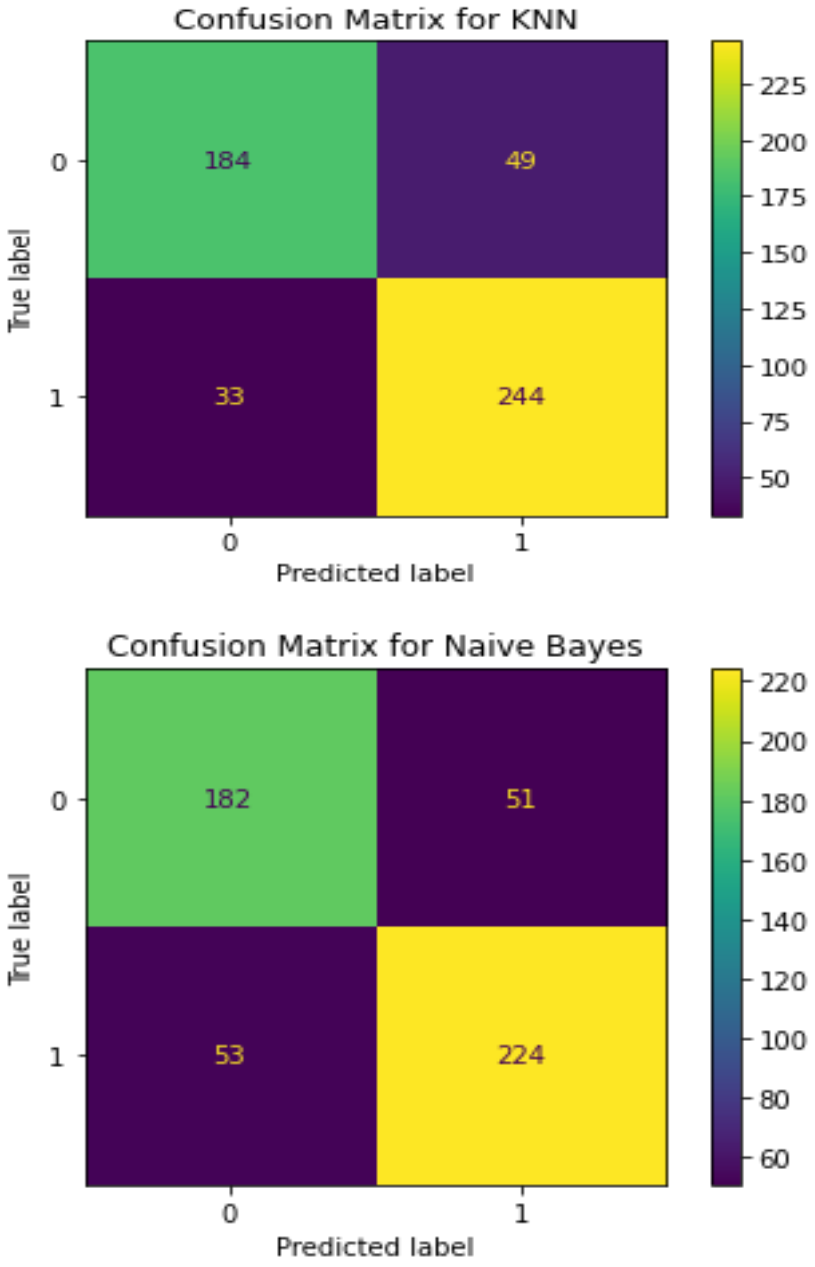
Figure 5 Variable importance plot for Random Forest

Table 2 Neutrosophic Based evaluation matrices for machine learning models.

Matrices	Navie Bays	95% CI	KNN	95% CI	SVM	95% CI	Logistic Regression	95% CI	Random Forest	95% CI
Accuracy	0.892	(0.865, 0.919)	0.839	(0.825, 0.929)	0.847	(0.834, 0.930)	0.804	(0.734, 0.820)	0.892	(0.834, 0.920)
Sensitivity	0.896	(0.861, 0.931)	0.881	(0.821, 0.931)	0.877	0.861, 0.941	0.834	0.811, 0.931	0.877	0.851, 0.931
Specificity	0.888	(0.847, 0.929)	0.790	(0.747, 0.919)	0.811	(0.734, 0.870)	0.768	(0.734, 0.820)	0.905	(0.834, 0.940)
F1 Score	0.9	(0.874, 0.926)	0.833	(0.824, 0.896)	0.862	(0.834, 0.920)	0.822	(0.814, 0.920)	0.883	(0.834, 0.910)
PPV (Precision)	0.905	(0.871, 0.939)	0.856	(0.811, 0.889)	0.847	(0.824, 0.920)	0.811	(0.804, 0.920)	0.889	(0.814, 0.916)
NPV	0.877	(0.834, 0.920)	0.848	(0.834, 0.920)	0.848	(0.824, 0.928)	0.796	(0.734, 0.820)	0.895	(0.834, 0.926)
MCC	0.713		0.769		0.690		0.303		0.78	


Confusion Matrix for Logistic Regression
179
54
46
231
0
1
True label
0
1
Predicted label
225
200
175
150
125
100
75
50


Confusion Matrix for SVM
189
44
34
243
0
1
True label
0
1
Predicted label
225
200
175
150
125
100
75
50


Confusion Matrix for KNN
184
49
33
244
True label
Predicted label
0
1
225
200
175
150
125
100
75
50
Confusion Matrix for Naive Bayes
182
51
53
224
True label
Predicted label
0
1
220
200
180
160
140
120
100
80
60

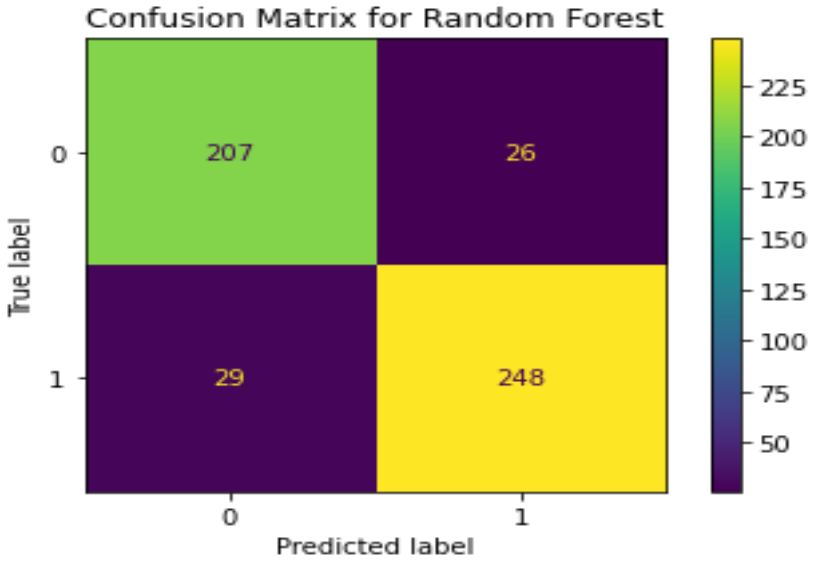


Figure 6 confusion matrices for all models

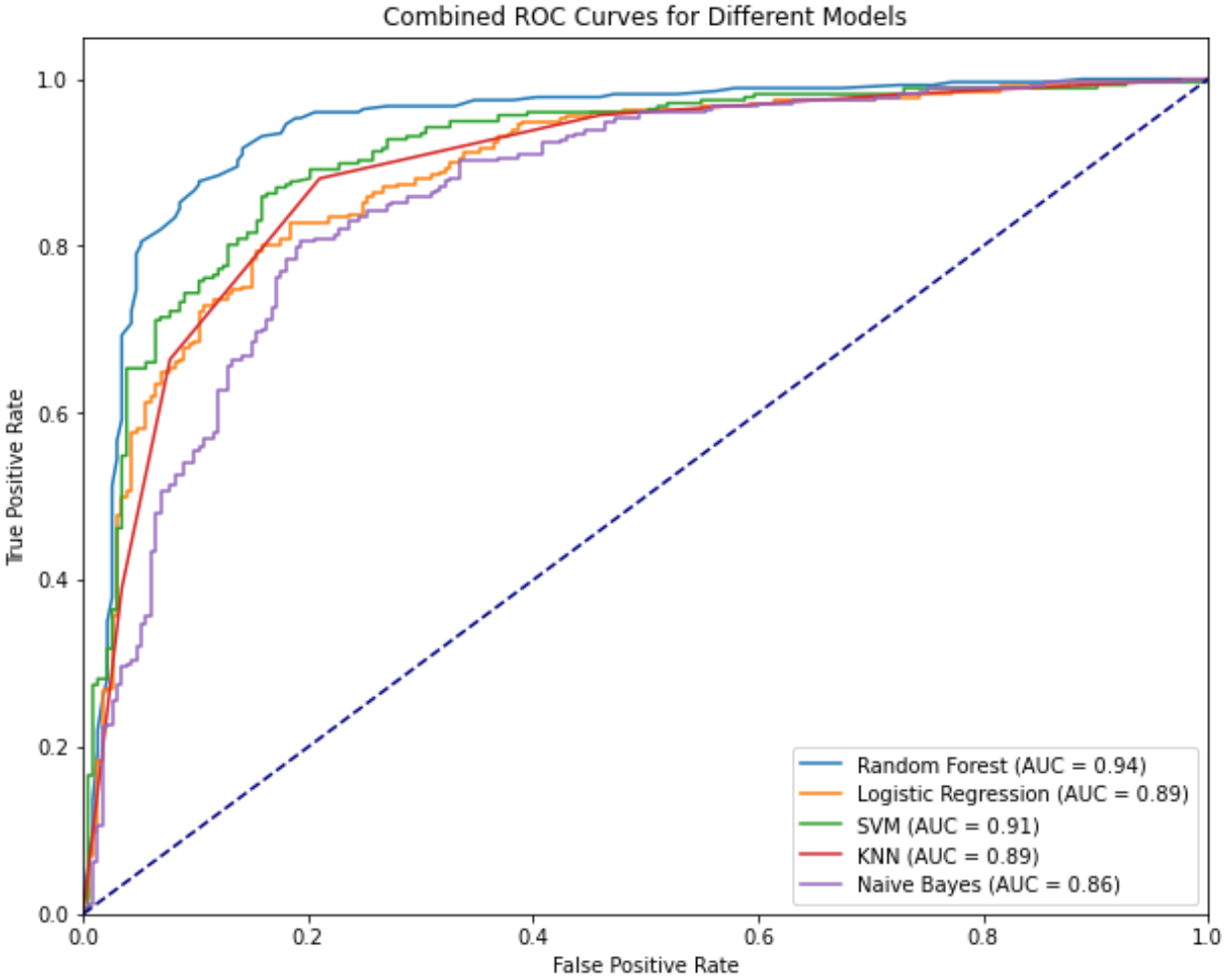


Figure 7 ROC Curve for all fuzzy based machine learning models

5. Results

The performance evaluation of Naïve Bayes, K-Nearest Neighbours (KNN), Support Vector Machine (SVM), Logistic Regression, and Random Forest across key metrics shows varying strengths. Figure 5 shows the variable importance of the best model. Naive Bayes and Random Forest lead with the highest accuracy at 0.892 (CIs: 0.865-0.919 for Naïve Bayes, 0.834-0.920 for Random Forest), strong sensitivity (Naïve Bayes: 0.896, Random Forest: 0.877), and high F1 scores (Naïve Bayes: 0.9, Random Forest: 0.883). Random Forest achieved the best specificity (0.905) and MCC (0.78), indicating its robustness in minimizing false positives. KNN shows decent accuracy (0.839) and sensitivity (0.881), which is suitable for cases prioritizing true positives. SVM and Logistic Regression lag in specificity and F1 scores, with Logistic Regression showing the lowest accuracy (0.804) and MCC (0.303). In summary, Random Forest and Naïve Bayes provide balanced, reliable performance, while KNN can be considered for applications focused on high sensitivity. Figure 6 and figure 7 represents the confusion matrix and roc curve of all models.

6. Conclusion

This paper suggests a neutrosophic logic-based machine learning method for figuring out the risk of chronic kidney disease (CKD) so that early detection is more accurate. Combining fuzzy logic's capability to manage uncertain data with machine learning's pattern recognition, the model effectively identifies high-risk individuals with greater precision. The interpretability of fuzzy rules offers clinicians a clearer view of risk factors, aiding in decision-making and timely intervention. The results show that this hybrid approach is more accurate at predicting CKD than traditional models. This shows that it could help with proactive healthcare strategies and improve patient outcomes by stepping in early.

7. Limitation

The limitation of this study is that neutrosophic logic is rather complex, requiring significant computational and specialized knowledge for its implementation. The effectiveness of this model also depends on the quality and completeness of input data, which can vary from one real-world healthcare setting to another. Additionally, achieving interpretability for clinical decision-making may present a challenge when compared to traditional methods.

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Neutrosophic Decision Framework for Performance Analysis in Construction Project Management Using the EVAMIX Method

Jiao Wang[1,2]*

[1]Department of Construction Engineering, Guizhou Communications Polytechnic University, Guizhou, China

[2]School of Housing, Building and Planning, Universiti Sains Malaysia, 11800 Pulau Pinang, Malaysia

*Corresponding author, E-mail: wangjiaoshirley@student.usm.my

Abstract: The performance evaluation of construction project management is the process of comprehensively assessing the effectiveness and efficiency of project management based on key indicators such as cost, schedule, quality, and safety. Through quantitative analysis and comprehensive evaluation, it identifies strengths and weaknesses in management, ensuring the achievement of project goals and providing references for future improvements. This evaluation not only helps optimize resource allocation but also enhances the overall level of project management, ensuring the project is completed on time, with quality, and efficiently. The performance evaluation of construction project management is multiple-attribute decision-making (MADM). Currently, the EVAMIX approach is employed to implement MADM. The type-2 neutrosophic sets (T2NSs) are administrated as an efficient approach for manipulating uncertain information during the performance evaluation of construction project management. In this paper, the type-2 neutrosophic number EVAMIX (T2NN-EVAMIX) approach is administrated for MADM. The average approach is to put forward weight information with T2NSs. Finally, a numerical example for performance evaluation of construction project management is administrated and some efficient comparisons are found to verify the T2NN-EVAMIX approach. The major contribution of this paper is administrated: (1) The average approach is utilized to manage the weight numbers based on T2NSs; (2) the T2NN-EVAMIX approach is administrated for MADM under T2NSs; (3) Finally, a numerical example for performance evaluation of construction project management is administrated and (4) some comparisons are administrated to show some advantages of T2NN-EVAMIX approach.

Keywords: Multiple attribute decision making (MADM); type-2 neutrosophic sets (T2NSs); T2NN-EVAMIX; performance evaluation of construction project management

1. Introduction

In recent years, rapid economic growth has fueled a continuous increase in railway transportation mileage and government investment in railway construction [1]. To achieve the desired outcomes of railway engineering projects, effective project management is crucial. It's a critical factor influencing the quality of railway infrastructure.

Railway engineering construction is a complex process influenced by various factors, including personnel, materials, equipment, technology, environment, and construction timelines.

Adverse conditions such as geographical constraints and weather patterns also pose significant challenges [2]. Implementing scientific and reasonable project management measures is essential to mitigate these challenges and enhance the quality of railway engineering projects.
Selecting the appropriate management model for specific railway projects requires careful comparison and analysis of existing project management methods to meet expected standards.
Railway construction offers substantial social and economic benefits, making it a key investment target for national and local governments [3]. Supported by national policies, railway construction has achieved remarkable progress. However, challenges persist:
The absence of a comprehensive theoretical system for railway project management contributes to inconsistencies in management capabilities, hindering overall effectiveness.
The mismatch between project management and construction needs: This disparity negatively impacts the quality of railway engineering projects and needs urgent attention from government departments and construction units.
The current investment management system needs better adaptability to accommodate diverse investment sources, channels, and methods. Some projects exhibit inappropriate management practices, and investment behaviors during construction are not accurately recognized as investment property rights [4, 5].
Factors such as the sense of responsibility among investors, slow improvement in management expertise, and the effectiveness of bidding management personnel and supervision engineers can negatively impact project quality [6, 7, 8].
Insufficient supervision and management can lead to projects failing to meet quality standards, hindering effectiveness [9, 10]. Furthermore, corruption, high investment costs, and low quality remain concerns in some railway engineering projects.

1.1 Addressing the Challenges

To enhance the quality and efficiency of railway engineering projects, several steps are crucial:
Develop a comprehensive theoretical framework: A robust theoretical system for railway project management will provide a standardized approach, improve consistency, and enhance effectiveness.
Align project management with construction requirements: Bridging the gap between management practices and construction needs is vital to ensure project quality and meet deadlines [11, 12].
Improve the investment management system: The system should be adapted to accommodate diverse investment patterns, ensure accurate recognition of investment property rights, and promote self-investment control, risk management, and sustainable development.
Strengthen oversight and accountability: Increased supervision and management are essential to ensure project quality, control costs, and prevent corruption.
By addressing these challenges, railway engineering projects can effectively contribute to improving China's railway transportation network and support economic growth.

1.2 Performance Evaluation

Performance evaluation of construction project management is a Multi-Attribute Decision Making (MADM) problem. The EVAMIX approach [15-17] is currently used to implement

MADM. Type-2 Neutrosophic Sets (T2NSs) [18] offer an efficient technique for handling fuzzy information in performance evaluation.

This study employs the T2NN-EVAMIX approach for MADM under T2NSs. The average approach is used to determine weight information with T2NSs. A numerical example for performance evaluation of construction project management is presented, and comparisons are made to validate the T2NN-EVAMIX approach.

1.3 Paper Organization

Section 2 introduces T2NSs.

Section 3 presents the MADM based on the T2NN-EVAMIX approach.

Section 4 outlines the development criteria.

Section 5 provides a numerical example of performance evaluation in construction project management.

Section 6 concludes the study.

2. Type-2 Neutrosophic Numbers Definitions

We define the operations of the Type-2 Neutrosophic numbers (T2NNs) [20,21] as:

Definition 2.1 T2NN can be defined as:

T2NN can be defined as

$Y = \{(x, A_Y(x), B_Y(x), C_Y(x)) | \, x \in X\};\ A_Y(x) = \left(A_{A_Y}(x), A_{B_Y}(x), A_{C_Y}(x)\right)$

$;B_Y(x) = \left(B_{A_Y}(x), B_{B_Y}(x), B_{C_Y}(x)\right);\ C_Y(x) = \left(C_{A_Y}(x), C_{B_Y}(x), C_{C_Y}(x)\right)$

$A_Y(x) = (\, A_Y^1(x), A_Y^2(x), A_Y^3(x));\ B_Y(x) = (\, B_Y^1(x), B_Y^2(x), B_Y^3(x));\ C_Y(x) = (\, C_Y^1(x), C_Y^2(x), C_Y^3(x))$

Where $A_Y(x), B_Y(x), C_Y(x) \to [0,1]$ and $0 \leq A_Y^1(x) + B_Y^1(x) + C_Y^1(x) \leq 3$

Definition 2.2. We can define two T2NNs as:

$$Y_1 = \left(\left(A_{A_{Y_1}}(x), A_{B_{I_{Y_1}}}(x), A_{C_{Y_1}}(x)\right), \left(B_{A_{Y_1}}(x), B_{B_{Y_1}}(x), B_{C_{Y_1}}(x)\right), \left(C_{A_{Y_1}}(x), C_{B_{Y_1}}(x), C_{C_{Y_1}}(x)\right)\right)$$

$$Y_2 = \left(\left(A_{A_{Y_2}}(x), A_{B_{Y_2}}(x), A_{C_{Y_2}}(x)\right), \left(B_{A_{Y_2}}(x), B_{B_{Y_2}}(x), B_{C_{Y_2}}(x)\right), \left(C_{A_{Y_2}}(x), C_{B_{Y_2}}(x), C_{C_{Y_2}}(x)\right)\right)$$

Definition 2.3. Some of two T2NNS as:

$$Y_1 \oplus Y_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} A_{A_{Y_1}}(x) + A_{A_{Y_2}}(x) - A_{A_{Y_1}}(x) A_{A_{Y_2}}(x), \\ A_{B_{Y_1}}(x) + A_{B_{I_{Y_2}}}(x) - A_{B_{Y_1}}(x) A_{B_{Y_2}}(x), \\ A_{C_{Y_1}}(x) + A_{C_{Y_2}}(x) - A_{C_{Y_1}}(x) A_{C_{Y_2}}(x) \end{array} \right), \\ \left(B_{A_{Y_1}}(x) B_{A_{Y_2}}(x), B_{B_{Y_1}}(x) B_{B_{Y_2}}(x), B_{C_{Y_1}}(x) B_{C_{Y_2}}(x)\right), \\ \left(C_{A_{Y_1}}(x) C_{A_{Y_2}}(x), C_{B_{Y_1}}(x) C_{B_{Y_2}}(x), C_{C_{Y_1}}(x) C_{C_{Y_2}}(x)\right) \end{array} \right\} \quad (1)$$

Definition 2.4. Multiplication of two T2NNS as:

$$Y_1 \otimes Y_2 = \left\{ \begin{array}{c} \left(A_{A_{Y_1}}(x)A_{A_{Y_2}}(x), A_{B_{Y_1}}(x)A_{B_{Y_2}}(x), A_{C_{Y_1}}(x)A_{C_{Y_2}}(x)\right), \\ \left(\begin{array}{c} B_{A_{Y_1}}(x) + B_{A_{Y_2}}(x) - B_{A_{Y_1}}(x)B_{A_{Y_2}}(x), \\ B_{B_{Y_1}}(x) + B_{B_{Y_2}}(x) - B_{B_{Y_1}}(x)B_{B_{Y_2}}(x), \\ B_{C_{Y_1}}(x) + B_{C_{Y_{12}}}(x) - B_{C_{Y_1}}(x)B_{C_{Y_2}}(x) \end{array}\right), \\ \left(\begin{array}{c} C_{A_{Y_1}}(x) + C_{A_{Y_2}}(x) - C_{A_{Y_1}}(x)C_{A_{Y_2}}(x), \\ C_{B_{Y_1}}(x) + C_{B_{Y_2}}(x) - C_{B_{Y_1}}(x)C_{B_{Y_2}}(x), \\ C_{C_{Y_1}}(x) + C_{C_{Y_2}}(x) - C_{C_{Y_1}}(x)C_{C_{Y_2}}(x) \end{array}\right) \end{array} \right\} \quad (2)$$

Definition 2.5. Multiplication of constant by one T2NNS as:

$$\omega Y_1 = \left\{ \begin{array}{c} \left(\begin{array}{c} \left(1 - \left(1 - A_{A_{Y_1}}(x)\right)^{\omega}\right), \\ \left(1 - \left(1 - A_{B_{Y_1}}(x)\right)^{\omega}\right), \\ \left(1 - \left(1 - A_{C_{Y_1}}(x)\right)^{\omega}\right) \end{array}\right), \\ \left(\left(B_{A_{Y_1}}(x)\right)^{\omega}, \left(B_{B_{Y_1}}(x)\right)^{\omega}, \left(B_{C_{Y_1}}(x)\right)^{\omega}\right), \\ \left(\left(C_{A_{Y_1}}(x)\right)^{\omega}, \left(C_{B_{Y_1}}(x)\right)^{\omega}, \left(C_{C_{Y_1}}(x)\right)^{\omega}\right) \end{array} \right\} \quad (3)$$

Definition 2.6. Power of consent by one T2NNS as:

$$Y_1^{\omega} = \left\{ \begin{array}{c} \left(\left(\left(A_{A_{Y_1}}(x)\right)^{\omega}, \left(A_{B_{Y_1}}(x)\right)^{\omega}, \left(A_{C_{Y_1}}(x)\right)^{\omega}\right)\right), \\ \left(\left(\left(\begin{array}{c} \left(1 - \left(1 - B_{A_{Y_1}}(x)\right)^{\omega}\right), \\ \left(1 - \left(1 - B_{B_{Y_1}}(x)\right)^{\omega}\right), \\ \left(1 - \left(1 - B_{C_{Y_1}}(x)\right)^{\omega}\right) \end{array}\right)\right)\right), \\ \left(\left(\left(\begin{array}{c} \left(1 - \left(1 - C_{A_{Y_1}}(x)\right)^{\omega}\right), \\ \left(1 - \left(1 - C_{B_{Y_1}}(x)\right)^{\omega}\right), \\ \left(1 - \left(1 - C_{C_{Y_1}}(x)\right)^{\omega}\right) \end{array}\right)\right)\right) \end{array} \right\} \quad (4)$$

Definition 2.7. The score function is:

$$S(Y_1) = \frac{1}{12} * \left(\begin{array}{c} 8 + \left(A_{A_{Y_1}}(x) + 2 * A_{B_{Y_1}}(x) + A_{C_{Y_1}}(x)\right) + \\ \left(B_{A_{Y_1}}(x) + 2 * B_{B_{Y_1}}(x) + B_{C_{Y_1}}(x)\right) + \\ \left(C_{A_{Y_1}}(x) + C_{B_{Y_1}}(x) + C_{C_{Y_1}}(x)\right) \end{array} \right) \quad (5)$$

3. T2NN- EVAMIX Approach

This section shows the steps of the T2NN- EVAMIX

Step 3.1. Evaluate the criteria and alternatives.

T2NN is used to evaluate the criteria and alternatives. Then we used the score function to obtain one value, then we aggregate these values into one matrix. Then we compute the criteria weights.

Step 3.2. Determine the superiority rate of alternatives.

The criteria are divided into quantitative and qualitative criteria. Then the superiority rate of alternatives is computed as:

$$u_{ii'} = \left[\sum_{j\in O}\left\{w_j \times sgn\left(t_{ij} - t_{i'j}\right)\right\}^c\right]^{1/c}; i, i' \in \{1, \dots, m\}, j = 1, \dots, n \quad (6)$$

$$n\left(t_{ij} - t_{i'j}\right) = \begin{Bmatrix} -1 & if t_{ij} < t_{i'j} \\ 0 & if t_{ij} = t_{i'j} \\ +1 & if t_{ij} > t_{i'j} \end{Bmatrix} \quad (7)$$

Step 3.3. Determine the differential matrix of ordinal criteria.

$$R_{ii'} = \frac{(u_{ii'} - \min u_{ii'})}{\max u_{ii'} - \min u_{ii'}} \quad (8)$$

Step 4.4. Determine the differential matrix of cardinal criteria.

$$F_{ii'} = \frac{(u_{ii'} - \min u_{ii'})}{\max u_{ii'} - \min u_{ii'}} \quad (9)$$

Step 5.5. Compute the total dominance

$$D_{ii'} = w_o R_{ii'} + w_c F_{ii'} \quad (10)$$

$$\text{Where } w_o = \sum_{j\in o} w_j \text{ and } w_o = \sum_{j\in c} w_j \quad (11)$$

$$N_i = \left[\sum_{i'} \frac{D_{i'i}}{D_{ii'}}\right]^{-1} \quad (12)$$

Step 3.6. Rank the alternatives.

4. Description of the Criteria

Below is a detailed list of 20 evaluation criteria for construction project management. Each criterion is marked as quantitative (Q) or qualitative (Ql) and ranked in order of importance, from highest to lowest.

1. Time Management (Q)

Ability to meet deadlines and complete milestones according to the project schedule.

2. Budget Adherence (Q)

The extent to which the project remains within the approved budget.

3. Quality of Work (Ql)

Compliance with design specifications and industry standards.

4. Risk Management (Ql)

Effectiveness in identifying, mitigating, and addressing risks throughout the project.

5. Safety Management (Ql)

Implementation of safety protocols and practices to ensure a secure working environment.

6. Stakeholder Satisfaction (Ql)

Degree of satisfaction among clients, contractors, and other stakeholders regarding project outcomes.

7. Resource Utilization Efficiency (Q)

Optimal use of manpower, materials, and machinery.

8. Communication Effectiveness (Ql)

Clarity, timeliness, and accuracy of information sharing between project teams and stakeholders.

9. Contract Management (Ql)

Adherence to contractual terms and conditions, including compliance with timelines and payment schedules.

10. Environmental Sustainability (Ql)

Integration of eco-friendly practices in construction, including waste reduction and energy efficiency.

11. Compliance with Regulations (Ql)

Adherence to local laws, codes, and regulatory requirements.

12. Innovation and Technology Integration (Ql)

Use of advanced tools and technology, such as Building Information Modeling (BIM) or project management software.

13. Supply Chain and Logistics Efficiency (Q)

Effectiveness in procurement, transportation, and delivery of materials to prevent delays.

14. Change Management (Ql)

Ability to handle changes in project scope or conditions without significantly affecting project goals.

15. Team Productivity (Q)

Assessment of workforce efficiency and output during project execution.

16. Cost-Effectiveness (Q)

Evaluates the balance between project expenditures and the value delivered.

17. Facility and Equipment Maintenance (Q)

Timely upkeep of machinery and equipment to prevent disruptions.

18. Feedback Mechanisms (Ql)

Systems in place to gather and act on feedback from stakeholders and teams.

19. Cultural and Workforce Inclusivity (Ql)

Measures to ensure diversity, equity, and inclusion in team composition and decision-making processes.

20. Post-Project Evaluation and Learning (Ql)

Review of project outcomes to identify successes, lessons learned, and areas for improvement.

Table 1. T2NN decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_2	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_3	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_4	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_5	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_6	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_8	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_9	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_{10}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{11}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{12}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{13}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{14}	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_{15}	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_{16}	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_{17}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{18}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{19}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{20}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))

	A_1	A_2	A_3	A_4	A_5
C_1	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_2	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_3	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_4	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_5	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_6	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_8	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_9	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_{10}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{11}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{12}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{13}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{14}	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_{15}	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_{16}	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{17}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{18}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{19}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{20}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))

	A_1	A_2	A_3	A_4	A_5
C_1	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_2	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_3	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_4	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_5	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_6	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_8	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_9	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{10}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{11}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{12}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{13}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{14}	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{15}	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))

C_{16}	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{17}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{18}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{19}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{20}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))

5. Application: Performance Evaluation of Construction Projects using the T2NN-EVAMIX Approach

This numerical example demonstrates the performance evaluation of five railway construction projects (A1 through A5) using the T2NN-EVAMIX approach. The evaluation considers multiple attributes (criteria) to determine the most favorable project.

Step 1: Evaluation and Weighting

Type-2 Neutrosophic Sets (T2NSs) were utilized to evaluate the projects against each criterion, as detailed in Table 1. A score function was applied to derive crisp values from the T2NSs, and these values were compiled into a decision matrix. Subsequently, the weights of each criterion were calculated as shown in Tabel 2.

Table .2 The crietria weights.

Synmbol	Weights	Rank
C_1	0.050170987	12
C_2	0.049797215	8
C_3	0.050038958	10
C_4	0.048900906	2
C_5	0.050749311	16
C_6	0.051887363	20
C_7	0.050747451	15
C_8	0.049706096	7
C_9	0.049213312	4
C_{10}	0.049581506	6
C_{11}	0.047491358	1
C_{12}	0.051085891	18
C_{13}	0.051106347	19
C_{14}	0.04930815	5
C_{15}	0.049853002	9
C_{16}	0.050068711	11
C_{17}	0.04896785	3
C_{18}	0.050170987	12
C_{19}	0.050957582	17
C_{20}	0.050197021	14

Step 2: Determining Superiority Rates

Pairwise comparisons of the five alternatives were conducted to determine the superiority rate of each alternative over the others. This involved assessing the performance of each alternative relative to the others for every criterion.

Step 3: Constructing the Differential Matrix for Ordinal Criteria

Ordinal criteria, which are qualitative and can be ranked but not precisely measured, were evaluated. A differential matrix was constructed to represent the performance differences between alternatives based on these ordinal criteria.

Step 4: Constructing the Differential Matrix for Cardinal Criteria

Cardinal criteria, which are quantitative and measurable, were analyzed. A differential matrix was created to illustrate the performance differences between alternatives based on these cardinal criteria.

Step 5: Computing Total Dominance

The information from the differential matrices of both ordinal and cardinal criteria was aggregated to compute the total dominance of each alternative. This measure reflects the overall performance superiority of an alternative considering all criteria.

Step 6: Ranking the Alternatives

Based on the calculated total dominance values, the alternatives were ranked in descending order of preference:

A4 > A3 > A1 > A2 > A5

5.1 Discussion of Results

The performance evaluation of construction project management is a systematic and comprehensive assessment of the key aspects of the management process within construction projects. Its primary purpose is to analyze and assess the effectiveness, efficiency, and outcomes of project management using scientific evaluation methods, ensuring that projects are completed on time, within budget, and meet quality standards, while also providing a basis for improving the management of future projects. Firstly, performance evaluation typically covers several core dimensions, including cost management, schedule control, quality assurance, safety management, and resource utilization. Cost management focuses on evaluating whether the project operates within the budget and whether resources are allocated and used efficiently. Schedule control assesses whether the project progresses as planned and if there are any delays or unreasonable timelines. In terms of quality assurance, the evaluation ensures that the project meets design standards and relevant regulations through inspections and acceptance testing. Safety management ensures that the construction process complies with safety regulations, minimizing the occurrence of accidents. Lastly, the assessment of resource utilization examines whether labor, materials, and financial resources are used efficiently and whether their utilization is maximized. Secondly, construction project management performance evaluation is not merely an assessment of the outcome but also serves as a tool for monitoring and providing feedback throughout the entire management process. By regularly tracking and evaluating the project, potential problems and deficiencies can be identified early, preventing them from accumulating and becoming more severe later in the project. For instance, by evaluating the schedule, adjustments can be made to the construction plan to ensure smooth progress; through cost control evaluation, the risk of budget overruns can be identified and corrective actions taken promptly. Moreover, performance evaluation is often based on quantitative indicators and standards, which can be developed from historical project data, market standards, or industry regulations. For example, cost variance, schedule completion rate, quality compliance rate, and safety incident rate are all important metrics for measuring project management performance. These quantitative indicators allow managers to objectively assess the project's actual situation and develop targeted improvement measures. Finally, the significance of performance evaluation in construction project management lies not only in controlling the current project but also in its guiding role for future projects. By evaluating existing projects, managers can accumulate valuable lessons and experiences that provide a reference for planning and executing future projects. Performance evaluation can also drive innovation in project management techniques, optimize management processes, and enhance the overall management capabilities of the team. In summary, the performance evaluation of construction project management is

a crucial tool for ensuring the successful implementation of projects. Through comprehensive analysis and assessment of the management process, it helps managers identify problems, optimize decision-making, and improve efficiency, thereby maximizing the project's economic and social benefits. However, the performance evaluation of construction project management is MADM.

5.2 Comparative analysis

Then, the T2NN-EVAMIX approach is compared, with the GRA approach [22], VIKOR approach[23], cross-entropy (CE) approach [24], and MABAC approach [19]. The order is administered in Table 3.

Table 3. The order

Approach	Order
MABAC approach	A4>A1>A3>A2>A5
VIKOR approach	A4>A3>A2>A1>A5
GRA approach	A4>A1>A3>A2>A5
CE approach	A4>A3>A2>A1>A5
Our approach	A4>A3>A1>A2>A5

Then, the order similarity coefficients between the GRA approach[22], VIKOR approach[23], cross-entropy (CE) approach [24], MABAC approach [19] and EVAMIX approach were administrated with WS coefficients [25, 26], the WS is administrated in Table 4.

Table 4. The WS coefficient

	Our approach	MABAC approach	VIKOR approach	GRA approach	CE approach
Coefficients	WS	0.9	0.9	0.9	0.9

The WS coefficient shows the order of the T2NN-EVAMIX approach are same as to order of the GRA approach[22], the VIKOR approach[23], the cross-entropy (CE) approach [24], the MABAC approach [19]; the WS coefficient shows the order of T2NN approach are slightly different to order of different approach. This verifies the T2NN-EVAMIX approach is effective. The advantages of the T2NN-EVAMIX approach are administrated by calculating the superiority rate from each alternative.

5. Conclusion

The performance evaluation of construction project management is a systematic and comprehensive assessment of the project management process. It aims to measure the actual effectiveness of management by analyzing multiple dimensions, such as cost control, schedule management, quality assurance, safety, and resource allocation. The evaluation not only focuses on whether the project is completed as planned but also examines the effectiveness of management strategies, the rationality of resource utilization, and the efficiency of team collaboration. Through quantitative indicators and standards, performance evaluation helps identify problems and shortcomings in project management, providing a basis for improving management methods and increasing efficiency. Additionally, it offers valuable insights and references for the planning and implementation of future projects. Ultimately, performance evaluation ensures that projects are completed on time, within budget, and to the desired quality, while also enhancing the management capabilities of the team and improving project operational efficiency. This maximizes both the economic and social benefits of the project. The performance evaluation of construction project management is MADM. Currently, the EVAMIX approach and average approach are employed to implement the MADM. The T2NSs are administrated as an efficient technique for manipulating the fuzzy information during the performance evaluation of construction project management. In this study, the T2NN-EVAMIX approach is administrated for MADM under T2NSs. The average approach is to put forward weight information with T2NSs. Finally, a numerical example for performance evaluation of construction project management is administrated and some efficient comparisons are found to verify the T2NN-EVAMIX approach. The major contribution of this paper is administrated: (1) The average approach is utilized to manage the weight numbers under T2NSs; (2) the T2NN-EVAMIX approach is administrated for MADM under T2NSs; (3) Finally, a numerical example for performance evaluation of construction project management is administrated and (4) some comparisons are administrated to show some advantages of T2NN-EVAMIX approach.

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Neutrosophic DUS Exponential Distribution

Megha.C.M [1, 4], Divya.P.R [2], and Sajesh.T.A [3,*]

[1] Department of Statistics, St Thomas College (Autonomous), Thrissur; meghacm@gmail.com
[2] Department of Statistics, Vimala College (Autonomous), Thrissur; divyastat@vimalacollege.edu.in
[3] Department of Statistics, St Thomas College (Autonomous), Thrissur; sajesh.t.abraham@gmail.com
[4] University of Calicut

* Correspondence: sajesh.t.abraham@gmail.com; Tel.: (+919747936823)

Abstract: The exponential distribution is a widely used lifetime model with applications spanning numerous disciplines. The DUS transformation extends the applicability of the exponential distribution by improving its flexibility for diverse real-world scenarios. This study proposes the Neutrosophic DUS Exponential Distribution (NDUS-ED), an extension of the DUS Exponential Distribution, designed to effectively address and quantify uncertainty, inconsistency, and indeterminacy in data. Key statistical properties of NDUS-ED, including quantiles, moments, moment-generating functions, and order statistics, are derived under neutrosophic conditions. The performance of estimated parameters is evaluated through simulation, revealing superior results with larger sample sizes, particularly in managing imprecise and indeterminate data. Finally, the proposed distribution is applied to an actual dataset, and the results are compared with those of the Neutrosophic Exponential Distribution and the DUS Exponential Distribution, demonstrating superior performance.

Keywords: Neutrosophy; DUS transformation; Probability distribution; Exponential distribution

1. Introduction

The idea of vagueness or ambiguity is often used nowadays to solve a variety of real-world issues. Classical statistics is the term used to describe statistics, which primarily works with numerical data. We can't always obtain numerical data in its precise form in the real world because of the indeterminacy of data. Numerous advancements have been made recently to model such imprecise situations by taking fuzzy logic and neutrosophy into consideration [1-4]. In order to address indeterminacy or partially ambiguous aspects in the data, Smarandache [5] proposed neutrosophic statistics. In 1995, he presented the idea of neutrosophic logic by denoting the components as T, I, and F, which stand for truth, indeterminate, and

falsehood, respectively. Neutrosophic is a multiple-valued logic that defines imprecise probability, fuzzy logic and classical logic. This logic has been applied in various domains of science and engineering.

Probability distributions have become an integral part of all scientific study. These probabilistic frameworks describe many real-world unforeseen occurrences [6]. Many studies have been conducted using probability distributions that incorporates neutrosophic logic [7]. Patro et al. [8] developed the neutrosophic probability distributions, such as the normal and binomial distributions and presented related case studies. Alhabib et al. [9] introduced neutrosophic Poisson, uniform, and exponential distributions by generalizing the respective classical distributions. Alhasan et al. [10] proposed and studied the neutrosophic Weibull distribution and the neutrosophic version of its related family of distributions. Aslam [11] introduced the neutrosophic Rayleigh distribution under the neutrosophic statistics and discussed its various properties.

Sherwani et al. [12] extended the classical beta distribution to a neutrosophic environment and proposed neutrosophic beta distribution. They have established many statistical characteristics of the distribution and the results were validated by applying the distribution on two real life datasets. Khan et al. [13] established gamma distribution under indeterminacy with applications to complex data analysis. Khan et al. [14] derived neutrosophic lognormal distribution and data from Nitrogen oxide emissions of Denmark was used to validate the established characteristics of the distribution. Ahsan-ul-Haq [15] proposed the neutrosophic Kumaraswamy distribution to analyze bounded data sets under an indeterminacy environment.

In 2023, more distributions pertaining to imprecise data were introduced. Norouzirad et al. [16] introduced neutrosophic generalized Rayleigh distribution and shown that the distribution is ideal to model skewed lifetime data. Rao [17] established neutrosophic log-logistic distribution model in complex alloy metal melting point applications. Alanaz et al. [18] proposed neutrosophic exponentiated inverse Rayleigh distribution to describe diverse survival data with indeterminacies.

Neutrosohic beta-Lindley distribution was established by Algamal et al. [18] and modeled bladder cancer data. Neutrosophic Birnbaum-Saunders distribution for imprecise data was introduced by Hassan et al. [19] and the results were validated through datasets based on alloy melting points and lifetime of batteries. Neutrosophic Topp-Leone distribution for interval-valued data analysis [20] was proposed by Ahsan-ul-Haq [20]. Under indeterminacy, discrete geometric distribution was introduced by Khan et al. [21] with emphasis to reliability functions and relevant case studies. Jamal et al. [22] proposed neutrosophic BURR III to model COVID 19 data. Recently, Aslam [23] established the neutrosophic negative

binomial distribution and developed algorithms for generating data based on this distribution.

A common lifetime distribution among the classical distributions is the exponential distribution, which is simple in terms of the closeness of its cumulative density function (cdf) and other statistics. Since exponential distribution have fixed failure rate constraint, transformations are applied to arrive at more robust and meaningful distribution. One such transformation is DUS transformation [24]. The DUS transformation yields a parsimonious distribution since it has no additional parameters, making it simple to understand. When the exponential distribution undergoes the DUS transformation, it leads to the DUS-exponential distribution (DUS-ED) which is a more robust distribution. Considering the exponential distribution using neutrosophic reasoning Duan et al. [25] presented neutrosophic exponential distribution. Later, in 2022 Khan et al. [26] developed the neutrosophic design of the exponential model with application to lifetime failures (in hours) of air conditioning instrument used in 720-Boeing planes.

This study proposes a new lifetime distribution the neutrosophic DUS-exponential distribution (NDUS-ED), an extension of DUS-exponential distribution intended to efficiently capture and quantify uncertainty, inconsistency, and indeterminacy. The primary purpose of the proposed distribution is to incorporate the indeterminate information about the variables under study into the existing classical distribution. It is necessary to consider indeterminacy of the study parameters in practical situations and integrate it with the proposed model to present a system.

This paper is structured as follows: Section 2 introduces the proposed model and corresponding plots. Statistical properties are derived in section 3. The parametric estimation is done in section 4. Simulation study is explained in section 5. Real data set study is included in section 6. Conclusions are given in section 7.

2. Proposed Model

Neutrosophic DUS-Exponential Distribution (NDUS-ED)

Let $f(x)$ and $F(x)$ be the pdf and cdf of the baseline distribution respectively, then the pdf $g(x)$ of the distribution obtained by DUS transformation of the baseline distribution is given by

$$g(x) = \frac{1}{(e^{(1)} - 1)} f(x) e^{F(x)} \tag{1}$$

Now, the corresponding cdf and hazard function are given by

$$G(x) = \frac{1}{(e^{(1)} - 1)} \left[e^{F(x)} - 1 \right] \tag{2}$$

and

$$h(x) = \frac{1}{\left(e^{(1)} - e^{F(x)}\right)} f(x) e^{F(x)} \tag{3}$$

respectively.

The neutrosophic random variable X, which equals the distance between successive events in a Poisson process, follows the neutrosophic exponential distribution (NED) model with the following neutrosophic density function

$$f_N(x) = \theta_N \, e^{-\theta_N x}, x > 0, \theta_N \in [\theta_L, \theta_U], \theta_N > 0 \tag{4}$$

The corresponding neutrosophic cumulative distribution function

$$F_N(x) = 1 - e^{-\theta_N x}, x > 0, \theta_N \in [\theta_L, \theta_U], \theta_N > 0 \tag{5}$$

Now, using (1), (2), and (3), neutrosophic- DUS exponential model is proposed. The neutrosophic density function (npdf) and neutrosophic cumulative distribution function (ncdf) of the proposed distribution respectively are

$$g_N(x) = \frac{1}{\left(e^{(1)} - 1\right)} \theta_N \, e^{-\theta_N x} \, e^{(1 - e^{-\theta_N x})}, x > 0, \theta_N \in [\theta_L, \theta_U], \theta_N > 0 \tag{6}$$

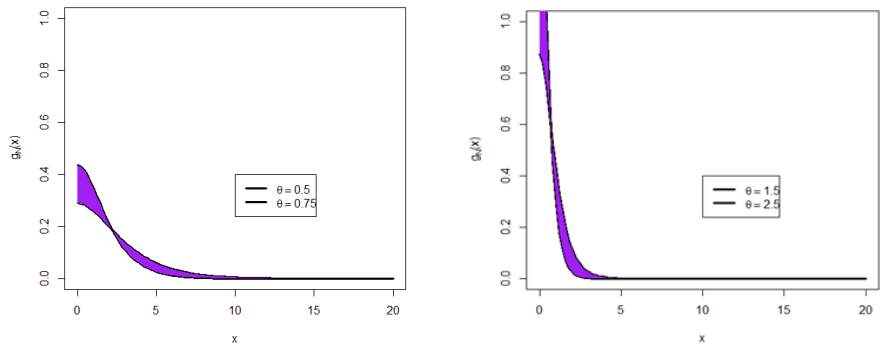
and

$$G_N(x) = \frac{1}{e^{(1)} - 1}\left[e^{(1 - e^{-\theta_N x})} - 1\right], \; x > 0, \theta_N \in [\theta_L, \theta_U], \theta_N > 0 \tag{7}$$

The hazard function is given by

$$h_N(x) = \frac{1}{e^{(1)} - e^{(1 - e^{-\theta_N x})}} \, \theta_N \, e^{-\theta_N x} \, e^{(1 - e^{-\theta_N x})}, x > 0, \theta_N \in [\theta_L, \theta_U], \theta_N > 0 \tag{8}$$

The plots of npdf are given below.



(a) (b)

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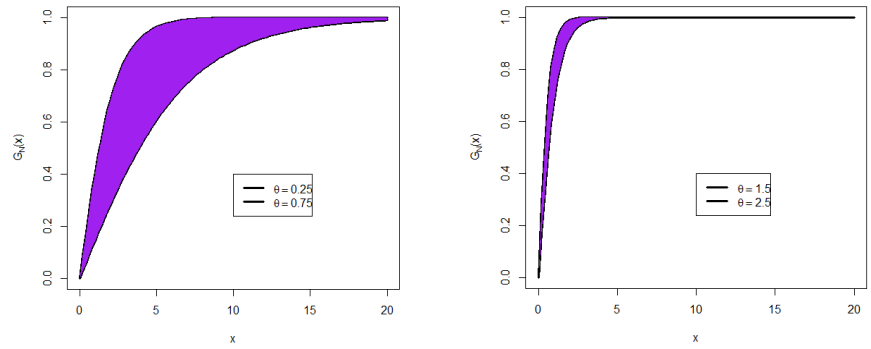
Figure 1 Density graph of neutrosophic DUS-exponential
(a) $\theta_N \in [1.5,2.5]$ (b) $\theta_N \in [3.5,4.5]$.

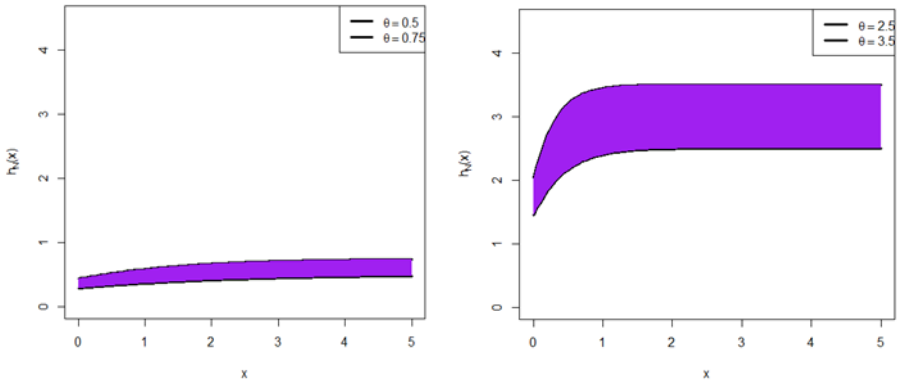
The plots of ncdf are given below.



(a) (b)

Figure 2 Cumulative Density graph of neutrosophic DUS-exponential
(a) $\theta_N \in [0.25,0.75]$ (b) $\theta_N \in [1.5,2.5]$.

The plots of hazard function are given below.



(a) (b)

Figure 3. Hazard function graph of neutrosophic DUS-exponential
(a) $\theta_N \in [0.5,0.75]$ (b) $\theta_N \in [2.5,3.5]$

3. Statistical Properties

The main properties like moments, moment generating function, cumulant generating function, characteristic function, quantiles and order statistics are studied below.

3.1. Moments

The r^{th} moment of the distribution, μ'_r is given by

$$\mu'_r = \int_0^\infty x^r \frac{1}{(e^{(1)}-1)} \theta_N e^{-\theta_N x} e^{(1-e^{-\theta_N x})} dx \tag{9}$$

To solve (9) we use exponential expansion $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$ and binomial series of expansion $(1-z)^a = \sum_{j=0}^{\infty}(-1)(aC_j)z^j$. Now Equation (9) takes the form

$$\mu'_r = \frac{1}{\theta_N^r(e^{(1)}-1)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n}{m!} (mC_n) \frac{\Gamma(r+1)}{(n+1)^{r+1}} \tag{10}$$

Put $r = 1$, then we get

$$E(X) = \frac{1}{\theta_N(e^{(1)}-1)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n}{m!} (mC_n) \frac{1}{(n+1)^2} \tag{11}$$

The expression for V(X) is given in equation (12)

$$V(X) = \frac{1}{\theta_N^2(e^{(1)}-1)}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{m!}\,(mC_n)\frac{\Gamma(3)}{(n+1)^3} - \frac{1}{(e^{(1)}-1)}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{m!}\,(mC_n)\frac{1}{(n+1)^2}\right\}^2\right] \quad (12)$$

3.2 Moment Generating Function

If X ~ NDUS-ED, the moment generating function is derived as

$$M_X(\mathrm{t}) = E(e^{tX}) = \frac{\theta_N}{e^{(1)}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{m!}\,(mC_n)\frac{\Gamma(1)}{(\theta_N(n+1)-t)} \quad (13)$$

3.3 Cumulant Generating Function

The cumulant generating function of X, where X ~NDUS-ED is given as

$$K_X(\mathrm{t}) = \log M_X(\mathrm{t}) = \log\left[\frac{\theta_N}{e^{(1)}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{m!}\,(mC_n)\frac{\Gamma(1)}{(\theta_N(n+1)-t)}\right] \quad (14)$$

3.4 Characteristic Function

The characteristic function of X, where X ~NDUS-ED is as follows

$$\varphi_X(t) = E\left(e^{itX}\right) = \frac{\theta_N}{e^{(1)}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{m!}\,(mC_n)\frac{\Gamma(1)}{(\theta_N(n+1)-it)} \quad (15)$$

3.5 Quantile Function

The p^{th} quantile function, denoted by $Q(p)$ of NDUS-ED is obtained by solving

$$G_N\left(Q(p)\right) = p, 0 < p < 1.$$

The quantile function is given by

$$Q(p) = \frac{-\ln\left(1-\ln\left(p\left(e^{(1)}-1\right)+1\right)\right)}{\theta_N} \tag{16}$$

While setting $p = \frac{1}{4}$, we get the first quartile Q_1, setting $p = \frac{1}{2}$, we get the second quartile/Median Q_2 and setting $p = \frac{3}{4}$, we get the third quartile Q_3.

3.6 Order Statistics

The hazard rate function plays a vital role in the domains of survival analysis and reliability theory. This makes the relevance of order statistics in the domains. Let $X_1, X_2, \dots, X_n$ be n independently and identically random variables with the corresponding order statistics $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ from NDUS-ED with the density function $g_N(x)$(6) and the distribution function $G_N(x)$(7). The pdf and cdf of the rth order statistics are given respectively by $g_r(x)$ and $G_r(x)$

$$\begin{aligned} g_r(x) &= \frac{n!}{(r-1)!\,(n-r)!} {G_N}^{(r-1)}(x)[1-G_N(x)]^{(n-r)}\, g_N(x) \\ &= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{e^{(1)}-1}\left[e^{\left(1-e^{-\theta_N x}\right)}-1\right]\right]^{(r-1)} \\ &\left[1-\frac{1}{e^{(1)}-1}\left[e^{\left(1-e^{-\theta_N x}\right)}-1\right]\right]^{(n-r)} \frac{1}{(e^{(1)}-1)}\theta_N\, e^{-\theta_N x}\, e^{\left(1-e^{-\theta_N x}\right)} \end{aligned} \tag{17}$$

$$\begin{aligned} G_r(x) &= \sum_{k=r}^{N} nC_k\,[G_N(x)]^k[1-G_N(x)]^{(n-k)} \\ &= \textstyle\sum_{k=r}^{N} nC_k \left[\frac{1}{e^{(1)}-1}\left[e^{\left(1-e^{-\theta_N x}\right)}-1\right]\right]^k \left[1-\frac{1}{e^{(1)}-1}\left[e^{\left(1-e^{-\theta_N x}\right)}-1\right]\right]^{(n-k)} \end{aligned} \tag{18}$$

4. Method of Estimation

Here, the parameters are estimated using the method of maximum likelihood is discussed. Consider $x_1, x_2, \dots, x_n$ are n observations taken from NDUS-ED. Then, the likelihood function is given by

$$\begin{aligned} L(x:\theta_N) &= \textstyle\prod_{i=1}^{n} g_N\,(x;\theta_N) \\ &= \prod_{i=1}^{n} \frac{1}{(e^{(1)}-1)}\theta_N\, e^{-\theta_N x}\, e^{\left(1-e^{-\theta_N x}\right)} \\ &= \left[\frac{e^{(1)}}{e^{(1)}-1}\right]^n {\theta_N}^n e^{-\left(\theta_N \sum_{i=1}^{n} x_i\right)} e^{-e^{-\left(\theta_N \sum_{i=1}^{n} x_i\right)}} \end{aligned}$$

Now, the log-likelihood function is given by

$$\log L = nlog\left(e^{(1)}\right) - n\log\left(e^{(1)} - 1\right) + n\log\theta_N - \theta_N\left(\sum_{i=1}^{n} x_i\right) - e^{-\left(\theta_N \sum_{i=1}^{n} x_i\right)}$$

The maximum likelihood estimators are obtained by maximizing the log-likelihood function with respect to the unknown parameter θ_N

$$\frac{\partial logL}{\partial \theta_N} = \frac{n}{\theta_N} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i\, e^{-\left(\theta_N \sum_{i=1}^{n} x_i\right)}$$

This equation has a non-linear form. Therefore, it can be numerically solved using the statistical software R by choosing arbitrary initial values.

5. Simulation study

A comprehensive evaluation has been conducted in order to assess the MLE performance of the NDUS-ED. The neutrosophic root mean square error ($RMSE_N$) shows how well a model predicts the data; lower $RMSE_N$ values are indicative of higher model performance. The following definition of neutrosophic root mean square error have been used to evaluate the effectiveness of the neutrosophic maximum likelihood estimator:

$$RMSE_N = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\theta}_{N_i} - \theta_N\right)^2}{N}}$$

The inversion approach is used to obtain the samples for the two interval measures $\theta_N \in [1,2]$ and $\theta_N \in [3,4]$ and the analysis is repeated for N=10,000 times. The analysis is carried out using R software. For sample sizes n= 10, 25, 40, 55, 70, $RMSE_N$ is computed. Table 1 displays the results of a study on the performance of measurements of the neutrosophic maximum likelihood (NML) estimator.

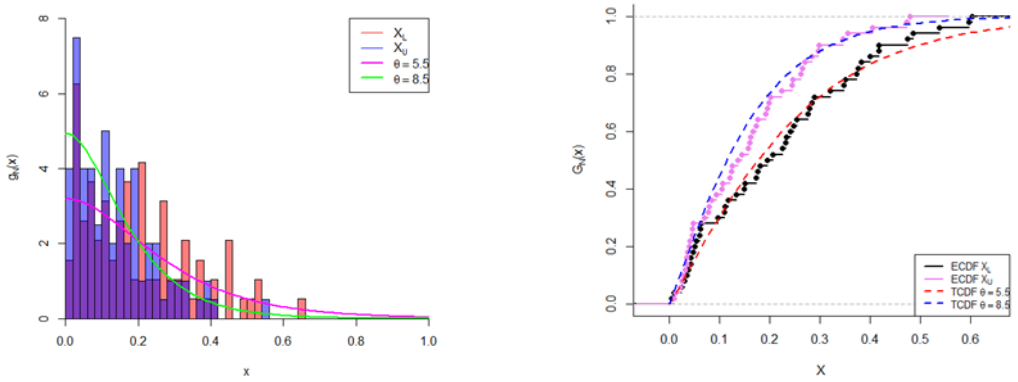
Table 1 Performance of NML estimate of the NDUS-ED for simulated neutrosophic data

Sample size	$RMSE_N$ for $\theta \in [1,2]$	$RMSE_N$ $\theta \in [3,4]$
10	[0.2912116, 0.5996537]	[0.8736341, 1.199315]
25	[0.1740633, 0.4682503]	[0.5221849, 0.936499]
40	[0.1463233, 0.4443628]	[0.4389749, 0.888727]
55	[0.134346, 0.4342111]	[0.4030462, 0.868416]
70	[0.127095, 0.4286541]	[0.3812866, 0.857305]

From the table, it can be seen that, as the sample size (n) increases $RMSE_N$ decreases for both $\theta \in [1,2]$ and $\theta \in [3,4]$. The study concludes that the NML estimator offers

reliable and accurate parameter estimates by showing a steady increase in estimation accuracy with increasing sample size.

The pdf and cdf plots of the simulated values are given below



(a) (b)

Figure 4. Graphical overview of simulated values (a) pdf plot (b) cdf plot

6. Real-World application

In this section, a real dataset is used to illustrate the application of the proposed model. The data is taken from [27]. We use two statistical criteria, Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) to test the performance of our proposed distribution. We have extracted the values of AIC and BIC of neutrosophic exponential distribution (NED) from [28] and DUS-exponential distribution (DUS-ED) from [24] and presented their values.

Table 2. AIC and BIC of NDUS-ED, NED and DUS-ED (DUS-exponential) distributions

Distributions	AIC	BIC
NDUS-ED	(823.8715, 827.376)	(826.7235, 830.2281)
NED	(825.8715, 829.3760)	(841.2796, 844.7842)
DUS-ED	834.044	836.896

Table 2 shows the comparison between the proposed distribution with that of the other distributions. Here, we can observe that the AIC (823.8715, 827.376) and BIC (826.7235, 830.2281) values of the proposed distribution are lower than those of the other distributions. Lower AIC and BIC values indicate that the neutrosophic framework is producing accurate and trustworthy parameter estimates by

effectively interpreting data uncertainty. The suggested model (6) is more accurate and adaptable in terms of imprecision and uncertainty when compared to the other distributions.

7. Conclusions

In contrast to classical probability distributions, neutrosophic probability distributions take an intrusive approach to explaining and resolving a variety of real-world scenarios. A wide range of disciplines, including engineering, actuarial sciences, health sciences, etc. heavily rely on the exponential distribution. A neutrosophic DUS-exponential distribution has been developed to better account for the imprecisions that are frequently present in real-world data, especially when it comes to interval form. The generating functions, order statistics, and hazard rate function are among the properties of the suggested neutrosophic DUS-exponential distribution that have been established. Using the neutrosophic maximum likelihood estimation method, parameters were estimated, and a simulation study was used to assess the distribution's efficiency. This model's adaptability was further illustrated by using it on a real-world dataset and contrasting its results with those of the neutrosophic DUS-exponential distribution, neutrosophic exponential distribution and DUS-exponential distribution. Results show that the suggested NDUS-ED yields trustworthy outcomes while managing data uncertainty and indeterminacy.

The NDUS-ED distribution could be extended or modified to accommodate different types of data distributions by introducing additional flexibility through parameterization or hybridization with other distributions. Adding shape parameters to account for skewness or heavy tails could make the distribution more adaptable to asymmetric or leptokurtic data. Developing multivariate versions of the NDUS-ED distribution could enable modeling of relationships and dependencies between multiple variables.

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Improved MAUT Framework for Quality Evaluation of University Party Building Work in the New Era Based on the Interval Neutrosophic Multi-Attribute Group Decision-Making

Yao Mo*
School of Marxism, Chongqing Technology and Business University, Chongqing, 400067, China
*Corresponding author, E-mail: moyao@ctbu.edu.cn

Abstract: In the new era, the evaluation of the quality of party-building work in universities focuses on strengthening political, ideological, and organizational construction. By using a scientific evaluation system, it ensures the implementation of Party theories and policies. It emphasizes innovative methods to enhance Party members' ideological awareness and organizational skills, promoting the integration of Party building with education. This ensures the Party's leadership plays a central role in all university activities, driving comprehensive development. The quality evaluation of university party building work in the new era is a multi-attribute group decision-making (MAGDM) problem. Recently, both the MAUT method and the average approach have been applied to solve MAGDM challenges. Interval Neutrosophic Sets (INSs) are utilized to represent uncertain data during the quality evaluation of university party-building work in the new era. In this study, the MAUT method is adapted for MAGDM with INSs. Furthermore, the Interval Neutrosophic Number MAUT (INN- MAUT) approach is developed for MAGDM. The average approach is used to determine the criteria weights within the INS framework. Finally, a numerical example is provided to demonstrate the application of the INN- MAUT approach in the quality evaluation of university party-building work in the new era. The key contributions of this study include: (1) the development of a MAGDM method using the INN- MAUT approach with INSs, (2) the application of the average method to compute weights under INSs, and (3) the proposal of a novel MAGDM approach for quality evaluation of university party building work in the new era using the INN- MAUT method.

Keywords: Multiple-attribute group decision-making (MAGDM); INSs; MAUT approach; quality evaluation of university party building work; Neutrosophic sets

1. Introduction

In the study of party-building in Chinese higher education institutions, scholars proposed various theoretical and practical innovations tailored to the needs of different times. Below is a chronological review of the literature. In 2004, Zhou [1] explored the current state of democratic life meetings within student party branches and proposed strategies to improve their quality, aiming to enhance the effectiveness of student party-building efforts. In 2008, Su [2] introduced the ISO9001 quality management system to university party-building, emphasizing process management to improve the scientific and standardized operation of party-building work. In 2012, Xu [3] analyzed the characteristics and challenges of student party-building in private universities and provided targeted solutions, highlighting the uniqueness of party-building in private institutions. In 2013, Li [4] further examined the application of the ISO9000 quality management system in party-building, offering a systematic approach to enhance university party work. In 2015, Liu [5] discussed the impact of changing domestic and international political environments on student party-building and proposed pathways to optimize the quality of party-building, emphasizing the deepened education of student party members. In 2016, Gao [6] argued that the core of university party-building was “fostering virtue through education,” stressing the importance of ideological guidance throughout talent cultivation processes. In 2018, Ma [7] reviewed the evolution of quality evaluation mechanisms in university party-building since the reform and opening-up period, emphasizing problem-oriented approaches in the new era to improve the quality of party-building efforts. In 2019, Lu [8] analyzed challenges in building online party platforms in universities and suggested developing more systematic, appealing, and efficient e-governance platforms for party-building. In 2020, Chen [9] emphasized the role of party-building in cultivating innovative talents, advocating for innovative approaches to foster students’ creativity and enhance educational quality. In 2021, Qi [10] explored the construction of a quality evaluation system for grassroots party-building in university departments, systematically addressing organizational functions, institutional guarantees, and other critical factors. In 2022, Mi [11] highlighted the practical applications of digital technology in enhancing party-building, analyzing its contributions to improving organizational quality, and identifying challenges such as technological bottlenecks and ethical issues. In 2023, Gao [12] studied the role of party-building in senior cadre work at universities, proposing strategies to leverage the political and experiential advantages of retired party members. In the same year, Wang, Cheng, and Chen [13] discussed the integration of party-building with operational work in private universities, constructing a quality evaluation system tailored to new era demands. In 2024, Zhao and Zhang [14] focused on building a high-quality university party-building system by integrating political and organizational functions while innovating party activities to achieve the “integration and dual excellence” goals. Liu, Xu and Dong [15] summarized the current status and challenges of grassroots party-building in universities and proposed the "Five Strengthenings" strategy to promote high-quality development of grassroots party-building.

MAGDM is a method used to address complex decision-making problems involving multiple decision-makers and multiple evaluation attributes. Its core objective is to find the optimal decision solution by comprehensively considering the preferences of different decision-makers and the impact of various attributes. Typically, the MAGDM process includes defining decision goals, selecting and defining evaluation attributes, collecting and standardizing data, assigning weights, and conducting comprehensive evaluation and ranking. Decision-makers can use various methods, such as the Analytic Hierarchy Process (AHP), fuzzy comprehensive evaluation, or the CoCoSo technique[16], to evaluate and compare alternatives. MAGDM is widely applied in fields such as engineering, economics, management, and social sciences, aiding organizations in making more scientific and rational choices in diverse and complex decision environments. By integrating multiple opinions and criteria, MAGDM effectively enhances the objectivity and fairness of decision-making. The quality evaluation of university party-building work in the new era falls within the framework of MAGDM.

Currently, both the MAUT methods [17] are employed to address various challenges in MAGDM. However, there has been limited application of these methods in combination with INSs [18], which are highly effective in representing uncertain and imprecise data, such as that encountered in evaluating virtual reality user experiences. Specifically, the integration of average with the MAUT technique within the context of INSs has not been extensively studied.

To address this gap, we propose the INN-MAUT approach, a novel method that integrates average solve MAGDM problems under INSs. This approach is designed to handle the complexity and uncertainty inherent in modern decision-making environments, such as quality evaluation of university party-building work in the new era.

By incorporating INSs, the INN-MAUT method provides a more flexible and comprehensive framework for representing the uncertain preferences of decision-makers. An illustrative example focusing on the evaluation of virtual reality user experiences is presented to demonstrate the effectiveness and reliability of the INN-MAUT approach.

A comparative analysis is also included to showcase the advantages of this technique over existing methods. In addition to this example, a numerical study on quality evaluation of university party building work in the new era is conducted to validate the practical application of the proposed approach.

This study highlights the robustness of the INN-MAUT technique in addressing real-world MAGDM problems. In this paper, the INN-MAUT approach is constructed to address MAGDM problems using INSs. Furthermore, an average-based approach is developed to determine the weight values of different attributes, within the INSs framework. This ensures that the decision-making process is both objective and efficient, with the average approach providing a systematic method to assign appropriate weights to each criterion.

The main objectives of this study are as follows: (1) To develop a MAGDM approach using the INN- MAUT method under INSs; (2) To construct an average-based approach for deriving weight values within INSs; and (3) To propose a novel MAGDM solution for quality evaluation of university party building work in the new era utilizing the INN- MAUT technique. By addressing these objectives, this study contributes to the advancement of decision-making methodologies in complex, uncertain environments like quality evaluation of university party-building work in the new era. Figure 1 shows the types of uncertainty models.

The structure of this study is organized as follows: Section 2 introduces the INSs, providing the

background necessary for understanding the subsequent methods. Section 3 describes the development of the INN- MAUT approach using INSs integrated with average, detailing the methodological framework. Section 4 presents a numerical example focused on quality evaluation of university party building work in the new era, accompanied by a comparative analysis to demonstrate the approach's effectiveness. Finally, Section 5 offers concluding remarks, summarizing the key findings and implications of the study.

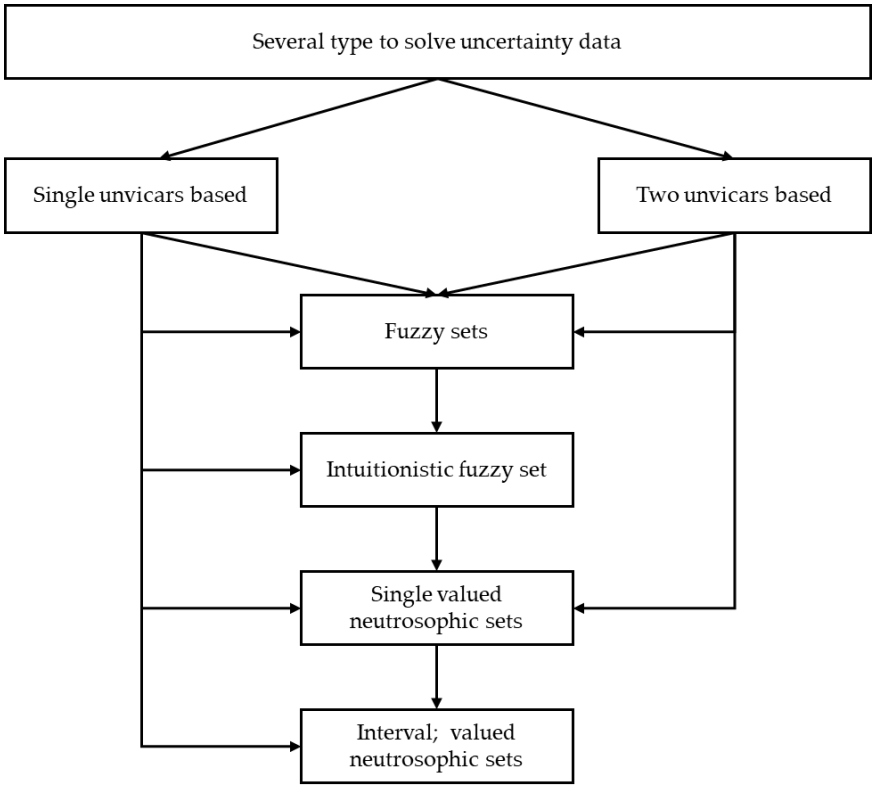

Figure 1. Types of uncertainty models.

2. Preliminaries

The neutrosophic set is defined with three elements such as truth, indeterminacy, and falsity functions [18].

The neutrosophic set is defined with three elements such as truth, indeterminacy, and falsity functions.

$0-\leq \sup T_A + \sup I_A + \sup F_A \leq 3+$

The IVNNs operations can be defined as:

$$\vartheta A = \left(\begin{bmatrix} 1-\left(1-T_A^L(x)\right)^{\vartheta}, \\ 1-\left(1-T_A^U(x)\right)^{\vartheta} \end{bmatrix}, \begin{bmatrix} \left(I_A^L(x)\right)^{\vartheta}, \left(I_A^U(x)\right)^{\vartheta} \end{bmatrix}, \begin{bmatrix} \left(F_A^L(x)\right)^{\vartheta}, \left(F_A^U(x)\right)^{\vartheta} \end{bmatrix} \right) \tag{1}$$

$$A+B = \left(\begin{bmatrix} T_A^L(x)+T_B^L(x)-T_A^L(x)T_B^L(x), \\ T_A^U(x)+T_B^U(x)-T_A^U(x)T_B^U(x) \end{bmatrix}, \begin{matrix} [I_A^L(x)I_B^L(x), I_A^U(x)I_B^U(x)], \\ [F_A^L(x)F_B^L(x), F_A^U(x)F_B^U(x)] \end{matrix} \right) \tag{2}$$

$$A-B = \left(\begin{matrix} [T_A^L(x)-T_A^U(x), T_A^U(x)-T_A^U(x)], \\ \begin{bmatrix} \max\left(I_A^L(x), I_B^L(x)\right), \\ \max\left(I_A^U(x), I_B^U(x)\right) \end{bmatrix}, \\ \begin{bmatrix} F_A^L(x)-F_A^U(x) \\ , F_A^U(x)-F_A^U(x) \end{bmatrix} \end{matrix} \right) \tag{3}$$

$$AB = \left(\begin{matrix} [T_A^L(x)T_B^L(x), T_A^U(x)T_B^U(x)], \\ \begin{bmatrix} I_A^L(x)+I_B^L(x)-I_A^L(x)I_B^L(x) \\ , I_A^U(x)+I_B^U(x)-I_A^U(x)I_B^U(x) \end{bmatrix}, \\ \begin{bmatrix} F_A^L(x)+F_B^L(x)-F_A^L(x)F_B^L(x), \\ F_A^U(x)+F_B^U(x)-F_A^U(x)F_B^U(x) \end{bmatrix} \end{matrix} \right) \tag{4}$$

$$\frac{A}{B} = \left(\begin{bmatrix} \min\left\{ \frac{T_A^L(x)}{T_B^L(x)}, \frac{T_A^L(x)}{T_B^U(x)}, \frac{T_A^U(x)}{T_B^L(x)}, \frac{T_A^U(x)}{T_B^U(x)} \right\}, \\ \max\left\{ \frac{T_A^L(x)}{T_B^L(x)}, \frac{T_A^L(x)}{T_B^U(x)}, \frac{T_A^U(x)}{T_B^L(x)}, \frac{T_A^U(x)}{T_B^U(x)} \right\} \end{bmatrix}, \begin{bmatrix} \min\left\{ \frac{I_A^L(x)}{I_B^L(x)}, \frac{I_A^L(x)}{I_B^U(x)}, \frac{I_A^U(x)}{I_B^L(x)}, \frac{I_A^U(x)}{I_B^U(x)} \right\}, \\ \max\left\{ \frac{I_A^L(x)}{I_B^L(x)}, \frac{I_A^L(x)}{I_B^U(x)}, \frac{I_A^U(x)}{I_B^L(x)}, \frac{I_A^U(x)}{I_B^U(x)} \right\} \end{bmatrix}, \begin{bmatrix} \min\left\{ \frac{F_A^L(x)}{F_B^L(x)}, \frac{F_A^L(x)}{F_B^U(x)}, \frac{F_A^U(x)}{F_B^L(x)}, \frac{F_A^U(x)}{F_B^U(x)} \right\}, \\ \max\left\{ \frac{F_A^L(x)}{F_B^L(x)}, \frac{F_A^L(x)}{F_B^U(x)}, \frac{F_A^U(x)}{F_B^L(x)}, \frac{F_A^U(x)}{F_B^U(x)} \right\} \end{bmatrix} \right) \tag{5}$$

3. The Proposed Study

This section shows the steps of the INN-MAUT for ranking the alternatives.

Step 1. Create the decision matrix.

Three experts used the neutrosophic numbers to evaluate the criteria and alternatives. Then we apply the score function to obtain one value. Then we combine these opinions into one matrix.

Step 2. Normalize the decision matrix

The decision matrix values are normalized based on the positive and negative kinds of criteria. The blow equation is used to normalize the decision matrix for the positive and negative criteria.

$$x_{ij} = \frac{y_{ij} - \min(y_{ij})}{\max(y_{ij}) - \min(y_{ij})}; i = 1, \dots, m; j = 1, \dots, n \quad (6)$$

$$x_{ij} = 1 + \frac{\min(y_{ij}) - y_{ij}}{\max(y_{ij}) - \min(y_{ij})}; i = 1, \dots, m; j = 1, \dots, n \quad (7)$$

Step 3. Determine the marginal utility score.

The below equation is used to compute the marginal utility score.

$$r_{ij} = \frac{e^{(x_{ij})^2} - 1}{1.71} \quad (8)$$

Step 4. Determine the final utility score

The final utility score of each alternative is computed using the below equation with the criteria weights.

$$P_i = \sum_{j=1}^{n} r_{ij} w_j \quad (9)$$

Step 5. Final rank of alternatives.

The final utility score of each alternative is ranked in descending order for the final rank of alternatives. The alternative is the highest rank is the best and the alternative is the lowest rank is the worst.

4. Actual DM Example

The quality evaluation of university party building work in the new era is MAGDM. Six potential local universities are evaluated with 9 attributes.

1. Ideological and Political Leadership

Assesses the effectiveness of Party organizations in guiding the ideological and political direction of the university. This includes fostering the correct political values among faculty and students, promoting patriotism, and ensuring adherence to Party principles in education and research.

2. Organizational Development

Evaluates the establishment and functioning of Party organizations within the university, including their structures, roles, and representation. It ensures that Party committees are active at various levels (department, student groups) and align with broader organizational goals.

3. Integration of Party Building with Core Educational Tasks

Measures how well Party building is integrated with teaching, research, and administrative tasks. This involves aligning Party-building activities with the university's strategic goals, such as talent cultivation, innovation, and community service.

4. Effectiveness of Ideological Education

Analyzes the quality of ideological and moral education imparted to students and staff. This includes curriculum design, extracurricular activities promoting Party values, and the impact of ideological courses on students' beliefs and actions.

5. Leadership in Social Responsibility and Service

Assesses the university's Party organization's role in contributing to societal development and serving the community. This includes outreach programs, social impact projects, and initiatives that showcase the Party's commitment to the public good.

6. Student Engagement in Party Activities

Examine the participation rate and quality of student involvement in Party-building activities. This includes membership drives, leadership training programs, and the formation of youth leagues to cultivate future Party leaders.

7. Party Member Development and Training

Evaluates the processes for identifying, recruiting, and training Party members among students and

staff. It focuses on the quality of training programs, mentorship opportunities, and the retention of highly committed members.

8. Ethical Governance and Anti-Corruption Measures

Monitors the transparency, accountability, and ethical practices within Party organizations in the university. This includes compliance with anti-corruption measures and fostering a culture of integrity and fairness in all Party-related activities.

9. Cultural Promotion and Party Branding

Assesses efforts to promote Party culture within the campus through events, publications, and media. This includes branding the university as a model for Party-led education and integrating cultural activities with Party themes.

Step 1. We created the decision matrix between criteria and alternatives by three experts.

For the first expert:

$$y_{ij} = \left\{ \begin{pmatrix} ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.2,0.3],[0.3,0.4],[0.7,0.8]), \\ ([0.1,0.2],[0.1,0.2],[0.8,0.9]), ([0.8,0.9],[0.1,0.2],[0.1,0.2]), ([0.7,0.8],[0.2,0.3],[0.2,0.3]), \\ ([0.6,0.7],[0.4,0.5],[0.3,0.4]), ([0.5,0.6],[0.5,0.6],[0.4,0.5]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]) \end{pmatrix}, \right.$$
$$\begin{pmatrix} ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), \\ ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), \\ ([0.1,0.2],[0.1,0.2],[0.8,0.9]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]) \end{pmatrix},$$
$$\begin{pmatrix} ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), \\ ([0.7,0.8],[0.2,0.3],[0.2,0.3]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), \\ ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]) \end{pmatrix},$$
$$\begin{pmatrix} ([0.6,0.7],[0.4,0.5],[0.3,0.4]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), \\ ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), \\ ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]) \end{pmatrix},$$
$$\begin{pmatrix} ([0.7,0.8],[0.2,0.3],[0.2,0.3]), ([0.8,0.9],[0.1,0.2],[0.1,0.2]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), \\ ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), \\ ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.4,0.5],[0.5,0.6],[0.5,0.6]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]) \end{pmatrix},$$
$$\left. \begin{pmatrix} ([0.1,0.2],[0.1,0.2],[0.8,0.9]), ([0.2,0.3],[0.3,0.4],[0.7,0.8]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), \\ ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.5,0.5],[0.6,0.7],[0.4,0.5]), ([0.5,0.6],[0.5,0.6],[0.4,0.5]), \\ ([0.6,0.7],[0.4,0.5],[0.3,0.4]), ([0.3,0.4],[0.4,0.5],[0.6,0.7]), ([0.8,0.9],[0.1,0.2],[0.1,0.2]) \end{pmatrix} \right\}$$

Then we convert these values into one value. Then we combine these values into one matrix.

Step 2. Then we used Eq. (6) to obtain the normalized decision matrix as:

$$x_{ij} = \left\{ \begin{array}{c} \left(\begin{array}{c} 0.398230088,0.12962963,0, \\ 0,1,1,\ 0,911949686,0.328767123,0.289719626 \end{array} \right), \\ \left(\begin{array}{c} 0.536873156,0.24382716,0.524590164, \\ 0.439597315,0.189655172,0,0,0.315068493,0.252336449 \end{array} \right), \\ \left(\begin{array}{c} 0.442477876,0.450617284,0.765027322,1, \\ 0.431034483,0.387931034,0.509433962,0.630136986,0.133956386 \end{array} \right), \\ \left(\begin{array}{c} 0.849557522,0.358024691,1,0.322147651, \\ 0.189655172,0.043103448,0.371069182,0,0.224299065 \end{array} \right), \\ \left(\begin{array}{c} 1,1,0.016393443, \\ 0.503355705,0,0.224137931,0.603773585,0.068493151,0 \end{array} \right), \\ \left(\begin{array}{c} 0,0,0.229508197, \\ 0.439597315,0.310344828,0.469827586,1,1,1 \end{array} \right), \end{array} \right\}$$

Then we compute the criteria weights as:

C1= 0.113226969
C2= 0.117279979
C3= 0.101646938
C4= 0.094698919
C5= 0.111747298
C6= 0.115864642
C7= 0.105378281
C8= 0.127959341
C9= 0.112197633

We show that criteria 8 is the best and criteria 4 has the lowest weight in nine criteria.

Step 3. Then we used Eq. (8) to determine the marginal utility score as:

$$r_{ij} = \left\{ \begin{array}{c} \left(\begin{array}{c} 0.712091716,0.173081727,0,0, \\ 3.736290116,3.736290116, \\ 3.038579999,0.543875725,0.459086079 \end{array} \right), \\ \left(\begin{array}{c} 1.126504039,0.367539074,1.084976457, \\ 0.823952529,0.269752319,0,0, \\ 0.513372985,0.383885149 \end{array} \right), \\ \left(\begin{array}{c} 0.832091919,0.855345926,2.116040461, \\ 3.736290116,0.800032195, \\ 0.68565154,1.035121096,1.477419584,0.179668484 \end{array} \right), \\ \left(\begin{array}{c} 2.613511891,0.61189061,3.736290116, \\ 0.529031788,0.269752319,0.052650192, \\ 0.643521745,0,0.331061496 \end{array} \right), \\ \left(\begin{array}{c} 3.736290116,3.736290116,0.019491402, \\ 1.015547771,0,0.330766391, \\ 1.371502458,0.085855227,0 \end{array} \right), \\ \left(\begin{array}{c} 0,0,0.340653009,0.823952529, \\ 0.503047078,0.911753695,3.736290116, \\ 3.736290116,3.736290116 \end{array} \right), \end{array} \right\}$$

Step 4. Then we compute the final utility score as:

$$P_{ij} = \left\{ \begin{array}{l} \left(\begin{array}{c} 0.080627986,0.020299021,0, \\ 0,0.417520325,0.432903918, \\ 0.320200337,0.06959398,0.051508371 = 1.392653938 \end{array} \right), \\ \left(\begin{array}{c} 0.127550637,0.043104975,0.110284534,0.078027414, \\ 0.030144093,0,0,0.065690869, \\ 0.043071005 = 0.497873527 \end{array} \right), \\ \left(\begin{array}{c} 0.094215246,0.100314953,0.215089033,0.353822636, \\ 0.089401436,0.07944277,0.109079282,0.189049637, \\ 0.020158379 = 1.250573371 \end{array} \right), \\ \left(\begin{array}{c} 0.295920029,0.071762518,0.379782449,0.050098739, \\ 0.030144093,0.006100296,0.067813215,0, \\ 0.037144316 = 0.938765654 \end{array} \right), \\ \left(\begin{array}{c} 0.423048804,0.438192028,0.001981241,0.096171276, \\ 0,0.03832413,0.144526571, \\ 0.010985978,0 = 1.153230029 \end{array} \right), \\ \left(\begin{array}{c} 0,0,0.034626335,0.078027414,0.056214152, \\ 0.105640016,0.39372383,0.478093222, \\ 0.419202905 = 1.565527874 \end{array} \right), \end{array} \right\}$$

Step 5. Then we rank alternatives.

A6>A1>A3>A5>A4>A2

4.1 Results and discussion

The evaluation of the quality of Party building work in universities in the new era is a crucial means to enhance the level of Party work, ensuring effective implementation of political, ideological, and organizational construction. The evaluation emphasizes political construction, requiring universities to consistently uphold Party leadership and enhance political awareness and responsibility among Party members and students through systematic education and training.

Ideological construction is another key aspect. Universities should innovate in ideological and political education, using modern technologies to make these efforts more appealing and effective. Diverse activities help deepen students' and faculty's understanding and acceptance of Party theories and policies. In terms of organizational construction, the evaluation focuses on institutionalizing and standardizing Party organizations to ensure their role as strongholds. By establishing sound organizational systems and scientific management mechanisms, the cohesion and effectiveness of Party organizations are enhanced. Additionally, the evaluation system emphasizes the integration of Party work with university education, teaching, and research management. Universities should leverage the exemplary role of Party members, guiding them to embody the Party's mission in their daily work and promoting steady development across all university initiatives. Finally, quality evaluation serves not only as an assessment of current efforts but also as guidance for future work. Through regular feedback and adjustments, the Party work mechanisms are continuously improved, ensuring universities remain at the forefront of the era, providing solid political and organizational

support for cultivating talented individuals with integrity and ability.

When implementing the proposed model, we can apply these steps.

Input: interval valued neutrosophic numbers with three opinions of experts and decision makers.
Computations:
Step 1. Obtain the criteria weights by the average method.
Step 2. Normalize the decision matrix.
Step 3. Then we compute the marginal utility score.
Step 4. Compute the utility score.
Step 5. Rank the alternatives to select the best one.

4.2. Comparative analysis

The INN-MAUT approach was thoroughly compared with the INN-EDAS approach [19] and the INN-TODIM approach [20]. The results of this comparison are presented in Table 1. This analysis shows that while the approaches may rank options slightly differently, they consistently identify the same best and worst local universities. This suggests that the INN-MAUT approach is a valid and effective method.

Compared to traditional methods and their improvements, the INN-MAUT approach has a key advantage: by using interval neutrosophic sets (INSs), it handles uncertainty more effectively. This is especially important for complex tasks like evaluating the quality of university party-building work, where there is a lot of ambiguity. INSs allow evaluators to express their opinions more accurately, even when faced with uncertainty. This approach provides a more nuanced way to deal with imprecise data, making sure that all uncertainties are considered during the evaluation process.

Table 1. Order for different approaches

Method	Order	Optimal	Worst
INN-EDAS approach [19]	A6>A1>A3>A4>A5>A2	A6	A2
INN-TODIM approach[20]	A6>A1>A4>A5>A3>A2	A6	A2
INN-TOPSIS approach	A6>A1>A3>A5>A4>A2	A6	A2
INN- MAUT approach	A6>A1>A3>A5>A4>A2	A6	A2

5. Conclusions and Future Scope

In the new era, the evaluation of the quality of party-building work in colleges focuses on the organic integration of political, ideological, and organizational aspects. Firstly, political construction is central, ensuring the effective implementation of the Party's leadership in universities by enhancing the political awareness of party members and faculty through education and training. Secondly, ideological construction emphasizes innovation and effectiveness, utilizing modern technology to enrich educational methods and enhance recognition of the Party's theories and policies. Organizational construction requires well-established grassroots party organizations and standardized systems to enhance organizational cohesion and combat effectiveness. Additionally, party-building work needs to be deeply integrated with education and teaching, leveraging the exemplary role of party members to promote comprehensive development in teaching and research. Through the comprehensive evaluation of these indicators, universities can continuously enhance the scientific and effective nature of party-building work. The quality evaluation of university party building work in the new era is a MAGDM problem. Recently, the MAUT method has been

employed to tackle MAGDM issues effectively. In this context, INSs are utilized to represent and manage uncertain data during the quality evaluation of university party-building work in the new era, providing a more flexible and comprehensive approach to handling uncertainty. In this study, we propose the development of an INN-MAUT approach to introduce an enhanced MAGDM technique under the framework of INSs. This approach allows for better decision-making in scenarios where uncertainty plays a significant role. Furthermore, the average method is applied to determine the weights of various decision criteria, utilizing the INSs framework to ensure that the influence of uncertainty is accurately captured in the weighting process. To demonstrate the practicality and effectiveness of the proposed INN-MAUT approach, a numerical example is provided, focusing on the application of this method to quality evaluation of university party building work in the new era. This example validates the approach by showing how it can improve decision-making and enhance the overall security assessment process.

In the future study, the proposed model can be applied to different decision-making problems to rank the alternatives and compute the criteria weights. Different methods can be applied to this kind of problem such as TOPSIS, VIKOR, and EDAS methods.

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Neutrosophic Dynamic Network DEA: Efficient Allocation of Carryover Variables in Organizational Processes

Maryam Heydar[1] and Hadi Bagherzadeh Valami[1*]

[1]Department of Applied Mathematics, Yadegar-e-Imam Khomeini (RAH), Shahre Rey Branch, Islamic Azad University, Tehran, Iran, hadibagherzade0@gmail.com.

Abstract

Carryover activities in dynamic DEA refer to the persistence of resources, inputs, or outputs across periods in organizational processes, reflecting the impact of past decisions on current and future performance. In practical applications, some carryover variables can extend beyond the immediate next period, and their allocation is discretionary, controlled by the Decision-Maker (DM). This paper introduces a novel dynamic network DEA (DNDEA) model aimed at optimizing the allocation of these carryovers and identifying inefficiencies within a network system across multiple evaluation periods.

Recognizing the uncertainties present in real-world data, we incorporate neutrosophic sets to effectively process uncertain information, which adds complexity to our analysis. To address this, we transform the Neutrosophic Dynamic Network Slack-Based Measure (NDNSBM) model into a two-stage framework. By leveraging the concept of Pareto efficiency, our model establishes boundaries for overall and period scores across varying levels of truth, indeterminacy, and falsity. The key contribution of this work is the introduction of discretionary carryover variables in DNDEA models, facilitating strategic allocation across future periods. Additionally, the integration of neutrosophic data provides a more realistic approach to dynamic decision-making contexts. We validate our methodology through a numerical example evaluating the performance of Iranian bank branches, demonstrating that our proposed model is more discriminative and offers deeper insights into resource allocation strategies compared to the DNSBM model. This comprehensive approach enhances understanding of resource management in dynamic environments, offering valuable implications for decision-makers in various sectors.

Keywords: Neutrosophic Set; Dynamic Network DEA; Decision Making; Carryover Variables.

1 Introduction

Data Envelopment Analysis (DEA) proposed by Charnes et al. [1] is known as an effective tool for measuring the performance of homogeneous Decision-Making Units (DMUs) [2]–[5]. One key application of DEA is resource allocation [6]–[8]. Most studies about the resource allocation in DEA literature can be generally categorized into two groups [9]. The first group encompasses studies which concentrate on allocating fixed costs to DMUs. Fixed costs represent the total overhead cost utilized for common infrastructures across the subunits of a DMU. This issue is a typical challenge encountered in organizational budgeting and costing, involving the distribution of an overhead cost among various departments [10], [11]. The second group includes studies discuss the allocation of input resources (such as funds and personnel) to DMUs. In these studies, the Decision Maker (DM) allocates available resources to units or sub-units in order to reach a definite goal [12].

In today's dynamic and competitive economic environment, it's evident that the market undergoes continuous changes [13]. However, in both groups mentioned above, these dynamic changes are not adequately addressed [14]. There is a need to open a discussion on how these allocation methods can adapt to the evolving

market conditions over time [15]. Dynamic DEA is recognized as an effective tool to capture temporal changes and fluctuations in market dynamics [16], [17]. Dynamic DEA models with network structure, DNDEA, analyze the internal processes of DMUs and monitor the fluctuation in both period efficiency and divisional efficiency of DMUs over time.

Integrating DNDEA principles into resource allocation frameworks could enhance the adaptability and responsiveness of allocation strategies to changing market conditionsn [18]. Carry-over activities are non-separable elements of dynamic DEA. They indicate how past decisions affect the current and future performance of organizations and provide a better understanding of performance evaluation over time periods by showing how resources and outcomes persist across different periods. In the literature of Dynamic DEA, carry-overs are typically viewed as variables or activities transferred from the current to the next period [19]. However, in real-world scenarios, some carryover variables in the production process can extend beyond the immediate next period [20]. In this paper, we refer to these variables as discretionary carryovers. For instance, in the banking industry, Loan Loss Reserves can be considered as such carryovers [21], [22]. Banks allocate reserves to cover potential losses from loans that may default in the future. If actual loan losses in a specific period are lower than expected, these reserves can be allocated to future periods to improve the bank's efficiency [23]. Another example is deferred tax assets or liabilities. Banks may have deferred tax assets or liabilities due to differences between accounting and tax laws. If the tax impact of certain transactions or events cannot be understood in the current period, these assets or liabilities can be carried forward to future periods when they are expected to be realized or addressed. Additionally, when a bank agrees with a borrower to postpone loan repayment to a future period with a specific interest rate, it's another instance of carryover allocation. Moreover, when a bank negotiates to extend the payment of taxes to a future period, it delays the tax obligation from the current period to a later one [24]. This adjustment affects financial planning and resource allocation over multiple periods, making it a form of carryover. Accrued revenues or expenses are also significant. Banks measure accumulated income from loans and investments and incurred expenses from deposits and borrowings over time. If the actual income or expenses differ from the accrued amounts in a specific period, the difference can be carried forward to future periods until reconciled [25]. These examples illustrate how carryover variables in the banking industry can extend to periods beyond the immediate next one and have implications for financial reporting, risk management, and regulatory compliance over multiple periods. There are also a lot of examples in other industries. For instance, in manufacturing, the carryover of raw materials inventory from one production cycle to the future can impact production efficiency and cost management [26], [27]. Therefore, strategic allocation of carryover activities and tactically distributing resources, considering both the present and the future, is crucial for managing resources effectively within organizations[2], [28].

On the other hand, in real-world problems, the uncertainty and vagueness present in available data often complicate performance evaluations [29], [30]. Neutrosophic sets proposed by Smarandache [31] offer innovative tools for modeling this uncertainty and incorporating it into calculations [32]–[34].

This paper introduces a novel approach within a neutrosophic environment to evaluate the overall, period, and divisional efficiencies of DMUs, incorporating excesses or shortfalls of carryover variables into the objective function. The key contribution is the introduction of discretionary carryover variables in DNDEA models, enabling strategic allocation across future periods. Additionally, the integration of neutrosophic data accounts for uncertainties, providing a more realistic analysis in dynamic decision-making environments. The methods in the paper were chosen based on their ability to address key challenges in DNDEA, particularly in handling carry-over activities that extend beyond consecutive periods. The SBM framework allows us to optimize efficiency while incorporating discretionary allocation of carryovers, a key feature of real-world decision-making. Additionally, the neutrosophic approach was introduced to handle uncertainty, as it provides a robust tool for dealing with indeterminate and vague data, which is often encountered in practical applications. This combination of methods ensures a comprehensive and flexible efficiency evaluation.

The rest of this paper is structured as follows. Section 2 presents an overview of previous studies and theoretical discussions on DNDEA. Section 3 provides an insightful exploration of the fundamentals of neutrosophic sets. Following this, Section 4 presents the proposed method. Subsequently, in Section 5, a case study is presented to illustrate the application of the proposed method, validate the proposed model and

compare the results with those of an existing approach. Finally, Section 6 summarizes the key findings and concludes the paper.

2 Literature Review

Dynamic DEA modeling has become increasingly significant in the literature, offering valuable insights into the sources of inefficiencies over time periods. Network DEA, in particular, is notable for its investigation of inefficiency sources within DMUs. However, integrating dynamic considerations into network systems presents significant challenges. While network modeling provides a theoretical framework for examining the internal structure of DMUs, dynamic modeling clarifies the connections between periods via carry-over activities. DNDEA models handle the complexity of evaluating efficiency by incorporating multiple dynamic stages connected through network structures in each period. This involves comparing a set number of static models [35], which enables us to conduct a thorough analysis. We can observe changes in overall efficiency, dynamic adjustments in divisional efficiency, potential improvements, and efficiency estimates derived from a comprehensive assessment considering interactions between periods and divisions [36]. The innovative approach of DNDEA enables us to look into the black box of the traditional approach and model the interactions among DMUs across different time periods.

The seminal study of Chen [37] represents a pioneering proposition in the research area of DNDEA. This development integrates dynamic effects into the network structure to assess the efficiency of subunits and the overall system. Afterward, scholars and researchers developed many theoretical studies, resulting in a wide range of practical applications in various sectors such as healthcare [38], banking [39], transportation [40], [41], education [42], [43], research and development [44], [45], and energy [46]. One of the most famous models in the DNDEA literature is the one proposed by Tone and Tsutsui [47]. They developed their model based on the SBM approach, which builds upon the NSBM and the dynamic SBM model developed by the same authors in 2009 and 2010, respectively [19], [48]. They assume continuity of link flows between divisions and between periods in their proposed models. Lozano [49] relaxes the intermediate product constraints of the NSBM free link model in their study. In this approach more intermediate products can be produced internally than consumed.

Kuo et al. [50] introduced a DNDEA model to evaluate the sustainability and profitability efficiency of 53 multinational mining companies from 2016 to 2019. They found that companies prioritizing corporate social responsibility are often characterized by high levels of integrity, granting them access to greater financial resources and attracting more customers, thereby achieving satisfactory profitability. Chao et al. [35] introduced a two-stage methodology to evaluate the efficiency of container shipping companies, where the stages represented operational and marketing divisions sharing expenses and personnel. An et al. [51] conducted a study concentrating on the Chinese high-tech industry, employing a framework that shared inputs and outputs across stages.

Wang et al. [52] proposed a novel DNDEA model to measure the sustainable development efficiency of Jiangxi Province's "Internet Plus Logistics" sector from 2002 to 2016. Their study utilized a comprehensive evaluation indicator system, with a particular emphasis on waterway, highway, and railway logistics outputs.

Liu et al. [53] proposed an evaluation indicator system from a stakeholder perspective and suggested a novel combined method applying DNDEA, cross-efficiency evaluation, and Shannon entropy method to evaluate the benefits of bus transit. They conducted an empirical study on bus transit systems in 33 major Chinese cities from 2016 to 2019. Samavati et al. [54] proposed a DNDEA model capable of measuring both optimistic and pessimistic efficiency and effectiveness. Their model accommodates undesirable outputs and enables the ranking of sustainable supply chains based on optimistic and pessimistic efficiency and effectiveness. Lobo et al. [55] introduced a DNDEA model to develop an evaluation framework for assessing the efficiency of university hospitals. Their model considers the coordination among teaching, patient care, and research activities, while also monitoring changes over time.

Table 1 presents a selection of studies in the DNDEA field, showcasing diverse methodologies and their applications across sectors such as banking, logistics, and public transportation. These studies examine various approaches to efficiency measurement, addressing key factors such as innovation, cost efficiency, and performance evaluation within dynamic environments, thereby improving decision-making processes in complex systems.

Table 1. Summary of studies on dynamic network data envelopment analysis.

Study	Methodology	Application	Key Contributions	Flow of carry overs
Tone & Tsutsui [19]	Dynamic DEA	Efficiency evaluation in various industries	Introduced carryover activities in dynamic DEA	Between Consecutive Periods
Fukuyama & Weber [56]	Dynamic Network DEA	Banking sector (Japanese banks)	Assessed production efficiency and financial stability, considering both period-specific and overall efficiency	Between Consecutive Periods
Herrera-Restrepo et al. [57]	Dynamic Network DEA with Slacks-based Model	Evacuation planning and transportation strategies	Integrated multi-perspective evacuation efficiency measurement.	Between Consecutive Periods
Soltanzadeh & Omrani [58]	Fuzzy Relational Dynamic Network DEA	Airline industry	Extended efficiency evaluation by considering interrelated processes over time in fuzzy environment	Between Consecutive Periods
Gharakhani et al. [59]	Dynamic Network DEA with Goal Programming	Performance assessment of insurance companies in Iran	Proposed common weights in dynamic network DEA.	Between Consecutive Periods
Samavati et al. [54]	Dynamic Network DEA	Performance evaluation in multiple sectors	Assessed both positive and negative performance aspects	Between Consecutive Periods
Wang et al. [52]	Enhanced DNDEA	Sustainable development in 'Internet Plus Logistics'	Proposed a custom algorithm in enhanced DNDEA to assess sustainable development in the logistics sector	Between Consecutive Periods
Tatlari et al. [44]	DNDEA with Game Theory	Stock selection in capital markets	Used DNDEA with game theory to rank stock performance and provided insights into efficient stock selection strategies	Between Consecutive Periods
Kaur & Puri [60]	Relational Dynamic Network DEA	Cost efficiency measurement in Indian banking sector	Developed cost-efficient targets in dynamic DEA.	Between Consecutive Periods
Anouze et al. [61]	DNDEA	National innovation system (NIS)	Developed a framework to evaluate NIS and proposed policy	Between Consecutive Periods

		in 23 countries	interventions for underperforming nations	
Liu et al. [53]	DNDEA, Cross-efficiency, Shannon entropy	Bus transit benefits in 33 Chinese cities	Showed inefficiencies in public transit services and suggested improvements focusing on service effectiveness	Between Consecutive Periods
Wu et al. [62]	DNDEA	Global airline performance during COVID-19	Evaluated airline performance across multiple dimensions with a focus on the COVID-19 pandemic impact	Between Consecutive Periods
Rashid et al. [63]	Dynamic Network DEA with Innovation Capital	Airline efficiency incorporating innovation capital	Assessed airline efficiency incorporating innovation capital.	Between Consecutive Periods
The current study	Neutrosophic Dynamic Network DEA	Banking industry	Developed a novel dynamic network SBM model allowing discretionary allocation of carry-overs and handling neutrosophic data	Discretionary

In the banking sector, numerous studies have investigated various aspects of performance and efficiency. In the following, we highlight some notable studies.

Lahouel et al. [64] evaluated the efficiency score of 114 European banks from 2010 to 2019 employing a three-stage DNSBM model. Their study examines both the carryover characteristics of non-performing loans in the first stage of the production process and the carryover characteristics of net operating income in the final stage. Deposits are presented as intermediate products in the first stage, while loans and securities are considered intermediate products in the second stage. Wanke et al. [65] evaluated the performance score of the Indian banking sector applying a static dynamic DEA model. Similar to the study of Lahouel et al. [64], they considered both desirable (net profits) and undesirable (non-performing loans) carryover variables. Interest expenses and operating expenses were used as inputs in the single-stage DEA, while deposits, loans, and investments were considered outputs. In the study by Fukuyama & Tan [66] overall efficiency is broken down into five distinct components: innovation efficiency, primary business stability efficiency, strategic management stability efficiency, profitability efficiency, and corporate social responsibility efficiency. In another study, Fukuyama et al. [66] decomposed overall efficiency into input efficiency, output efficiency, and stability efficiency. Alongside their contributions to efficiency decomposition, their study analytically introduces market power in deposits and loans as intermediate products. Additionally, loan loss provisions are, for the first time, incorporated into the production process as a beneficial or good intermediate product Boussemart et al. [67], by considering carryover activities in the production process, proposed an approach where overall efficiency is decomposed into two sub-efficiencies, reflecting different aspects of banking operations. One aspect relates to economic performance, termed economic efficiency, and the other pertains to stability performance, termed credit risk efficiency. Zhou et al. [68] presented a comprehensive three-stage DEA model with network structure to evaluate bank efficiency in China. Their model includes the stages of capital organization, capital allocation, and profitability generation. They considered both carryover variables and shared inputs in their study to facilitate connections across different periods and production stages. Fukuyama et al. [66] analyzed efficiency in two significant ways: Firstly, they presented a novel model in DNDEA by proposing a sequential structure that incorporates the dual-role characteristics of production factors. Secondly, they improved their innovative sequential DNDEA proposal with a behavioral-causal analysis. To validate their model, they applied these two analyses to the banking industry, using a sample of

43 Chinese commercial banks between 2010 and 2018. Kweh et al. [69] proposed a DNDEA model to evaluate resource management and profitability efficiencies of 287 US commercial banks from 2010 to 2020. They assessed the efficiency score by incorporating five variables: capital adequacy, asset quality, management quality, earning ability, and liquidity, referred to as the CAMEL ratings. Their study contributes significantly to the discussion on measuring performance in the banking industry.

All studies mentioned above rely on accurate data. However, real-world data often lack clarity, and the gathered information is uncertain and indeterminate. Recognizing this challenge, many scholars in the literature of Decision-Making have begun to address uncertainty within their models by employing Zadeh's fuzzy sets [70] and developing various fuzzy approaches [71]–[76]. To handle imprecise data, Intuitionistic Fuzzy Sets (IFSs) proposed by Atanassov [77], an extension of Zadeh's fuzzy sets, have been frequently utilized to address imprecision and uncertainty, particularly in the DEA literature [78]–[81]. Al-Omeri [82] proposed a novel method for constructing a topological space by integrating two distinct fuzzy topologies. Ghareeb & Al-Omeri [83] introduced semiopenness, semicontinuity, preopenness, and related notions in (L, M)-fuzzy topological spaces using graded concepts. They explored their relationships with semi(pre)-compactness, semi(pre)-connectedness, and fuzzy separation axioms like Semi-T1 and Pre-T2. AL-Omeri et al. [84] investigated the complement of the maximum product of two neutrosophic graphs, analyzing vertex degrees. Additionally, they presented an application for locating online streaming services using normalized Hamming distance.

Smarandache [31] proposed the neutrosophic set to address the issues containing imprecise, incomplete, uncertain and indeterminate data. However, without a detailed definition, neutrosophic sets are challenging to apply in real-world scientific and technical applications. Wang et al. [85] defined Single-Valued Neutrosophic Sets (SVNSs), which are a special case of neutrosophic sets. In recent years, numerous research efforts have focused on scientific problems based on SVNSs and their extensions [84]. Al-Omeri et al. [86] presented common fixed-point theorems in neutrosophic cone metric spaces, introducing the concept of (Φ, Ψ)-weak contraction via altering distance functions. Several examples of cone metric spaces are reviewed to verify key properties. In the field of Multi-Criteria Decision-Making (MCDM), scholars have proposed various methodologies to tackle complex decision processes across different industries [87], [88]. Recent research has focused on handling uncertainty, especially through neutrosophic frameworks. Hesami [89] applied a hybrid ANP-TOPSIS method for supplier selection in Reverse Logistics, while Alharbi et al. [90] used the WASPAS method to evaluate leadership challenges in the energy sector. El-Douh et al. [91] employed a neutrosophic MCDM model to manage uncertainties in health sustainability during COVID-19, improving decision accuracy.

Edalatpanah [34] was the first to extend the DEA model using Single-Value Neutrosophic Numbers (SVNNs) in order to enhance the efficiency of private institutions. Abdelfattah [92] proposed a parametric approach to solve neutrosophic linear programming models. Kahraman et al. [93] developed the Neutrosophic Analytic Hierarchy Process (NAHP) and introduced a hybrid NAHP-NDEA algorithm for performance evaluation. Mao et al. [94] and Yang et al. [95] explored and enhanced the neutrosophic DEA model in various real-world applications; see also [96]. Rasinojehdehi & Valami [97] proposed a neutrosophic network SBM model to measure the relative efficiencies of Iranian Airlines. Their approach not only addresses the issue caused by the dual role of intermediate measures but also incorporates the inefficiency associated with intermediate measures into the objective function.

In the literature of DEA, researchers and scholars have increasingly extended DEA models with neutrosophic data. However, there are notable research gaps in this intriguing field. Neutrosophic sets, which encompass more uncertainty than other fuzzy numbers and are considered the most acceptable form of a fuzzy number, have rarely been employed in DNDEA with neutrosophic inputs, outputs, intermediate measures and carry over variables. All of the studies mentioned in this section and in Table 1 restrict carry-over variables to only two consecutive periods, limiting the flexibility in resource management across time. In the following, we propose a novel DNDEA model within the SBM framework, where carryover activities can be allocated to specific periods to enhance performance evaluation. The proposed model not only determines the optimal allocation of carryovers but also provides divisional, period, and overall efficiency within a unified framework. However, before presenting the proposed model, we review some fundamental background of Neutrosophic sets for a better understanding the idea of this paper in the next section.

3 Preliminaries

In this section, we will review some fundamental backgrounds of Neutrosophic logic required in this paper.

Definition 1 ([31]). Suppose X is a space of points with a generic element denoted by x. Then, a neutrosophic set A in X is indicated by $A = \{< x, T_A(x), I_A(x), F_A(x) | x \in X >\}$, where $T_A(x),\ I_A(x),\ F_A(x) \in\ ]0^-, 1^+[$ signify the truth, indeterminacy and falsity-membership functions, respectively, such that $0^- \leq sup\, T_A(x) + sup\, I_A(x) + sup\, F_A(x) \leq 3^+$.

Wang et al. [85] recognizing the limitations in applying neutrosophic sets to practical problems, simplified the approach by converting nonstandard interval numbers into SVNSs of standard interval numbers.

Definition 2 ([85]). Let X be a space of points with a generic element in X, denoted by x. Then, An SVNS in A in X is denoted by $A = \{< x, T_A(x), I_A(x), F_A(x) | x \in X >\}$, where $T_A(x),\ I_A(x),\ F_A(x) \in [0,1]$, such that $0 \leq sup\, T_A(x) + sup\, I_A(x) + sup\, F_A(x) \leq 3$.

For an SVNS $\{< x, T_A(x), I_A(x), F_A(x) | x \in X >\}$, the ordered triple components $< T_A(x), I_A(x), F_A(x) >$ are described as an SVNN, and each SVNN can be characterized as $a =< T_a, I_a, F_a >$, where $T_a, I_a, F_a \in [0,1]$ and $0 \leq T_a + I_a + F_a \leq 1$.

Definition 3 ([98]). The Triangular Neutrosophic Number (TNNs) $\tilde{\Upsilon} =\prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ have the truth, indeterminacy, and falsehood membership functions of x as presented respectively in Equation (1) to Equation (3):

$$T_{\tilde{\Upsilon}}(x) = \begin{cases} \dfrac{(x - r_1)}{(r_2 - r_1)}\omega & r_1 \leq x < r_2, \\ \omega & x = r_2, \\ \dfrac{(r_3 - x)}{(r_3 - r_2)}\omega & r_2 \leq x < r_3, \\ 0 & otherwise. \end{cases} \tag{1}$$

$$I_{\tilde{\Upsilon}}(x) = \begin{cases} \dfrac{(r_2 - x)}{(r_2 - r_1)}\theta, & r_1 \leq x < r_2, \\ \theta, & x = r_2, \\ \dfrac{(x - r_3)}{(r_3 - r_2)}\theta, & r_2 \leq x < r_3, \\ 1, & otherwise. \end{cases} \tag{2}$$

$$F_{\tilde{\Upsilon}}(x) = \begin{cases} \dfrac{(r_2 - x)}{(r_2 - r_1)}\chi, & r_1 \leq x < r_2, \\ \chi, & x = r_2, \\ \dfrac{(x - r_3)}{(r_3 - r_2)}\chi, & r_2 \leq x < r_3, \\ 1, & otherwise. \end{cases} \tag{3}$$

Where, $0 \leq T_{\tilde{\Upsilon}}(x) + I_{\tilde{\Upsilon}}(x) + F_{\tilde{\Upsilon}}(x)(x) \leq 3, x \in \tilde{\Upsilon}$.

Definition 4 ([99]). Let $\tilde{\Upsilon} =\prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ is a TNN. Then the aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)}$ is described as:

$$\varphi_{(\alpha,\beta,\gamma)} = [\tilde{\Upsilon}^L_{(\alpha,\beta,\gamma)}, \tilde{\Upsilon}^U_{(\alpha,\beta,\gamma)}] = [r_1 + (r_2 - r_3)\hbar, r_3 - (r_3 - r_2)\hbar] \tag{3}$$

Where, $\alpha \in [0, \eta], \beta \in [\lambda, 1], \gamma \in [\kappa, 1]$, and $\hbar = \frac{1}{4}(\frac{\alpha}{\omega} + 2\frac{(1-\beta)}{1-\theta} + \frac{(1-\gamma)}{1-\chi})$ is a variation degree of the TNNs.

In the upcoming section, we will introduce our innovative model and provide a detailed insight into the process of allocating carryovers.

4 The Proposed Neutrosophic Dynamic Network SBM Model

In this section, we introduce our novel neutrosophic DNDEA model within the SBM framework. As discussed previously, our proposed model allocates carryover activities to one of the subsequent periods in a manner that maximizes overall inefficiencies while the data are available in neutrosophic numbers. Equations

(4-17) illustrate the proposed model in which $\tilde{x}_{ij}^{tk}$,$\tilde{y}_{rj}^{tk}$,$\tilde{z}_{cj}^{tcarry_\alpha}$and $z_{dp}^{t(k,h)\beta}$ are neutrosophic numbers. This paper utilizes the notations for the data and variables as presented in Table 2 and Table 3, respectively:

Table 2. Data notation.

data	symbol	definition
input	x_{ip}^{tk}	Input resource i to Division k (Divk) of DMUj at time period t
output	y_{rp}^{tk}	Output product r from Divk of DMUj at time period t
link	$z_{dp}^{t(k,h)\beta}$	link d from category β between interconnected Divk and h of DMUp in period t. β stands for free, fixed, as-input, and as-output
Carry-over	$z_{ckj}^{tcarry\alpha}$	Carry-over c from category α of DMUj at Divk from period t

Table 3. Variable notation.

Variable	Symbol	Definition
Input slack	s_{ip}^{tk-}	Slack of input i of DMU_p for Divk at period t.
Output slack	s_{rp}^{tk+}	Slack of output r of DMUp for Divk at period t.
Link slack	$s_{dp}^{t(k,h)\beta}$	Slack of link d between Divk and Divh of DMUp at period t. β stands for "fixed", "free", "as-input", and" as-output".
Carry-over slack	$s_{cp}^{(t,f)k\alpha}$	Slack of carry-over c for Divk of DMUp from period t to period f. α stands for free, fixed, good and bad.
intensity	λ_j^{tk}	Intensity of DMUj corresponding to Divk at period t.
Carry-over allocator	$t_{kc\alpha}^{(t,f)}$	a binary variable that equals 1 if the carry-over c in category α from Divk in period t is assigned to the corresponding division in period f; otherwise, it equals zero.
Carry-over value control	$zz_{cj}^{tcarry\_bad}$	A free variable which controls the value of the corresponding carry over.
symbol	$h_k^{(a,t)}$	The number of all carry overs from Divk in period a to Divk in period t ($\forall a < t$).
symbol	$h_k^{(t,a)}$	The number of all carry overs from Divk in period t to Divk in period a ($\forall a$, $t < a \leq T$).

The objective function presented in the Equation (4) measures non-oriented overall efficiency, where W^t and W^k denote the weights for period t and *Divk*, respectively, determined exogenously by the DM.

Equations (5) and (6) represent input and output constraints, respectively, for *Divk* of DMUj in period t.

For linking activities, we present four scenarios, each offering distinct characteristics: the as-input value case (Equation (7)), the as-output value case (Equation (8)), the fixed-link value case (Equation (9)), and the free-link value case (Equation (10)). For a comprehensive understanding, please refer to the DNSBM model in [47].

In this study, discretionary carryover activities are defined as those that can be allocated not only to the next period but also extended beyond it. Figure 1 illustrates the carry-over activities in a multi-stage dynamic network system across several periods. Outputs from earlier periods (such as Period t) are carried over as inputs to subsequent periods (like Period t+1 and beyond), ensuring the interconnection between stages and the influence of past decisions on future performance. We categorize these activities into four distinct types, following the framework outlined by Tone and Tsutsui (2014). However, unlike the carryovers in their study, which were limited to the next period, the carryovers in our study can be allocated at the discretion of the DM, to the future periods beyond the immediate next one. The categories are presented as follows;

a. Good or Desirable Carry-over:

This category encompasses favorable carry-over such as profit transferred to the one of the subsequent periods. In our model, desirable carry-overs are considered outputs from the corresponding period, and their magnitudes are controlled to not fall below the observed values. Any shortfall in comparative carry-over within this category is considered as an inefficiency.

b. Bad or Undesirable Carry-over:

This category comprises unfavorable instances of carry-over such as losses carried forward, bad debts, and dead stock. In our model, undesirable carry-overs are treated as inputs to one of the next periods determined by the model, with their magnitudes constrained to not exceed the observed values. Any surplus in comparative carry-over within this category is considered as an inefficiency.

c. Free Carry-over:

The carry-overs in this category are managed freely by the DMU. Their value can be lower or higher than the observed value, and this variation does not directly influence the efficiency. They indirectly impact the efficiency score through the continuity assumption between periods.

It's important to highlight that in the study of Tone and Tsutsui (2014), free carryovers are also referred to as discretionary. To simplify matters in our study, we reserve the term "discretionary" for carryovers whose allocation is determined freely by the DM. Meanwhile, we simply refer to the other carryovers as free carryovers, indicating that their values can be managed without restrictions.

d. Fixed Carry-over:

The carry-overs in this category are beyond the control of the DMU and their values are fixed at the observed level. Like free carry-overs, fixed carry-overs influence the efficiency score through the continuity assumption between periods.

The carry-over constraints are formulated as presented in Equation (11) to Equation (17). Equation (11) maintains the continuity assumption between the periods t and f if there exists a connection between them. In the case $t_{kc\alpha}^{(t,f)} = 1$ where, indicating the allocation of carry-over *c* from category α and *Divk* in period t to period f, the replacement of $t_{kc\alpha}^{(t,f)} = 1$ results in:

$$\sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{kcj}^{tcarry\_\alpha} = \sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{kcj}^{tcarry\_\alpha} \tag{3}$$

In the scenario where $t_{kc\alpha}^{(t,f)} = 0$, Equation (11) becomes redundant.

Equation (12) ensures that the carry over $\tilde{z}_{kcj}^{tcarry\_\alpha}$ from period t is allocated to only one of the subsequent periods.

Equation (13) ensures that the carryovers in “fixed” category do not deviate from their observed values.

Equation (14) illustrates that the carry over $\tilde{z}_{kcj}^{tcarry\_good}$ is considered as an output from period t and its shortfall from the observed values is measured by $s_{cp}^{(t,f)kgood}$.

Equations (15) and (16) demonstrate that if the undesirable carry over $\tilde{z}_{kcj}^{tcarry\_bad}$ is allocated to period f ($t_{kc\alpha}^{(t,f)} = 1$), it is regarded as an input to that period and any excess from the observed values is quantified by $s_{cp}^{(t,f)kbad}$.

$$\tilde{\psi}_p^* = Min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk-}}{\tilde{x}_{ip}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{\mathrm{t}(f,k)as-input}}{\tilde{z}_{dp}^{\mathrm{t}(f,k)as-input}} + \sum_{a=1}^{t} \sum_{\mathrm{c}=1}^{h_k^{(\mathrm{a},t)}} \frac{s_{cp}^{(t}}{\tilde{z}_{cj}^{tc}}\right.\right.}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk+}}{\tilde{y}_{rp}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{\mathrm{t(k,h)}as-output}}{\tilde{z}_{dp}^{\mathrm{t(k,h)}as-output}} + \sum_{a=t}^{T} \sum_{\mathrm{c}=1}^{h_k^{(\mathrm{t},a)}} \frac{s_{c}^{(}}{\tilde{z}_{c_j}^{t}}\right.\right.} \tag{4}$$

$$\text{s.t.} \sum_{j=1}^{n} \lambda_j^{tk} \tilde{x}_{ij}^{tk} + s_{ip}^{tk-} = \tilde{x}_{ip}^{tk}, \ (k = 1, \ldots, K), (i = 1, \ldots, m_k), \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{y}_{rj}^{tk} - s_{rp}^{tk+} = \tilde{y}_{rp}^{tk}, \ \ (k = 1, \ldots, K), (r = 1, \ldots, r_k), \tag{6}$$

$$\sum_{j=1}^{n} \lambda_j^{th} \tilde{z}_{dj}^{t(k,h)in} + s_{dp}^{t(k,h)as-input} = \tilde{z}_{dp}^{t(k,h)as-input}, \left(d \in l_{(k,h)in}\right), \forall (k,h), \tag{7}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{z}_{dj}^{t(k,h)out} - s_{dp}^{t(k,h)as-output} = \tilde{z}_{dp}^{t(k,h)as-output}, \quad \left(d \in l_{(k,h)out}\right), \forall (k,h), \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{z}_{dj}^{t(k,h)fixed} = \tilde{z}_{dp}^{t(k,h)fixed}, \left(d \in l_{(k,h)fix}\right), \forall (k,h), \tag{9}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{z}_{dj}^{t(k,h)} = \sum_{j=1}^{n} \lambda_j^{th} \tilde{z}_{dj}^{t(k,h)}, \left(d = 1, \ldots, l_{(k,h)}\right), \forall (k,h), \tag{10}$$

$$\sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{kcj}^{tcarry\_\alpha} - M\left(1 - t_{kc\alpha}^{(t,f)}\right) \leq \sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{cj}^{tcarry_\alpha} \leq \sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{cj}^{tcarry_\alpha} + M\left(1 - t_{kc\alpha}^{(t,f)}\right), f > t, \alpha \in \{bad, good, free, fixed\}, \tag{11}$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1, \forall f > t, (k = 1, \ldots, K), \alpha \in \{bad, good, free, fixed\}, \tag{12}$$

$$\sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{cj}^{tcarry\_fixed} = \tilde{z}_{cp}^{tfixed}, \forall t, \forall c, \forall k, \tag{13}$$

$$\sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{cj}^{tcarry\_good} - s_{cp}^{(t,f)kgood} = \tilde{z}_{cj}^{tcarry_{good}}, \forall t, \forall c, \forall k, \tag{14}$$

$$\sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{cj}^{tcarry\_bad} + s_{cp}^{(t,f)kbad} = \widetilde{zz}_{cj}^{tcarry\_bad}, \forall t, \forall c, \forall k, \tag{15}$$

$$\tilde{z}_{cj}^{tcarry\_bad} - M\left(1 - t_{kcbad}^{(t,f)}\right) \leq \widetilde{zz}_{cj}^{tcarry_{bad}} \leq \tilde{z}_{cj}^{tcarry_{bad}} + M\left(1 - t_{kcbad}^{(t,f)}\right), \forall t, \forall c, \forall k, \forall f > t, \tag{16}$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \geq 0, s_{rp}^{k+} \geq 0, s_{ip}^{k-} \geq 0, s_{dp}^{(k,h)-} \geq 0, s_{dp}^{(k,h)+} \geq 0, s_{cp}^{(t,f)kgood} \geq 0, s_{cp}^{(t,f)kbad} \geq 0. \tag{17}$$

In the following, we propose a theorem to compare the results of the proposed model with those of the DNSBM. The aim is to demonstrate that the scores obtained by the proposed model do not exceed those of the DNSBM.

Theorem: The scores produced by the proposed model are not greater than those obtained by the DNSBM.

Proof: By substituting $f = t + 1$ in the proposed model, it simplifies to the DNSBM model. This implies that the proposed model can, under certain conditions, replicate the behavior of the DNSBM. Moreover, the feasible region of the proposed model fully contains the feasible region of the DNSBM, meaning that the proposed model has at least as many feasible solutions as the DNSBM. The objective function of the proposed model, when $f = t + 1$, can achieve values identical to those of the DNSBM objective function. Consequently, the scores obtained by the proposed model cannot exceed those produced by the DNSBM.

In the proposed model, given that all inputs, outputs, intermediate products, and carry-overs are of the neutrosophic data type, the efficiency obtained from the model is also expressed as a neutrosophic number. In the next section, we aim to transform the uncertain model into a deterministic model.

4.1 Transformation of the proposed model into deterministic model

As we mentioned earlier the efficiency obtained from the proposed model is a neutrosophic number. The neutrosophic values of the model are denoted by the convex neutrosophic numbers $\tilde{z}_{ckj}^{tcarry\alpha}$, $\tilde{x}_{ij}^{tk}$, $\tilde{y}_{rj}^{tk}$ and $\tilde{z}_{dj}^{t(k,h)\beta}$ with the following truth, indeterminacy, and falsity membership functions presented in Equation (18) to Equation (21):

$$\tilde{X}_{ij}^{tk} = \left\{ \tilde{x}_{ij}^{tk}, T_{\tilde{x}_{ij}^{tk}}, I_{\tilde{x}_{ij}^{tk}}, F_{\tilde{x}_{ij}^{tk}} | \tilde{x}_{ij}^{k} \in S(\tilde{x}_{ij}^{tk}) \right\}, \tag{18}$$

$$\tilde{Z}_{ckj}^{tcarry\alpha} = \left\{ \tilde{z}_{ckj}^{tcarry\alpha}, T_{\tilde{z}_{ckj}^{tcarry\alpha}}, I_{\tilde{z}_{ckj}^{tcarry\alpha}}, F_{\tilde{z}_{ckj}^{tcarry\alpha}} | \tilde{z}_{ckj}^{tcarry\alpha} \in S(\tilde{z}_{ckj}^{tcarry\alpha}) \right\}, \tag{19}$$

$$\tilde{Y}_{rj}^{tk} = \left\{ \tilde{y}_{rj}^{tk}, T_{\tilde{y}_{rj}^{tk}}, I_{\tilde{y}_{rj}^{tk}}, F_{\tilde{y}_{rj}^{tk}} | \tilde{y}_{rj}^{tk} \in S(\tilde{y}_{rj}^{tk}) \right\}, \tag{20}$$

$$\tilde{Z}_{dj}^{t(k,h)\beta} = \left\{ \tilde{z}_{dj}^{t(k,h)\beta}, T_{\tilde{z}_{dj}^{t(k,h)\beta}}, I_{\tilde{z}_{dj}^{t(k,h)\beta}}, F_{\tilde{z}_{dj}^{t(k,h)\beta}} | \tilde{z}_{dj}^{t(k,h)\beta} \in S(\tilde{z}_{dj}^{t(k,h)\beta}) \right\}. \quad (21)$$

Where $S(.)$ denotes the support of the set. Therefore, it is adequate to determine the left shape and the right shape function of the aggregate coefficient of Definition 4. Hence, for all $i, j, r, d, (k, h)$, the upper and the lower limits for each neutrosophic variable in a φ variation degree are obtained as Equations (22) to (25):

$$(X_{ij}^{tk})_{\varphi}^{L} \le x_{ij}^{tk} \le (X_{ij}^{tk})_{\varphi}^{U}, \quad (22)$$

$$(Y_{rj}^{tk})_{\varphi}^{L} \le y_{rj}^{tk} \le (Y_{rj}^{tk})_{\varphi}^{U}, \quad (23)$$

$$(Z_{ckj}^{tcarry\alpha})_{\varphi}^{L} \le z_{ckj}^{tcarry\alpha} \le (Z_{ckj}^{tcarry\alpha})_{\varphi}^{U}, \quad (24)$$

$$(Z_{dj}^{t(k,h)\beta})_{\varphi}^{L} \le z_{dj}^{t(k,h)\beta} \le (Z_{dj}^{t(k,h)\beta})_{\varphi}^{U}. \quad (25)$$

Using φ variation degree of neutrosophic numbers, the proposed neutrosophic DNSBM model easily can be converted into an interval problem. To do this, we created two-stage mathematical programming that measures the upper efficiency of the *DMUp* as presented in Equations (26) to (39).

$$(\psi_p^*)_{\varphi}^{U} = max$$
$$(X_{ij}^{tk})_{\varphi}^{L} \le x_{ij}^{tk} \le (X_{ij}^{tk})_{\varphi}^{U}$$
$$(Y_{rj}^{tk})_{\varphi}^{L} \le y_{rj}^{tk} \le (Y_{rj}^{tk})_{\varphi}^{U} \quad =$$
$$(Z_{ckj}^{tcarry\alpha})_{\varphi}^{L} \le z_{ckj}^{tcarry\alpha} \le (Z_{ckj}^{tcarry\alpha})_{\varphi}^{U}$$
$$(Z_{dj}^{t(k,h)\beta})_{\varphi}^{L} \le z_{dj}^{t(k,h)\beta} \le (Z_{dj}^{t(k,h)\beta})_{\varphi}^{U} \quad (26)$$

$$\begin{cases} Min \dfrac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk-}}{x_{ip}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(f,k)as-input}}{z_{dp}^{t(f,k)as-input}} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{s_{cp}^{(t,f)kbad}}{z_{cj}^{tcarry_{bad}}} \right) \right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk+}}{y_{rp}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(k,h)as-output}}{z_{dp}^{t(k,h)as-output}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{s_{cp}^{(t,a)kgood}}{z_{cj}^{tcarry\_good}} \right) \right]} \end{cases},$$

s.t. $\sum_{j=1}^{n} \lambda_j^{tk} x_{ij}^{tk} + s_{ip}^{tk-} = x_{ip}^{tk}, (k = 1, \dots, K), (i = 1, \dots, m_k),$ (27)

$\sum_{j=1}^{n} \lambda_j^{tk} y_{rj}^{tk} - s_{rp}^{tk+} = y_{rp}^{tk}, (k = 1, \dots, K), (r = 1, \dots, r_k),$ (28)

$\sum_{j=1}^{n} \lambda_j^{th} z_{dj}^{t(k,h)in} + s_{dp}^{t(k,h)as-input} = z_{dp}^{t(k,h)as-input}, \left(d \in l_{(k,h)in}\right), \forall (k, h),$ (29)

$\sum_{j=1}^{n} \lambda_j^{tk} z_{dj}^{t(k,h)out} - s_{dp}^{t(k,h)as-output} = z_{dp}^{t(k,h)as-output}, (d \in l_{(k,h)out}), \forall (k, h)$ (30)

$\sum_{j=1}^{n} \lambda_j^{tk} z_{dj}^{t(k,h)fixed} = z_{dp}^{t(k,h)fixed}, \left(d \in l_{(k,h)fix}\right), \forall (k, h),$ (31)

$\sum_{j=1}^{n} \lambda_j^{tk} z_{dj}^{t(k,h)} = \sum_{j=1}^{n} \lambda_j^{th} z_{dj}^{t(k,h)}, \left(d = 1, \dots, l_{(k,h)}\right), \forall (k, h),$ (32)

$$\sum_{j=1}^{n} \lambda_j^{kf} z_{kcj}^{tcarry\_\alpha} - M\left(1 - t_{kc\alpha}^{(t,f)}\right) \le \sum_{j=1}^{n} \lambda_j^{kt} z_{cj}^{tcarry_{\alpha}} \le \sum_{j=1}^{n} \lambda_j^{kf} z_{cj}^{tcarry_{\alpha}} + M\left(1 - t_{kc\alpha}^{(t,f)}\right), \quad (33)$$
$$\forall f > t\ , \alpha \in \{bad, good, free, fixed\},$$

$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1, \forall f > t, (k = 1, \dots, K), \alpha \in \{bad, good, free, fixed\},$ (34)

$$\sum_{j=1}^{n} \lambda_j^{kt} z_{cj}^{tcarry\_fixed} = z_{cp}^{tfixed}, \forall t, \forall c, \forall k, \quad (35)$$

$$\sum_{j=1}^{n} \lambda_j^{kt} z_{cj}^{tcarry\_good} - s_{cp}^{(t,f)kgood} = z_{cj}^{tcarry_{good}}, \forall t, \forall c, \forall k, \forall f > t, \quad (36)$$

$$\sum_{j=1}^{n} \lambda_j^{kf} z_{cj}^{tcarry\_bad} + s_{cp}^{(t,f)kbad} = zz_{cj}^{tcarry\_bad}, \forall t, \forall c, \forall k, \forall f > t, \quad (37)$$

$$z_{cj}^{tcarry\_bad} - M\left(1 - t_{kcbad}^{(t,f)}\right) \le zz_{cj}^{tcarry_{bad}} \le z_{cj}^{tcarry_{bad}} + M\left(1 - t_{kcbad}^{(t,f)}\right), \forall t, \forall c, \forall k, \forall f > t, \quad (38)$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \ge 0, s_{rp}^{k+} \ge 0, s_{ip}^{k-} \ge 0, s_{dp}^{(k,h)-} \ge 0, s_{dp}^{(k,h)+} \ge 0, s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0. \quad (39)$$

Similarly, we developed a two-step mathematical programming model to capture the lower efficiencies of the DMUp, as presented in Equation (40).

$$
\begin{gathered}
(\psi_p^*)_\varphi^L = min \\
(X_{ij}^{tk})_\varphi^L \le x_{ij}^{tk} \le (X_{ij}^{tk})_\varphi^U \\
(Y_{rj}^{tk})_\varphi^L \le y_{rj}^{tk} \le (Y_{rj}^{tk})_\varphi^U \\
(Z_{ckj}^{tcarry\alpha})_\varphi^L \le z_{ckj}^{tcarry\alpha} \le (Z_{ckj}^{tcarry\alpha})_\varphi^U \\
(Z_{dj}^{t(k,h)\beta})_\varphi^L \le z_{dj}^{t(k,h)\beta} \le (Z_{dj}^{t(k,h)\beta})_\varphi^U
\end{gathered}
=
\begin{cases}
Min \dfrac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk-}}{x_{ip}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(f,k)as-input}}{z_{dp}^{t(f,k)as-input}} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{s_{cp}^{(t,f)kbad}}{z_{cj}^{tcarry_{bad}}}\right)\right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk+}}{y_{rp}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(k,h)as-output}}{z_{dp}^{t(k,h)as-output}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{s_{cp}^{(t,f)kgood}}{z_{cj}^{tcarry_{good}}}\right)\right]}, \\
s.t. \\
constarints \quad (24-36) \\
s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0.
\end{cases}
\tag{40}
$$

By comparing the two-stage models, it becomes evident that $(\psi_p^*)_\varphi^L \le (\psi_p^*)_\varphi^U$ .Consequently, based on this finding, the range $\left[(\psi_p^*)_\varphi^L, (\psi_p^*)_\varphi^U\right]$ is deemed as the variation degree for φ in assessing DMUp performance. We will simplify by merging the two-stage models into one stage. Following the Pareto efficiency idea, for DMUp with a specific aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)} = \varphi$, maximum efficiency occurs when its inputs are minimized and outputs are maximized. Conversely, for other units, inputs should be maximized while outputs should be minimized. Likewise, for DMUp, the lowest performance will happen when inputs are at their maximum and outputs are at their minimum. Conversely, for other units, inputs should be minimized while outputs should be maximized. The model presented in Equations (41) to (54) computes the upper level for efficiency of DMUp at aggregate coefficient of φ.

$$
(\psi_p^*)_\varphi^U
=
Min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{(s_{ip}^{tk-})_\varphi^L}{(x_{ip}^{tk})_\varphi^L} + \sum_{d=1}^{l_{(f,k)}} \frac{(s_{dp}^{t(f,k)as-input})_\varphi^L}{(z_{dp}^{t(k,h)as-input})_\varphi^L} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{(s_{cp}^{(t,f)kbad})_\varphi^L}{(z_{cp}^{tcarry_{bad}})_\varphi^L}\right)\right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{(s_{rp}^{tk+})_\varphi^U}{(y_{rp}^{tk})_\varphi^U} + \sum_{d=1}^{l_{(f,k)}} \frac{(s_{dp}^{t(k,h)as-output})_\varphi^U}{(z_{dp}^{t(k,h)as-output})_\varphi^U} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{(s_{cp}^{(t,f)kgood})_\varphi^U}{(z_{cp}^{tcarry\_good})_\varphi^U}\right)\right]}
\tag{41}
$$

$$
\text{s.t.} \sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j^{tk} (x_{ij}^{tk})_\varphi^U + \lambda_p^{tk} (x_{ip}^{tk})_\varphi^L + (s_{ip}^{tk-})_\varphi^L = (x_{ip}^{tk})_\varphi^L
\tag{42}
$$

$$
(k = 1, \dots, K), (i = 1, \dots, m_k)
$$

$$
\sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j^{tk} (y_{ij}^{tk})_\varphi^L + \lambda_p^{tk} (y_{ip}^{tk})_\varphi^U + (s_{rp}^{tk+})_\varphi^U = (y_{rp}^{tk})_\varphi^U
\tag{43}
$$

$$
(k = 1, \dots, K), (r = 1, \dots, r_k)
$$

$$
\sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j^{th} (z_{dj}^{t(k,h)in})_\varphi^U + \lambda_p^{th} (z_{dj}^{t(k,h)in})_\varphi^L + (s_{dp}^{t(k,h)as-input})_\varphi^L = (z_{dp}^{t(k,h)as-input})_\varphi^L
\tag{44}
$$

$$
(d \in l_{(k,h)in}), \forall (k,h)
$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{tk}(z_{dp}^{t(k,h)as-output})_{\varphi}^{L} + \lambda_p^{tk}(z_{dp}^{t(k,h)as-output})_{\varphi}^{U} - (s_{dp}^{t(k,h)as-output})_{\varphi}^{U} = (z_{dp}^{t(k,h)as-output})_{\varphi}^{U} \tag{45}$$

$$(d \in l_{(k,h)out}), \forall (k,h)$$

$$\sum_{j=1}^{n} \lambda_j^{tk}(z_{dj}^{t(k,h)fixed})_{\varphi}^{U} = (z_{dp}^{t(k,h)fixed})_{\varphi}^{U} \tag{46}$$

$$(d \in l_{(k,h)fix}), \forall (k,h)$$

$$\sum_{j=1}^{n} \lambda_j^{tk}(z_{dj}^{t(k,h)})_{\varphi}^{L} = \sum_{j=1}^{n} \lambda_j^{th}(z_{dj}^{t(k,h)})_{\varphi}^{U} \tag{47}$$

$$(d = 1, \dots, l_{(k,h)}), \forall (k,h)$$

$$\sum_{j=1}^{n} \lambda_j^{kf}\,(z_{kcj}^{tcarry\_\alpha})_{\varphi}^{L} - M(1 - t_{kc\alpha}^{(t,f)}) \leq \sum_{j=1}^{n} \lambda_j^{kt}(z_{cj}^{tcarry\_\alpha})_{\varphi}^{U} \tag{48}$$

$$\leq \sum_{j=1}^{n} (\lambda_j^{kf} z_{cj}^{tcarry\_\alpha})_{\varphi}^{L} + M(1 - t_{kc\alpha}^{(t,f)})$$

$$\forall f > t\ ,\ \alpha \in \{bad, good, free, fixed\}$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1 \tag{49}$$

$$\forall f > t,\ (k = 1, \dots, K),$$

$$\alpha \in \{bad, good, free, fixed\}$$

$$\sum_{j=1}^{n} \lambda_j^{kt}\,(z_{cj}^{tcarry\_fixed})_{\varphi}^{U} = (z_{cp}^{tfixed})_{\varphi}^{U} \tag{50}$$

$$\forall t, \forall c, \forall k$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{kt}\,(z_{cj}^{tcarry\_good})_{\varphi}^{L} + \lambda_p^{kt}(z_{pj}^{tcarry\_good})_{\varphi}^{U} - (s_{cp}^{(t,f)kgood})_{\varphi}^{U} = (z_{cp}^{tcarry\_good})_{\varphi}^{U} \tag{51}$$

$$\forall t, \forall c, \forall k, \forall f > t$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{kf}\,(z_{cj}^{tcarry\_bad})_{\varphi}^{U} + \lambda_p^{kf}(z_{cj}^{tcarry\_bad})_{\varphi}^{L} + (s_{cp}^{(t,f)kbad})_{\varphi}^{L} = zz_{cj}^{tcarry\_bad} \tag{52}$$

$$\forall t, \forall c, \forall k, \forall f > t$$

$$(z_{cj}^{tcarry\_bad})_{\varphi}^{L} - M\left(1 - t_{kcbad}^{(t,f)}\right) \leq zz_{cj}^{tcarry_{bad}} \leq (z_{cj}^{tcarry_{bad}})_{\varphi}^{L} + M\left(1 - t_{kcbad}^{(t,f)}\right) \tag{53}$$

$$\forall t, \forall c, \forall k, \forall f > t$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \geq 0, s_{rp}^{k+} \geq 0, s_{ip}^{k-} \geq 0, s_{dp}^{(k,h)-} \geq 0, s_{dp}^{(k,h)+} \geq 0 \tag{54}$$

The lower score can be derived from the model outlined in Equations (55) to (68).

$$(\psi_p^*)_{\varphi}^{L} = Min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k}\left(\sum_{i=1}^{m_k} \frac{(s_{ip}^{tk-})_{\varphi}^{U}}{(x_{ip}^{tk})_{\varphi}^{U}} + \sum_{d=1}^{l_{(f,k)}} \frac{(s_{dp}^{t(f,k)as-input})_{\varphi}^{U}}{(z_{dp}^{t(k,h)as-input})_{\varphi}^{U}} + \sum_{a=1}^{t} \sum_{c}^{h}\right.\right.}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k}\left(\sum_{i=1}^{m_k} \frac{(s_{rp}^{tk+})_{\varphi}^{L}}{(y_{rp}^{tk})_{\varphi}^{L}} + \sum_{d=1}^{l_{(f,k)}} \frac{(s_{dp}^{t(k,h)as-output})_{\varphi}^{L}}{(z_{dp}^{t(k,h)as-output})_{\varphi}^{L}} + \sum_{a=t}^{T} \sum_{}\right.\right.} \tag{55}$$

$$\text{s.t.}\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^{tk}(x_{ij}^{tk})_\varphi^L+\lambda_p^{tk}(x_{ip}^{tk})_\varphi^U+(s_{ip}^{tk-})_\varphi^U=(x_{ip}^{tk})_\varphi^U \tag{56}$$

$$(k=1,\dots,K),(i=1,\dots,m_k)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^{tk}(y_{ij}^{tk})_\varphi^U+\lambda_p^{tk}(y_{ip}^{tk})_\varphi^L+(s_{rp}^{tk+})_\varphi^L=(y_{rp}^{tk})_\varphi^L \tag{57}$$

$$(k=1,\dots,K),(r=1,\dots,r_k)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^{th}\,(z_{dj}^{t(k,h)in})_\varphi^L+\lambda_p^{th}(z_{dj}^{t(k,h)in})_\varphi^U+(s_{dp}^{t(k,h)as-input})_\varphi^U=(z_{dp}^{t(k,h)as-input})_\varphi^U \tag{58}$$

$$(d\in l_{(k,h)in}),\forall(k,h)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^{tk}(z_{dp}^{t(k,h)as-output})_\varphi^U+\lambda_p^{tk}(z_{dp}^{t(k,h)as-output})_\varphi^L-(s_{dp}^{t(k,h)as-output})_\varphi^L=(z_{dp}^{t(k,h)as-output})_\varphi^L \tag{59}$$

$$(d\in l_{(k,h)out}),\forall(k,h)$$

$$\sum_{j=1}^{n}\lambda_j^{tk}(z_{dj}^{t(k,h)fixed})_\varphi^L=(z_{dp}^{t(k,h)fixed})_\varphi^L \tag{60}$$

$$(d\in l_{(k,h)fix}),\forall(k,h)$$

$$\sum_{j=1}^{n}\lambda_j^{tk}(z_{dj}^{t(k,h)})_\varphi^U=\sum_{j=1}^{n}\lambda_j^{th}(z_{dj}^{t(k,h)})_\varphi^L \tag{61}$$

$$(d=1,\dots,l_{(k,h)}),\forall(k,h)$$

$$\begin{gathered}\sum_{j=1}^{n}\lambda_j^{kf}\,(z_{kcj}^{tcarry\_\alpha})_\varphi^U-M(1-t_{kc\alpha}^{(t,f)})\le\sum_{j=1}^{n}\lambda_j^{kt}(z_{cj}^{tcarry\_\alpha})_\varphi^L\\ \le\sum_{j=1}^{n}(\lambda_j^{kf}z_{cj}^{tcarry\_\alpha})_\varphi^U+M(1-t_{kc\alpha}^{(t,f)})\end{gathered} \tag{62}$$

$$\forall f>t\ ,\alpha\in\{bad,good,free,fixed\}$$

$$\sum_{f=t+1}^{T}t_{kc\alpha}^{(t,f)}=1 \tag{63}$$

$$\forall f>t,(k=1,\dots,K),\ \alpha\in\{bad,good,free,fixed\}$$

$$\sum_{j=1}^{n}\lambda_j^{kt}\,(z_{cj}^{tcarry\_fixed})_\varphi^L=(z_{cp}^{tfixed})_\varphi^L \tag{64}$$

$$\forall t,\forall c,\forall k$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^{kt}\,(z_{cj}^{tcarry\_good})_\varphi^U+\lambda_p^{kt}(z_{pj}^{tcarry\_good})_\varphi^L-(s_{cp}^{(t,f)kgood})_\varphi^L=(z_{cp}^{tcarry\_good})_\varphi^L \tag{65}$$

$$\forall t,\forall c,\forall k,\forall f>t$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^{kf}\,(z_{cj}^{tcarry\_bad})_\varphi^L+\lambda_p^{kf}(z_{cj}^{tcarry\_bad})_\varphi^U+(s_{cp}^{(t,f)kbad})_\varphi^U=zz_{cj}^{tcarry\_bad} \tag{66}$$

$$\forall t, \forall c, \forall k, \forall f > t$$
$$(z_{cj}^{tcarry\_bad})_{\varphi}^{U} - M\left(1 - t_{kcbad}^{(t,f)}\right) \leq zz_{cj}^{tcarry_{bad}} \leq (z_{cj}^{tcarry_{bad}})_{\varphi}^{U} + M\left(1 - t_{kcbad}^{(t,f)}\right) \quad (67)$$
$$\forall t, \forall c, \forall k, \forall f > t$$
$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \geq 0, s_{rp}^{k+} \geq 0, s_{ip}^{k-} \geq 0, s_{dp}^{(k,h)-} \geq 0, s_{dp}^{(k,h)+} \geq 0, s_{cp}^{(t,f)kgood} \geq 0, s_{cp}^{(t,f)kbad} \geq 0. \quad (68)$$

Due to the nonlinear nature of objective function presented in Equation (4) and the binary variables $t_{kc\alpha}^{(t,f)}$, the proposed model is a Mixed-integer nonlinear programming (MINLP) problems, which can be computationally complex. To address this, in the next subsection, we apply the Charnes-Cooper transformation, which linearizes the model, converting it into a linear programming problem. This transformation significantly reduces the computational burden, allowing for more efficient solving of the model while maintaining the accuracy of the results.

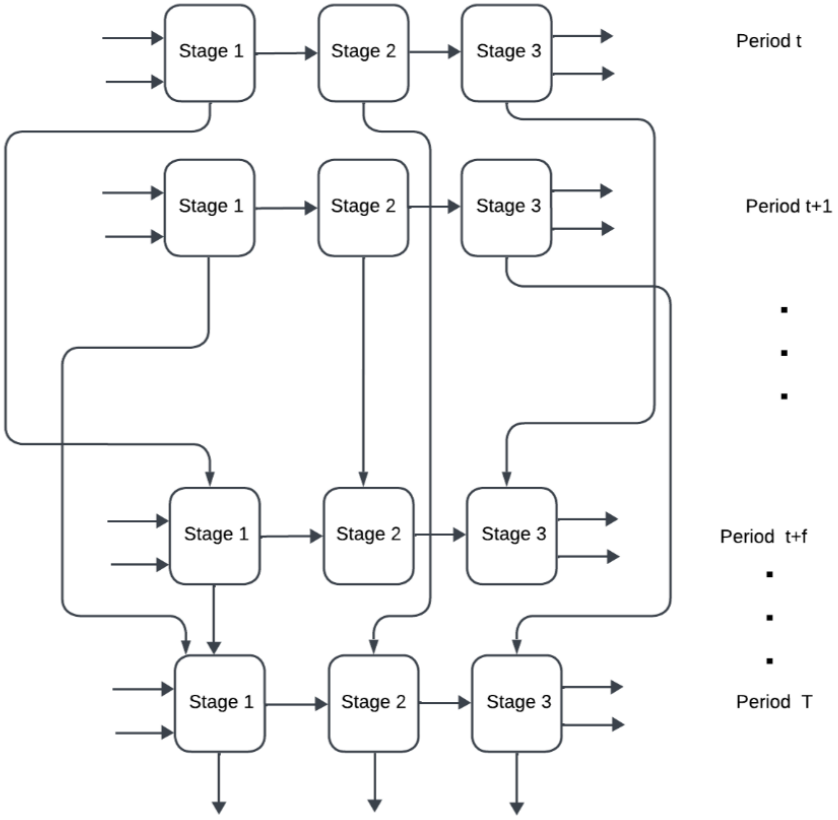


Figure 1. Allocation of carry overs.

4.2 Linearization of the proposed model

The models described by equations (41-54) and (55-68) are nonlinear programming because of its nonlinear objective function. To simplify and linearize the models, we utilize the Charnes-Cooper transformation method (Charnes et al., 1978). To enhance clarity, we'll concentrate on the linearization of the model delineated in equations (41) - (54), computing the upper level of *DMUp* efficiency at the aggregate coefficient of φ. Given its similarity, we'll omit the linearization of the lower efficiency model.

Equation (69) introduces u as a factor modifying the original expression to achieve linearity.

$$u = \frac{1}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k}\left(\sum_{i=1}^{m_k} \frac{(s_{rp}^{tk+})_{\varphi}^{U}}{(y_{rp}^{tk})_{\varphi}^{U}} + \sum_{d=1}^{l_{(f,k)}} \frac{(s_{dp}^{t(k,h)as-output})_{\varphi}^{U}}{(z_{dp}^{t(k,h)as-output})_{\varphi}^{U}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{(s_{cp}^{(t,f)kgood})_{\varphi}^{U}}{(z_{cp}^{tcarry\_good})_{\varphi}^{U}}\right)\right]}, \quad (69)$$
$$s_{cp}^{(t,f)kgood} \geq 0, s_{cp}^{(t,f)kbad} \geq 0.$$

Applying $u > 0$ to both the numerator and denominator of the objective function (38) keeps the variable $(\psi_p^*)_\varphi^U$ unchanged, while adjusting u effectively standardizes the denominator to 1. This leads to the reformulation of the model, resulting in Equations (70) to (84).

$$(\psi_p^*)_\varphi^U = Min \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[u - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{u(s_{ip}^{tk-})_\varphi^L}{(x_{ip}^{tk})_\varphi^L} + \sum_{d=1}^{l_{(f,k)}} \frac{u(s_{dp}^{t(f,k)as-input})_\varphi^L}{(z_{dp}^{t(k,h)as-input})_\varphi^L} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{(s_{cp}^{(t,f)kbad})_\varphi^L}{(z_{cp}^{tcarry_{bad}})_\varphi^L} \right) \right]. \quad (70)$$

$$\text{s.t.} \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[u + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{u(s_{rp}^{tk+})_\varphi^U}{(y_{rp}^{tk})_\varphi^U} + \sum_{d=1}^{l_{(f,k)}} \frac{u(s_{dp}^{t(k,h)as-output})_\varphi^U}{(z_{dp}^{t(k,h)as-output})_\varphi^U} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{u(s_{cp}^{(t,f)kgood})_\varphi^U}{(z_{cp}^{tcarry\_good})_\varphi^U} \right) \right] = 1 \quad (71)$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} u\lambda_j^{tk}(x_{ij}^{tk})_\varphi^U + u\lambda_p^{tk}(x_{ip}^{tk})_\varphi^L + u(s_{ip}^{tk-})_\varphi^L = u(x_{ip}^{tk})_\varphi^L \quad (72)$$

$$(k = 1, \dots, K), (i = 1, \dots, m_k)$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} u\lambda_j^{tk}(y_{ij}^{tk})_\varphi^L + u\lambda_p^{tk}(y_{ip}^{tk})_\varphi^U + u(s_{rp}^{tk+})_\varphi^U = u(y_{rp}^{tk})_\varphi^U \quad (73)$$

$$(k = 1, \dots, K), (r = 1, \dots, r_k)$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} u\lambda_j^{th}(z_{dj}^{t(k,h)in})_\varphi^U + u\lambda_p^{th}(z_{dj}^{t(k,h)in})_\varphi^L + u(s_{dp}^{t(k,h)as-input})_\varphi^L = u(z_{dp}^{t(k,h)as-input})_\varphi^L \quad (74)$$

$$(d \in l_{(k,h)in}), \forall (k,h)$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} u\lambda_j^{tk}(z_{dp}^{t(k,h)as-output})_\varphi^L + u\lambda_p^{tk}(z_{dp}^{t(k,h)as-output})_\varphi^U - u(s_{dp}^{t(k,h)as-output})_\varphi^U = u(z_{dp}^{t(k,h)as-output})_\varphi^U \quad (75)$$

$$(d \in l_{(k,h)out}), \forall (k,h)$$

$$\sum_{j=1}^{n} u\lambda_j^{tk}(z_{dj}^{t(k,h)fixed})_\varphi^U = u(z_{dp}^{t(k,h)fixed})_\varphi^U \quad (76)$$

$$(d \in l_{(k,h)fix}), \forall (k,h)$$

$$\sum_{j=1}^{n} u\lambda_j^{tk}(z_{dj}^{t(k,h)})_\varphi^L = \sum_{j=1}^{n} u\lambda_j^{th}(z_{dj}^{t(k,h)})_\varphi^U \quad (77)$$

$$(d = 1, \dots, l_{(k,h)}), \forall (k,h)$$

$$\sum_{j=1}^{n} u\lambda_j^{kf} (z_{kcj}^{tcarry\_\alpha})_{\varphi}^{L} - M(u - ut_{kc\alpha}^{(t,f)}) \le \sum_{j=1}^{n} \lambda_j^{kt} u(z_{cj}^{tcarry\_\alpha})_{\varphi}^{U} \le \sum_{j=1}^{n} u\lambda_j^{kf} (z_{cj}^{tcarry\_\alpha})_{\varphi}^{L} + M(u - ut_{kc\alpha}^{(t,f)}) \quad (78)$$

$$\forall f > t\ ,\ \alpha \in \{bad, good, free, fixed\}$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1 \quad (79)$$

$$\forall f > t, (k = 1, \dots, K),\ \alpha \in \{bad, good, free, fixed\}$$

$$\sum_{j=1}^{n} u\lambda_j^{kt} (z_{cj}^{tcarry\_fixed})_{\varphi}^{U} = u(z_{cp}^{tfixed})_{\varphi}^{U} \quad (80)$$

$$\forall t, \forall c, \forall k$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} u\lambda_j^{kt} (z_{cj}^{tcarry\_good})_{\varphi}^{L} + u\lambda_p^{kt} (z_{pj}^{tcarry\_good})_{\varphi}^{U} - u(s_{cp}^{(t,f)kgood})_{\varphi}^{U} = u(z_{cp}^{tcarry\_good})_{\varphi}^{U} \quad (81)$$

$$\forall t, \forall c, \forall k, \forall f > t$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} u\lambda_j^{kf} (z_{cj}^{tcarry\_bad})_{\varphi}^{U} + u\lambda_p^{kf} (z_{cj}^{tcarry\_bad})_{\varphi}^{L} + u(s_{cp}^{(t,f)kbad})_{\varphi}^{L} = u.zz_{cj}^{tcarry\_bad} \quad (82)$$

$$\forall t, \forall c, \forall k, \forall f > t$$

$$u(z_{cj}^{tcarry\_bad})_{\varphi}^{L} - M\left(u - ut_{kcbad}^{(t,f)}\right) \le uzz_{cj}^{tcarry_{bad}} \le u(z_{cj}^{tcarry_{bad}})_{\varphi}^{L} + M\left(u - ut_{kcbad}^{(t,f)}\right) \quad (83)$$

$$\forall t, \forall c, \forall k, \forall f > t$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \ge 0, s_{rp}^{k+} \ge 0, s_{ip}^{k-} \ge 0, s_{dp}^{(k,h)-} \ge 0, s_{dp}^{(k,h)+} \ge 0, s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0. \quad (84)$$

Given that $zz_{cj}^{tcarry\_bad}$ is an unrestricted variable and u is positive ($u > 0$), the variable $ZZ_{cj}^{tcarry\_bad}$, defined as $ZZ_{cj}^{tcarry\_bad} = u(zz_{cj}^{tcarry\_bad})$, maintains the same sign as $zz_{cj}^{tcarry\_bad}$. To transform Equation (83) into a linear form, we utilize the relationships outlined in Equations (85) to (88).

$$u.t_{kc\alpha}^{(t,f)} = h_{kc\alpha}^{(t,f)} \quad (85)$$

$$h_{kc\alpha}^{(t,f)} \le u \quad (86)$$

$$h_{kc\alpha}^{(t,f)} \le M.t_{kc\alpha}^{(t,f)} \quad (87)$$

$$h_{kc\alpha}^{(t,f)} \ge u - M(1 - t_{kC\alpha}^{(t,f)}) \quad (88)$$

Now, in the last stage of linearization of the model, we define $u(s_{ip}^{tk-})_{\varphi}^{L} = (S_{ip}^{tk-})_{\varphi}^{L}$, $u(s_{dp}^{t(f,k)as-input})_{\varphi}^{L} = (S_{dp}^{t(f,k)as-input})_{\varphi}^{L}, u(s_{rp}^{tk+})_{\varphi}^{U} = (S_{rp}^{tk+})_{\varphi}^{U}, u(s_{dp}^{t(k,h)as-output})_{\varphi}^{U} = (S_{dp}^{t(k,h)as-output})_{\varphi}^{U}$ ،$u(s_{cp}^{(t,f)kgood})_{\varphi}^{U} = (S_{cp}^{(t,f)kgood})_{\varphi}^{U}, u(s_{cp}^{(t,f)kbad})_{\varphi}^{L} = (S_{cp}^{(t,f)kbad})_{\varphi}^{L}$, $u\lambda_j^{kf} = \Lambda_j^{kf}$.

As a result, we obtain the equivalent linear representation of the model, as described in Equation (89).

$(\psi_n^*)_{\omega}^{U}$

$$
\begin{aligned}
&= Min \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \Bigg[u \\
&\qquad - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \Big(\sum_{i=1}^{m_k} \frac{(S_{ip}^{tk-})_{\varphi}^{L}}{(x_{ip}^{tk})_{\varphi}^{L}} \\
&\qquad + \sum_{d=1}^{l_{(f,k)}} \frac{(S_{dp}^{t(f,k)as-input})_{\varphi}^{L}}{(z_{dp}^{t(k,h)as-input})_{\varphi}^{L}} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{(S_{cp}^{(t,f)kbad})_{\varphi}^{L}}{(z_{cp}^{tcarry_{bad}})_{\varphi}^{L}} \Big) \Bigg], \\
\text{s.t.} \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k &\Bigg[u \\
&\qquad + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \Big(\sum_{i=1}^{m_k} \frac{(S_{rp}^{tk+})_{\varphi}^{U}}{(y_{rp}^{tk})_{\varphi}^{U}} \\
&\qquad + \sum_{d=1}^{l_{(f,k)}} \frac{(S_{dp}^{t(k,h)as-output})_{\varphi}^{U}}{(z_{dp}^{t(k,h)as-output})_{\varphi}^{U}} + \sum_{a=t}^{T} \sum_{\mathrm{c}=1}^{h_k^{(\mathrm{t,a})}} \frac{(S_{cp}^{(t,f)kgood})_{\varphi}^{U}}{(z_{cp}^{tcarry\_good})_{\varphi}^{U}} \Big) \Bigg] = 1, \\
&\sum_{\substack{j=1 \\ j \neq p}}^{n} \Lambda_j^{kt} (x_{ij}^{tk})_{\varphi}^{U} + \Lambda_p^{kt} (x_{ip}^{tk})_{\varphi}^{L} + (S_{ip}^{tk-})_{\varphi}^{L} = u(x_{ip}^{tk})_{\varphi}^{L}, (k = 1, \dots, K), (i = 1, \dots, m_k), \\
&\sum_{\substack{j=1 \\ j \neq p}}^{n} \Lambda_j^{th} (y_{ij}^{tk})_{\varphi}^{L} + \Lambda_p^{kt} (y_{ip}^{tk})_{\varphi}^{U} + (S_{rp}^{tk+})_{\varphi}^{U} = u(y_{rp}^{tk})_{\varphi}^{U}, (k = 1, \dots, K), (r = 1, \dots, r_k), \\
&\sum_{\substack{j=1 \\ j \neq p}}^{n} \Lambda_j^{th} (z_{dj}^{t(k,h)in})_{\varphi}^{U} + \Lambda_p^{th} (z_{dj}^{t(k,h)in})_{\varphi}^{L} + (S_{dp}^{t(k,h)as-input})_{\varphi}^{L} \\
&\qquad = u(z_{dp}^{t(k,h)as-input})_{\varphi}^{L}, \left(d \in l_{(k,h)in}\right), \forall (k,h), \\
&\sum_{\substack{j=1 \\ j \neq p}}^{n} \Lambda_j^{tk} (z_{dp}^{t(k,h)as-output})_{\varphi}^{L} + \Lambda_p^{tk} (z_{dp}^{t(k,h)as-output})_{\varphi}^{U} - \left(S_{dp}^{t(k,h)as-output}\right)_{\varphi}^{U} \\
&\qquad = u \left(z_{dp}^{t(k,h)as-output}\right)_{\varphi}^{U}, \left(d \in l_{(k,h)out}\right), \forall (k,h), \\
&\sum_{j=1}^{n} \Lambda_j^{tk} (z_{dj}^{t(k,h)fixed})_{\varphi}^{U} = u(z_{dp}^{t(k,h)fixed})_{\varphi}^{U}, \left(d \in l_{(k,h)fix}\right), \forall (k,h), \\
&\sum_{j=1}^{n} \Lambda_j^{tk} (z_{dj}^{t(k,h)})_{\varphi}^{L} = \sum_{j=1}^{n} \Lambda_j^{tk} \left(z_{dj}^{t(k,h)}\right)_{\varphi}^{U}, \left(d = 1, \dots, l_{(k,h)}\right), \forall (k,h), \\
&\textstyle\sum_{j=1}^{n} \Lambda_j^{fk} (z_{kcj}^{tcarry\_\alpha})_{\varphi}^{L} - M \left(u - ut_{kc\alpha}^{(t,f)}\right) \leq \sum_{j=1}^{n} \Lambda_j^{tk} \left(z_{cj}^{tcarry_{\alpha}}\right)_{\varphi}^{U} \leq \\
&\textstyle\sum_{j=1}^{n} \Lambda_j^{fk} \left(z_{cj}^{tcarry_{\alpha}}\right)_{\varphi}^{L} + M \left(u - ut_{kc\alpha}^{(t,f)}\right), \forall f > t \ , \alpha \in \{bad, good, free, fixed\}, \\
&\textstyle\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1 \ , \forall f > t, (k = 1, \dots, K), \\
&\qquad \alpha \in \{bad, good, free, fixed\}
\end{aligned}
\tag{89}
$$

$$\sum_{j=1}^{n} \Lambda_j^{tk} \left(z_{cj}^{tcarry\_fixed}\right)_{\varphi}^{U} = u\left(z_{cp}^{tfixed}\right)_{\varphi}^{U}, \forall t, \forall c, \forall k,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^{tk} \left(z_{cj}^{tcarry\_good}\right)_{\varphi}^{L} + \Lambda_p^{tk} \left(z_{pj}^{tcarry\_good}\right)_{\varphi}^{U} - \left(S_{cp}^{(t,f)kgood}\right)_{\varphi}^{U} = u\left(z_{cp}^{tcarry\_good}\right)_{\varphi}^{U},$$

$$\forall t, \forall c, \forall k, \forall f > t,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^{fk} \left(z_{cj}^{tcarry\_bad}\right)_{\varphi}^{U} + \Lambda_p^{fk} \left(z_{cj}^{tcarry\_bad}\right)_{\varphi}^{L} + \left(S_{cp}^{(t,f)kbad}\right)_{\varphi}^{L} = u.zz_{cj}^{tcarry\_bad},$$

$$\forall t, \forall c, \forall k, \forall f > t,$$

$$u\left(z_{cj}^{tcarry\_bad}\right)_{\varphi}^{L} - M\left(u - h_{kCbad}^{(t,f)}\right) \leq ZZ_{cj}^{tcarry_{bad}} \leq u\left(z_{cj}^{tcarry_{bad}}\right)_{\varphi}^{L} + M\left(u - h_{kCbad}^{(t,f)}\right),$$

$$\forall t, \forall c, \forall k, \forall f > t,$$

$$h_{kc\alpha}^{(t,f)} \leq u, \forall t, \forall c, \forall k, \forall f > t,$$

$$h_{kc\alpha}^{(t,f)} \leq M.t_{kc\alpha}^{(t,f)}, \forall t, \forall c, \forall k, \forall f > t,$$

$$h_{kc\alpha}^{(t,f)} \geq \mathrm{u} - M(1 - t_{kC\alpha}^{(t,f)}), \forall t, \forall c, \forall k, \forall f > t,$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \geq 0, s_{rp}^{k+} \geq 0, s_{ip}^{k-} \geq 0, s_{dp}^{(k,h)-} \geq 0, s_{dp}^{(k,h)+} \geq 0, s_{cp}^{(t,f)kgood} \geq 0, s_{cp}^{(t,f)kbad} \geq 0.$$

In this section, we derived linear models to measure the upper and lower efficiency scores at specific levels of truth, indeterminacy, and falsity. The mathematical contribution of the presented model lies in proposing a novel DNDEA model that incorporates discretionary carryover variables extending beyond the immediate next period, with binary variables determining their allocation. The model utilizes neutrosophic numbers to handle uncertainties in efficiency scores, transforming these into deterministic forms through a two-stage mathematical programming approach that captures both upper and lower efficiency bounds for decision-making units. The use of Pareto efficiency principles simplifies the two-stage model into a single-stage framework for performance evaluation. Additionally, the model addresses the computational complexity of Mixed-Integer Nonlinear Programming (MINLP) problems by applying the Charnes-Cooper transformation, converting the problem into a linear programming model, thus significantly improving computational efficiency while maintaining accuracy. These contributions advance resource allocation and efficiency analysis in dynamic environments.

In the next section, we will present an empirical study to validate the proposed model.

5 Empirical Study

The proposed DNDEA model has broad applications across various sectors where carryover activities extend beyond immediate periods. In the banking sector, the model can optimize loan portfolios, risk management, and capital allocation by considering how loans or financial risks carry over from one fiscal period to the next. Similarly, in supply chain management, it can address inventory levels and production capacity, ensuring efficient resource allocation over time to minimize costs. In healthcare, the model can help optimize resource distribution (staff, equipment, and patient care) by factoring in the carryover effects of ongoing treatments and future healthcare demands. The energy sector can benefit from the model in optimizing the distribution and storage of energy resources across periods, taking into account fluctuations in supply and demand. Additionally, manufacturing industries can use the model to manage work-in-progress inventories and production schedules, ensuring better efficiency and reduced waste. Finally, in public infrastructure planning, the model can strategically allocate resources across multi-year construction projects, accounting for delays, budget fluctuations, and other uncertainties. These applications highlight the versatility of the DNDEA model in optimizing resource allocation across various dynamic and uncertain environments.

In this section, we apply the proposed DNDEA model to compute the overall, divisional, and period performance of ten major Iranian private banks. The DMUs are represented by the following banks: Bank Pasargad (DMU1), Parsian Bank (DMU2), Saman Bank (DMU3), Karafarin Bank (DMU4), Tejarat Bank (DMU5), Sina Bank (DMU6), Bank Mellat (DMU7), Eghtesad Novin Bank (DMU8), Ansar Bank (DMU9),

and Shahr Bank (DMU10), over the years 1396, 1397, 1398, 1399, 1400, 1401, and 1402. Firstly, based on what recorded in the literature and consulting experts, we consider the banks as a network structure with three divisions. Each division has inputs, outputs, links and carry-overs presented in Table 4. Loan loss reserves and non-performing loans are considered as carry over variables. As we discussed earlier, this carry-over can be allocated to future periods to improve the bank's overall efficiency. This carry-over is considered as a good carry over and its value is under control of the current period. The model allocates this carry-over to one of the future periods in a way that enhances the objective function. In contrast, the carry-over Non-Performing Loans (NPL) is considered as a bad link carried to the next period and its allocation is not discretionary.

The related data are collected from various sources including the bank's records unit, the statistical center of Iran[1], online resources, insights from experts, and the Codel website[2]. During data collection, inconsistencies, indeterminacy, and incompleteness were observed. Our investigation revealed that several banking reforms and other factors have led to considerable uncertainty and indeterminacy in the data. As a result, these uncertainties were identified and modeled using Neutrosophic Numbers.

Table 4. Factors of production utilized in this study.

Input	Input 1 consumed at the first stage in period t	Personnel expenses
Input	Input 2 consumed at the first stage in period t	Fixed assets
Intermediate variable	Intermediate variable which is output from the first stage in period t and input to the second stage in the same period	Total deposits
Intermediate variable	Intermediate variable which is output from the second stage in period t and input to the third stage in the same period	Gross loans
Intermediate variable	Intermediate variable which is output from the second stage in period t and input to the third stage in the same period	Total securities investment
Output	Output from the third stage in period t	Income
carry over	The generated carry over form the 3rd stage in period t and is considered as good carry over to one of the subsequent periods.	Loan loss reserves
carry over	The generated carry over form the 3rd stage in period t and is considered as bad carry over to one of the subsequent periods.	Non-Performing Loans (*NPLs*)-bad loans

5.1 Performance evaluation of bank branches

In this section, we utilize our proposed model to assess the relative efficiency of bank branches. Employing the Generalized Algebraic Modeling System (GAMS), we solve the proposed model to measure efficiency bounds for branches and allocate carryovers. The subsequent results obtained are presented in Tables 5 and 6, illustrating the lower and upper performance limits of 10 branches at φ (0,1.0,1.0). Table 5 also represents the optimum allocation of carryovers during 1396-1402.

Table 5. Lower bound of efficiency at $\varphi_{(0,1.0,1.0)}$.

DMU	Overall score	Rank	Overall Efficiency in 1396	Overall Efficiency in 1397	Overall Efficiency in 1398	Overall Efficiency in 1399	Overall Efficiency in 1400	Overall Efficiency in 1401	Overall Efficiency in 1402	$\mathbf{h}_{kc\alpha}^{*(96,f)}=\mathbf{1}$	$\mathbf{h}_{kc\alpha}^{*(97,f)}=\mathbf{1}$	$\mathbf{h}_{kc\alpha}^{*(98,f)}=\mathbf{1}$	$\mathbf{h}_{kc\alpha}^{*(99,f)}=\mathbf{1}$	$\mathbf{h}_{kc\alpha}^{*(1400,f)}=\mathbf{1}$	$\mathbf{h}_{kc\alpha}^{*(1401,f)}=\mathbf{1}$
DMU1	0.868	5	0.88	1	0.96	0.78	0.75	0.69	0.82	$h_{kc}^{*(96,99)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU2	0.874	4	0.85	0.89	0.88	0.92	0.77	1	0.81	$h_{kc}^{*(96,97)}$	$h_{kc}^{*(97,1401)}$	$h_{kc}^{*(98,99)}$	$h_{kc}^{*(99,1402)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU3	0.738	7	0.77	0.87	0.74	0.61	0.92	0.57	0.45	$h_{kc}^{*(96,99)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}$
DMU4	0.960	1	0.985	1	0.95	0.86	0.95	1	0.95	$h_{kc}^{*(96,1401)}$	$h_{kc}^{*(97,99)}$	$h_{kc}^{*(98,1401)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU5	0.954	2	0.85	0.87	0.95	1	1	1	1	$h_{kc}^{*(96,1400)}$	$h_{kc}^{*(97,98)}$	$h_{kc}^{*(98,1402)}$	$h_{kc}^{*(99,1400)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}=1$

[1] https://amar.org.ir
[2] www.codal.ir

DMU6	0.578	9	0.66	0.695	0.78	0.45	0.58	0.36	0.58	$h_{kc}^{*(96,1402)}$	$h_{kc}^{*(97,98)}$	$h_{kc}^{*(98,1402)}$	$h_{kc}^{*(99,1400)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}$
DMU7	0.611	8	0.81	0.635	0.61	0.56	0.61	0.5	0.61	$h_{kc}^{*(96,1400)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,1401)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}$
DMU8	0.900	3	0.96	0.925	0.95	0.89	0.85	0.97	0.85	$h_{kc}^{*(96,1400)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,99)}$	$h_{kc}^{*(99,1402)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU9	0.771	6	0.96	0.829	0.79	0.77	0.74	0.62	0.74	$h_{kc}^{*(96,99)}$	$h_{kc}^{*(97.98)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU10	0.568	10	0.69	0.49	0.68	0.64	0.45	0.64	0.44	$h_{kc}^{*(96,97)}$	$h_{kc}^{*(97,99)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1402)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
Average	0.785	_	0.84	0.82	0.82	0.77	0.76	0.75	0.72	_	_	_	_	_	_

As seen in Table 5, DMU10 has the lowest overall score, whereas DMU4 obtained the highest overall score at φ (0,1.0,1.0). Specifically, for DMU4, the allocated carryover loan loss reserve is assigned from the year 1396 to 1401, from 1397 to 1399, from 1398 to 1401, from 1399 to 1401, from 1400 to 1402, and from 1401 to 1402. Based on the results derived from the proposed model, the allocation of carryover activities, particularly evident in the case of DMU4, sheds light on significant insights. These findings indicate not only the temporal distribution of the carryover loan loss reserve but also highlight potential patterns and trends within the financial operations of the organization. Understanding these patterns can empower managers to make informed decisions regarding resource allocation, enabling them to optimize efficiency and effectiveness in managing financial risks and obligations over time. Figure 2 compares the lower and upper bounds of efficiencies at φ (0,1.0,1.0). The results show significant variations between lower and upper efficiency bounds for certain DMUs, such as DMU6 and DMU10, indicating a higher level of uncertainty or inefficiency in these units. Conversely, DMUs like DMU4 and DMU5 demonstrate minimal difference between lower and upper efficiencies, reflecting more stable and optimal performance.

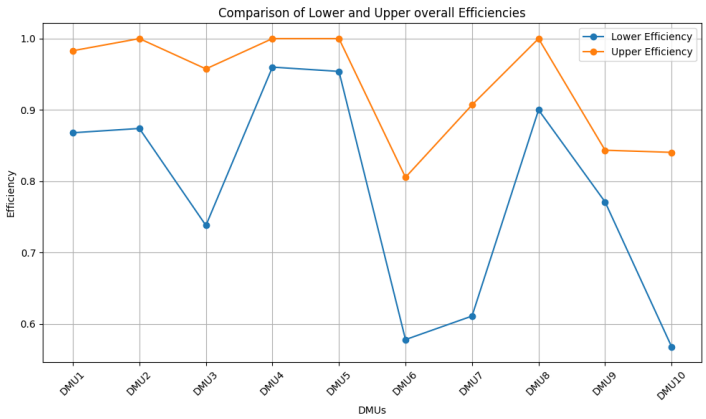


Figure 2. Comparison of the lower and upper bounds of scores at φ (0,1.0,1.0).

Table 6. Upper bound of efficiency at $\varphi_{(0,1.0,1.0)}$.

DMU	Overall Efficiency	Overall Efficiency in 1396	Overall Efficiency in 1397	Overall Efficiency in 1398	Overall Efficiency in 1399	Overall Efficiency in 1400	Overall Efficiency in 1401	Overall Efficiency in 1402
DMU1	1	1.00	1.00	1.00	0.94	0.96	1.00	1.00
DMU2	0.97	1.00	1.00	1.00	1.00	0.89	1.00	1.00
DMU3	0.89	0.96	1.00	0.95	0.85	1.00	0.76	0.65
DMU4	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU5	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU6	0.86	0.89	0.91	0.90	0.69	0.87	0.52	0.87
DMU7	0.81	1.00	0.79	0.89	0.77	0.81	0.65	0.85
DMU8	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU9	0.95	1.00	1.00	0.96	0.95	0.94	0.82	0.94
DMU10	0.84	0.94	0.68	0.68	0.81	0.65	0.84	0.65
average	0.932	0.979	0.938	0.938	0.901	0.913	0.859	0.896

Table 7 represents the overall scores of DMUs at different levels of truth, indeterminacy and falsity. According to the results DMUs with higher overall scores tend to have a narrower range between their lower and upper scores, indicating higher levels of efficiency and consistency in performance across different levels of uncertainty. For instance, DMU5 maintains an almost consistent perfect score across all levels, indicating exceptionally high efficiency.

Table 7. Scores of the DMUs at different levels of truth, indeterminacy and falsity.

DMU	Overall lower Score at $\boldsymbol{\varphi_{(0.2,0.7,0.9)}}$	Overall upper Score at $\boldsymbol{\varphi_{(0.2,0.7,0.9)}}$	Overall lower Score at $\boldsymbol{\varphi_{(0.5,0.8,0.9)}}$	Overall upper Score at $\boldsymbol{\varphi_{(0.5,0.8,0.9)}}$
DMU1	0.88	0.97	0.93	1.00
DMU2	0.89	0.97	0.93	1.00
DMU3	0.75	0.89	0.78	1.00
DMU4	1.00	1.00	1.00	1.00
DMU5	1.00	1.00	1.00	1.00
DMU6	0.68	0.82	0.78	0.8
DMU7	0.64	0.78	0.71	0.78
DMU8	0.94	1.00	0.97	1.00
DMU9	0.85	0.95	0.87	0.95
DMU10	0.64	0.79	0.64	0.74
Average	0.827	0.917	0.861	0.927

5.2 Comparison of the Results between Proposed Model and DNSBM

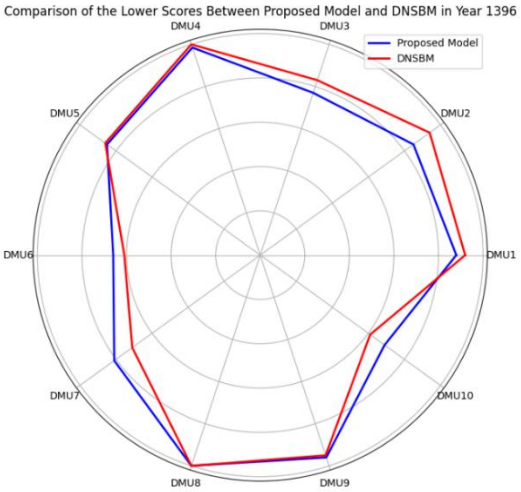
In this subsection, to validate our proposed model, we conduct a comparative analysis between the outcomes generated by our proposed model and those derived from the DNSBM model. Table 8 represents the overall and period efficiencies for the DMUs under consideration with the data used for determination of the lower score at $\varphi_{(0,1.0,1.0)}$ for the proposed model.

Table 8. The period and overall lower scores by DNSBM at $\varphi_{(0,1.0,1.0)}$.

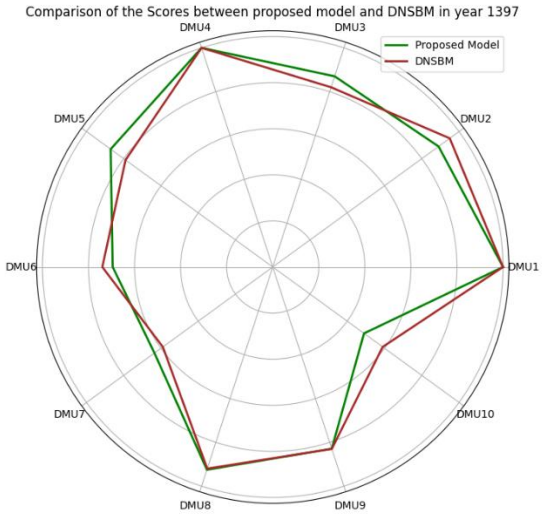
DMU	Overall score	Efficiency in 1396	Efficiency in 1397	Efficiency in 1398	Efficiency in 1399	Efficiency in 1400	Efficiency in 1401	Efficiency in 1402
DMU1	0.89	0.92	1.00	0.95	0.81	0.79	0.9	0.88
DMU2	0.94	0.94	0.95	0.92	0.95	0.88	1.00	0.91
DMU3	0.76	0.83	0.82	0.86	0.78	0.95	0.5	0.49
DMU4	0.98	1.00	1.00	1.00	0.95	0.97	1.00	0.98
DMU5	0.94	0.86	0.79	1.00	1.00	1.00	1.00	1.00
DMU6	0.59	0.61	0.74	0.69	0.56	0.46	0.38	0.59
DMU7	0.63	0.71	0.56	0.56	0.68	0.72	0.51	0.62
DMU8	0.92	1.00	0.92	0.87	1.00	0.89	0.89	0.86
DMU9	0.78	0.95	0.83	0.86	0.8	0.78	0.54	0.75
DMU10	0.57	0.61	0.59	0.51	0.72	0.46	0.65	0.44
average	0.801	0.843	0.823	0.822	0.825	0.804	0.737	0.752

According to the results, the overall scores obtained by the proposed model are lower than those of the DNSBM. The findings suggest that the proposed model exhibits greater discrimination than DNSBM. Across the years 1396-1402, DNSBM achieved 13 perfect scores, while the proposed model attained 8 perfect scores. DMU5 achieves a perfect score in both models from 1398 to 1402. In the year 1397, DMU4, and in 1399, DMU5 obtained perfect scores in both models. In the year 1396, both models recorded the lowest score for DMU6 and the highest score for DMU8. However, in the year 1397, the lowest score was achieved by DMU10 in the proposed model, while DMU7 obtained the lowest score in DNSBM. Notably, there is a sharp difference between the scores obtained by the proposed model and DNSBM during 1398 and 1399, contributing significantly to the difference in overall scores. This difference must be caused by the different assumption on allocation of carry over variables. Figure 3 compares the and period scores obtained by the proposed model and DNSBM at $\varphi_{(0,1.0,1.0)}$. The overall score of the proposed model is lower than that of the DNSBM (see

Figure 3(h)). However, for specific periods and DMUs, such as DMU6, DMU8, and DMU10 in Figure 3(c), and DMU10, DMU7, and DMU6 in Figure 3(a), the period scores of the DNSBM are lower than those of the proposed model. This difference likely arises from the distinct assumptions regarding the allocation of carryovers. It should be noted that the discretionary assumption for carryover variables in the proposed model provides flexibility, allowing the model to minimize overall efficiency rather than period efficiency. Therefore, these results are consistent with the objectives of the proposed model.



(a)



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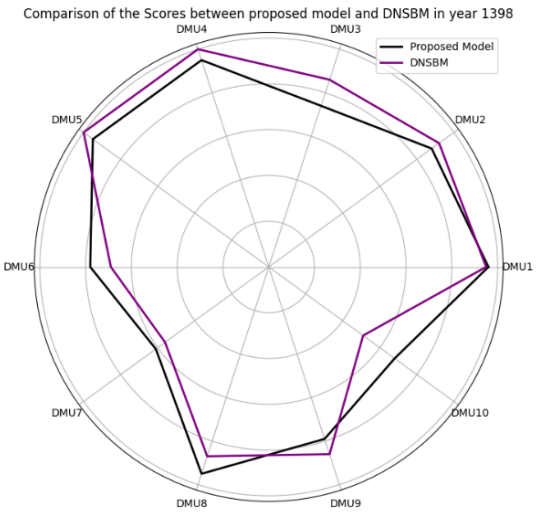
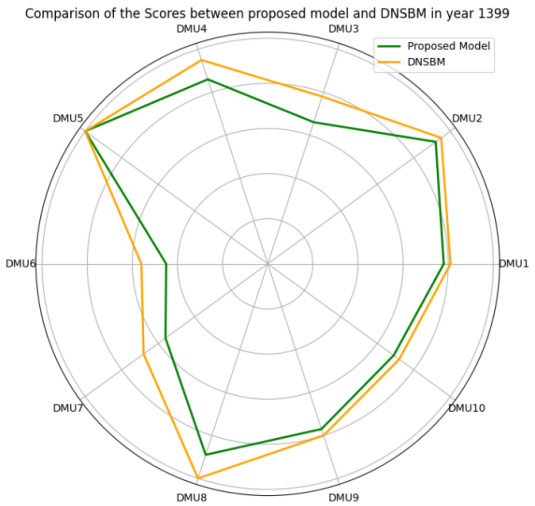


(c)



(d)

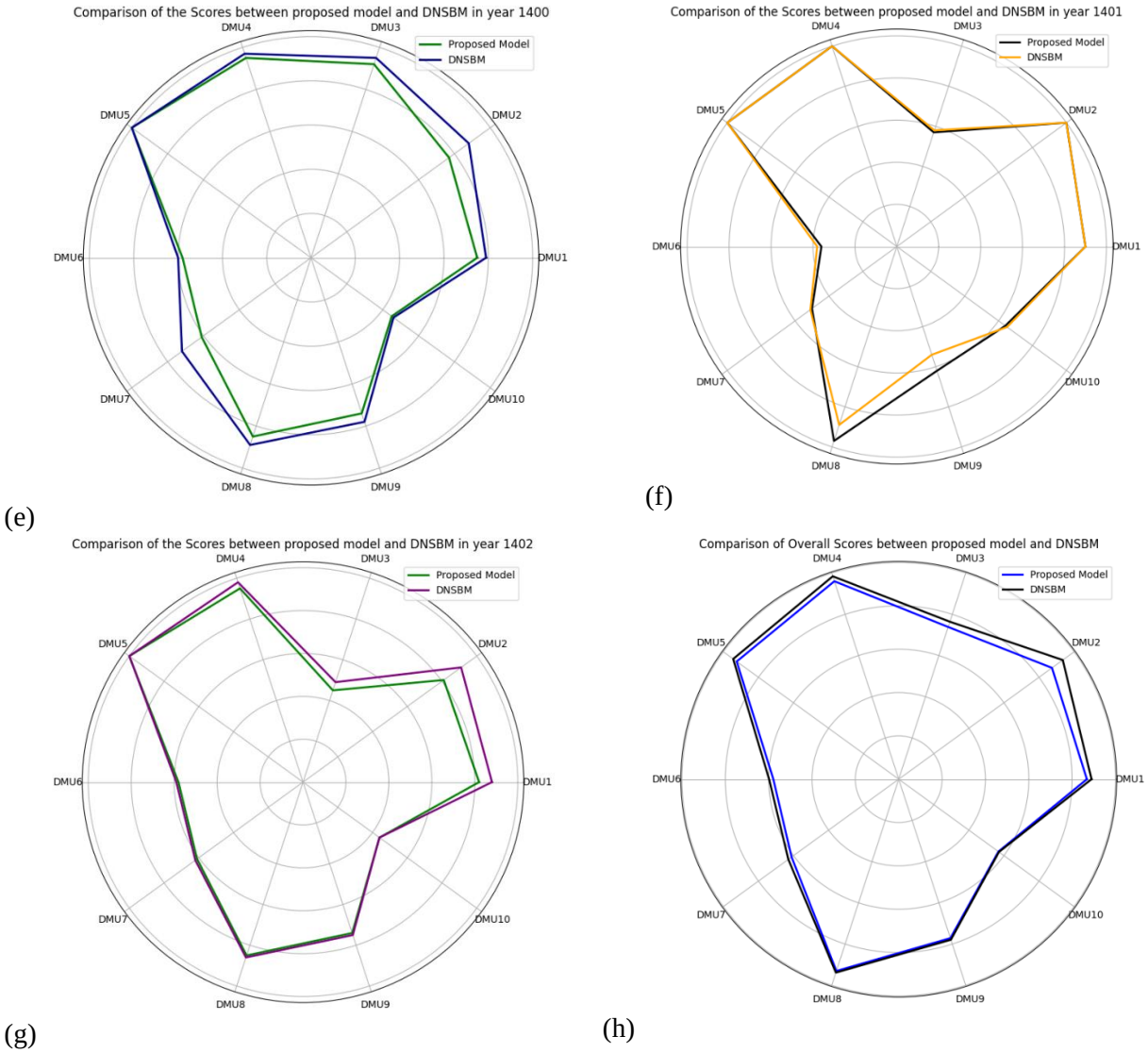


Figure 3. Comparison of the overall and period scores obtained by the proposed model and DNSBM.

Figure 4 demonstrates the trend of average period and overall scores obtained by the proposed model and DNSBM across the years 1396 to 1402 at $\varphi_{(0,1.0,1.0)}$. The trends are roughly similar, but there exist some gaps between the scores, likely caused by different assumptions on allocating carry-over variables. However, as discussed earlier, the overall scores obtained from the proposed model are lower than those of DNSBM. Notably, the lowest average score in both models is observed in 1402, while the highest average score is recorded in 1396.

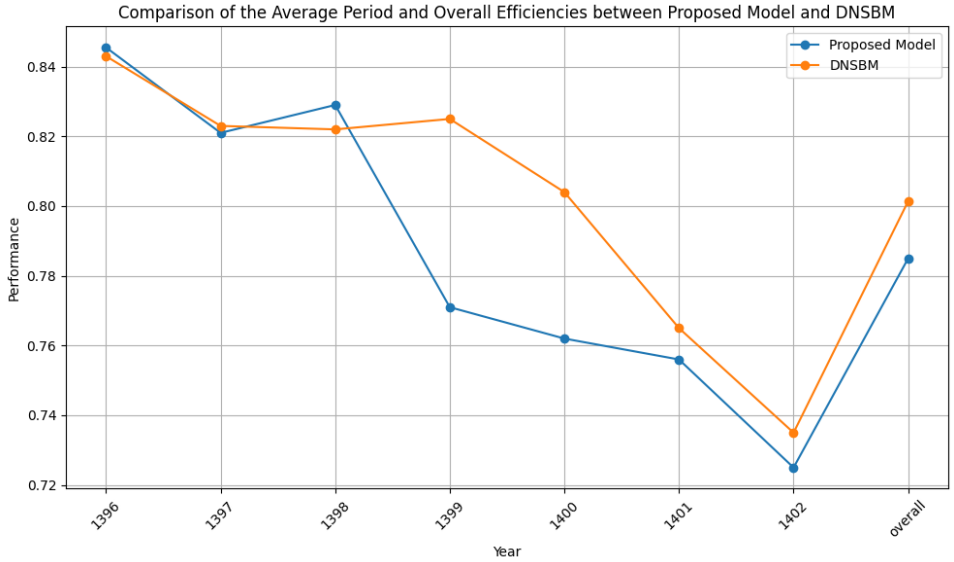


Figure 4. Comparisons of the average scores between the proposed and DNSBM models.

Table 9 provides insights into the overall lower and upper scores of the DMUs under evaluation at different levels of truth, indeterminacy, and falsity, as calculated by the DNSBM model. Similar to the results of the proposed model, DMU4 and DMU5 consistently achieve perfect scores, indicating optimal performance across all scenarios. On the other hand, DMUs like DMU6 and DMU10 exhibit significant inefficiencies, with lower scores dropping as low as 0.68 and 0.64, respectively, under specific conditions. These variations suggest that while some DMUs perform at a near-optimal level, others face challenges in maintaining efficiency, particularly when subject to uncertainty. The gap between the lower and upper scores in several DMUs, such as DMU6 and DMU10, highlights the potential for improvement in resource allocation and decision-making processes.

Table 9. Scores of the DMUs by the DNSBM model at different levels of truth, indeterminacy and falsity.

DMU	Overall lower Score at $\boldsymbol{\varphi 1} = \boldsymbol{\varphi}_{(0.2,0.7,0.9)}$	Overall upper Score at $\boldsymbol{\varphi 1} = \boldsymbol{\varphi}_{(0.2,0.7,0.9)}$	Overall lower Score at $\boldsymbol{\varphi 2} = \boldsymbol{\varphi}_{(0.5,0.8,0.9)}$	Overall upper Score at $\boldsymbol{\varphi 2} = \boldsymbol{\varphi}_{(0.5,0.8,0.9)}$
DMU1	0.92	1.00	1.00	1.00
DMU2	0.91	1.00	1.00	1.00
DMU3	0.84	0.94	0.84	1.00
DMU4	1.00	1.00	1.00	1.00
DMU5	1.00	1.00	1.00	1.00
DMU6	0.68	0.92	0.91	1.00
DMU7	0.81	1.00	0.89	1.00
DMU8	1.00	1.00	1.00	1.00
DMU9	1.00	1.00	0.87	1.00
DMU10	0.76	0.91	0.64	0.84
average	0.892	0.977	0.915	0.984

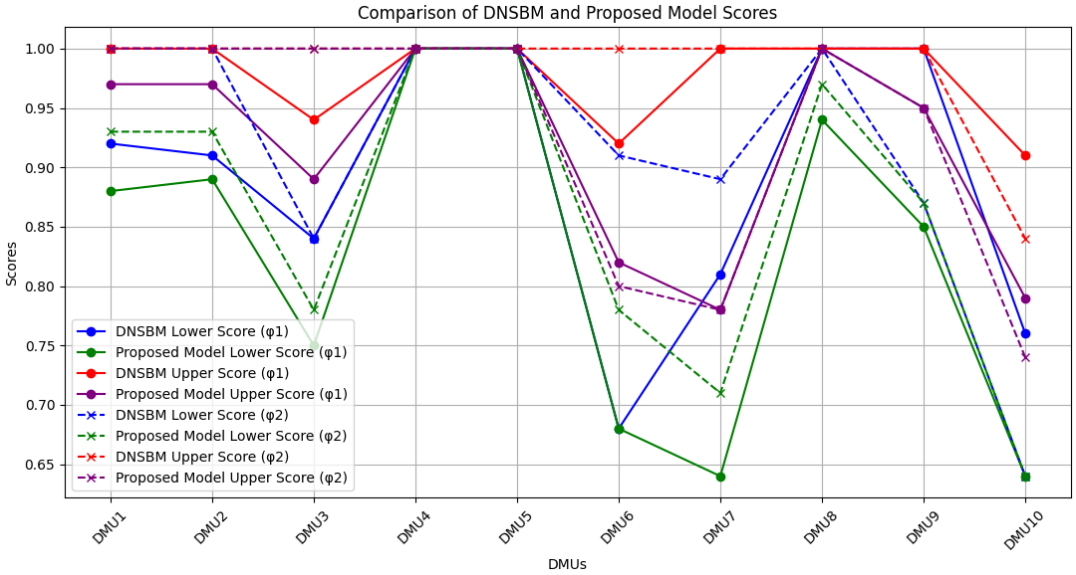


Figure 5. No caption.

Figure 5 compares the results obtained by the proposed model and DNSBM model at different variation of truth, indeterminacy and falsity. As can be seen from the figure the scores of the proposed model are not greater than those of DNSBM at the corresponding level of φ. When comparing the lower scores between the DNSBM and the proposed model across all DMUs, several differences are evident. For DMUs such as DMU1, DMU2, and DMU8, the proposed model slightly underperforms compared to DNSBM in both levels of $\boldsymbol{\varphi}$. For instance, DMU1's lower score in the proposed model is 0.88, while in DNSBM it is 0.92. This trend is consistent across DMUs where the proposed model generally yields lower scores, indicating more inefficiencies at the lower bound. Particularly, DMU3, DMU6, DMU7, and DMU10 exhibit significant differences. For example, DMU7 has a lower score of 0.64 in the proposed model compared to 0.81 in DNSBM, highlighting a pronounced inefficiency in the proposed model's performance under these conditions. Overall, lower scores tend to show that the proposed model introduces greater inefficiencies compared to DNSBM. In contrast, the comparison of upper scores shows more alignment between the two models. For many DMUs, such as DMU4, DMU5, and DMU8, both DNSBM and the proposed model achieve the highest possible score of 1.00 across both conditions, indicating optimal performance. However, in a few cases, such as DMU6 and DMU7, there are notable gaps in the upper scores. For example, DMU6's upper score in the proposed model is 0.82, compared to 0.92 in DNSBM under the first condition, and 0.80 compared to 1.00 under the second condition. While these discrepancies are not as large as those observed in the lower scores, they still suggest that the proposed model can provide lower scores in certain scenarios, particularly in terms of upper-bound efficiency under uncertainty. In summary, the results show that the scores of the proposed model exhibit more pronounced inefficiencies compared to those of the DNSBM model, while also demonstrating greater discriminatory power.

6 Conclusions

This paper has explored the dynamic allocation of carry-over variables within the context of resource management using a novel DNDEA model. Through a comprehensive review of existing literature, we have shed light on the significance of allocating discretionary carry-over activities in enhancing organizational efficiency over time periods. the proposed dynamic network DEA model not only computes the overall, divisional, and period efficiencies of DMUs with network structure but also adeptly incorporates inefficiencies regarding carry-over activities, especially in the presence of neutrosophic data. By strategically distributing carry-over variables across time periods, our proposed model optimizes overall efficiency scores. This dynamic allocation not only enhances organizations' adaptability and responsiveness to changing market conditions but also strengthens their competitive edge. To clarify the idea of the proposed approach and illustrate its capability, an empirical study for performance evaluation of 10 Iranian bank branches across years 1396 to 1402 is conducted in this paper. The results obtained from the model reveal the inefficient units in time periods

and identify their sources of inefficiencies. Comparing the results obtained from the proposed model with those from DNSBM reveals that the proposed model exhibits greater discriminative power and can better identify the sources of inefficiency. Moreover, it is able to reveal the optimal allocation of resources, thereby providing valuable insights for enhancing organizational efficiency. The proposed model measures the scores lower than those of DNSBM. The feasible region of the proposed model encompasses that of DNSBM and same objective function with DNSBM.

Future research could focus on extending the application of the proposed DNDEA model to other sectors, such as manufacturing, healthcare, or education, to evaluate its generalizability across different industries. Additionally, further studies could integrate other forms of uncertainty, such as stochastic data or interval data, to enhance the model's robustness. Exploring advanced optimization techniques or algorithms may also improve the model's scalability and computational efficiency, particularly when applied to larger and more complex datasets. Finally, incorporating real-time data or multi-objective optimization approaches could provide deeper insights into dynamic decision-making and resource allocation in rapidly changing environments.

Statements & Declarations

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The authors have no relevant financial or non-financial interests to disclose.

All authors contributed to the study conception and design. Material preparation performed by Maryam Heydar and data collection and analysis were performed by Hadi Bagerzadeh valami and Maryam Heydar. The first draft of the manuscript was written by Maryam Heydar and Hadi Bagerzadeh valami commented on previous versions of the manuscript. All authors read and approved the final manuscript.

The datasets generated during the current study are available from the corresponding author on reasonable request.

Compliance with Ethical Standards

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest:

Maryam Heydar declares that she has no conflict of interest.

Hadi Bagherzadeh valami declares that he has no conflict of interest.\

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Research on Mobile Advertising Users' Propagation Ability Evaluation based on the Neutrosophic Cubic Number Multiple-Attribute Decision-Making

Minming Weng[1,2], **Ying Long**[3,4]*
[1]Lampang Rajabhat University of Thailand, Mueang Lampang District, 52100, Lampang, Thailand
[2] School of Humanities and Teacher Education, Wuyi University, Wuyishan, 354300, Fujian, China
[3]Fujian Jiangxia University, Fuzhou 350108, Fujian, China
[4]Faculty of Social Sciences and Liberal Arts, UCSI University, Malaysia
*Corresponding author Email: wysly1022@gmail.com

Abstract: The evaluation of mobile advertising users' propagation ability assesses how effectively users can share or spread advertisements within their networks. It involves analyzing factors such as user engagement, social influence, sharing frequency, and network reach. By understanding these metrics, advertisers can identify highly influential users who can amplify campaign impact. This process helps optimize targeting strategies, improve ad performance, and maximize return on investment by leveraging users with strong propagation potential in mobile advertising ecosystems. The mobile advertising users' propagation ability evaluation is multiple-attribute decision-making (MADM). In recent years, the MOORA approach has been widely utilized to address multi-attribute decision-making (MADM) problems. To handle the uncertainty inherent in such problems, neutrosophic cubic sets (NCSs) have emerged as a powerful tool for expressing and analyzing uncertain information. This research introduces a novel approach, the neutrosophic cubic number MOORA (NCN-MOORA) approach, to solve MADM problems under the framework of NCSs. Specifically, the approach is applied to evaluate the propagation ability of mobile advertising users, a critical factor in optimizing advertising strategies. To demonstrate the validity of the proposed approach, a numerical case study is conducted. The primary objectives of this research are as follows: (1) to propose the NCN-MOORA approach for solving MADM problems within the NCS environment; (2) to determine attribute weight information using average method, ensuring an unbiased weighting process; (3) to design and apply the NCN-MOORA approach for evaluating mobile advertising users' propagation ability and compare its performance with existing approaches; and (4) to validate the effectiveness of the proposed approach through comparative analysis. The results confirm that the NCN-MOORA approach offers a reliable and efficient solution for evaluating mobile advertising users' propagation ability in uncertain environments.

Keywords: Multiple-attribute decision-making (MADM); Neutrosophic cubic sets (NCSs); MOORA approach; Mobile advertising users' propagation ability evaluation.

1. Introduction

Mobile advertising users' propagation ability refers to their capacity to share and disseminate advertising content within their networks, thereby amplifying the reach and impact of marketing campaigns. In the mobile advertising ecosystem, users act as key drivers of content distribution, influencing how effectively promotional messages spread through social platforms and digital communities. Propagation ability is crucial because it determines the extent to which advertisements can engage a larger audience through user interactions. This ability is influenced by various factors, including the user's level of activity, their influence within their network, and the size and quality of their connections. Users with strong propagation abilities often act as "key influencers," capable of encouraging others to engage with and further share the content. As mobile platforms continue to dominate the advertising landscape, understanding and leveraging user propagation ability has become a strategic priority for marketers. Evaluating propagation ability allows advertisers to identify high-value users who can maximize the visibility of their campaigns. By targeting these users, businesses can achieve better engagement rates, reduced costs, and improved conversion outcomes. Moreover, the dynamic and unpredictable nature of mobile advertising environments makes propagation ability a critical metric, as it ensures that campaigns stay relevant and effective in competitive markets. The earliest research focused on the technological applications of mobile advertising. In 2008, the partnership between mInfo and Nokia introduced a mobile advertising model centered on lifestyle information searches. By providing Nokia users with integrated search services, this study pioneered technological innovation in mobile advertising[1]. In 2012, Facebook launched a strategy to embed advertisements into users' news feeds through "sponsored stories," leveraging social data to increase ad revenue. However, this approach faced challenges due to users' resistance [2]. Also in 2012, Xiao and Wu [3] constructed a theoretical model for mobile ad adoption based on the Technology Acceptance Model (TAM). They found that factors such as entertainment, credibility, and informativeness significantly influenced users' willingness to adopt mobile ads. Later, Xiao [4] proposed a user attitude model, demonstrating that interactivity is a critical factor in shaping user attitudes, with personalization also having a significant positive impact. With the explosive growth of the mobile advertising market, the focus shifted to user experience and innovation. In 2014, Hong and Li [5] explored user-centric innovation models for mobile advertising,

emphasizing that user experience is a key factor for the future development of mobile advertising. By 2016, attention turned to the design and implementation of context-aware advertising. Wang and Zhao [6] proposed a context-aware service attraction model based on user behavior, improving ad relevance and appeal by analyzing the user's current environment and interests. In 2017, research moved toward optimizing the efficiency of mobile ad propagation. Xu and Yang [7] introduced a location-based mobile ad distribution strategy. By studying users' spatiotemporal characteristics and budget constraints, they designed a heuristic algorithm that significantly improved advertising propagation efficiency. Zhu and Wang [8] investigated user attitudes and preferences toward mobile ad visual design, providing targeted principles for ad designers. Finally, in 2021, Li [9] proposed a precision ad-push method based on user profiling and collaborative recommendation. By predicting the rating matrix using RBF neural networks, the method improved recommendation accuracy and user interest estimation. In summary, from early technological implementations to user behavior analysis, and later to precision targeting and propagation optimization, research in mobile advertising has gradually advanced toward user-oriented and intelligent approaches.

The evaluation of mobile advertising users' propagation ability is a MADM problem. Recently, the MOORA approach [14] has been widely applied to address MADM issues. NCSs [15] have emerged as a powerful tool for characterizing uncertain data, making them particularly suitable for evaluating users' propagation ability in mobile advertising. However, existing approaches often fail to address situations where attribute weight information is completely unknown. This motivates us to design a weight determination approach based on average and score values under NCSs. Furthermore, while many decision-making algorithms apply MOORA approaches independently to identify optimal solutions, limited studies have explored integrating these two approaches under the framework of NCSs. To the best of our knowledge, no or very few studies have focused on developing a MOORA approach using weighted averages under NCSs. Therefore, this research proposes a novel NCN-MOORA approach, which integrates the strengths of MOORA while employing an average-based weighting approach. The NCN-MOORA approach is specifically designed to deal with the MADM problems under the uncertain environment of NCSs. This paper dealt with the NCN-MOORA approach and applied it to the evaluation of mobile advertising users' propagation ability. To validate the proposed approach, a numerical case study is conducted. The

study demonstrates the practical applicability and effectiveness of the NCN-MOORA approach in real-world decision-making scenarios.

1.1 Criteria Definitions

Below are 10 criteria to evaluate the propagation ability of mobile advertising users, ranked in order of importance from highest to lowest with full descriptions.

1.1.1 Influence (High Priority)

The extent to which a user can impact others' opinions, behaviors, or purchasing decisions. Users with a high number of followers or connections who actively engage with their content are more influential.

1.1.2. Engagement Rate

The level of interaction a user generates on their posts, such as likes, shares, comments, and clicks. High engagement indicates a strong connection with their audience and greater propagation potential.

1.1.3. Reach

The total number of people a user can potentially reach through their posts or shares. This includes both direct followers and extended networks accessed through shares or reposts.

1.1.4. Content Relevance

How closely does the user's content align with the themes or goals of the advertising campaign. Relevance ensures that the message resonates with the target audience.

1.1.5. Sharing Frequency

The consistency and frequency with which a user shares or interacts with mobile advertisements. Frequent sharing enhances the probability of an ad reaching a wider audience.

1.1.6. Credibility

The trustworthiness of the user as perceived by their audience. Users with a reputation for providing authentic and accurate information are more likely to propagate ads effectively.

1.1.7. Virality Potential

The likelihood of a user's shared content going viral. This is influenced by the user's creativity, timing, and ability to capture trending topics or sentiments.

1.1.8. Social Network Centrality

The user's position within their social network, such as whether they are a hub for information sharing. Central users are better positioned to propagate messages effectively.

1.1.9. Platform Diversity

The number of platforms the user is active. Greater diversity increases the spread of the advertisement across different audience segments.

1.1.10. Demographic Alignment

The similarity between the user's demographic profile and the target audience of the advertisement. Alignment ensures that the propagated message reaches the intended recipients.

The primary contributions of this research are summarized:

- Proposing the NCN-MOORA approach for solving MADM problems under NCSs.
- Determining attribute weight information using an average-based approach.
- Designing and applying the NCN-MOORA approach to evaluate mobile advertising users' propagation ability and comparing it with existing approaches.
- Demonstrating through comparative analysis that the NCN-MOORA approach is both effective and reliable for evaluating mobile advertising users' propagation ability.

The research structure of this paper is outlined: Section 2 introduces the NCSs, which serve as a foundational tool for addressing uncertainty. In Section 3, the NCN-MOORA approach is developed under the framework of NCSs. Section 4 presents an illustrative case study to evaluate the propagation ability of mobile advertising users, along with a comparative analysis to validate the proposed approach. Section 5 provides concluding remarks and insights constructed from the study.

2. Preliminaries

Ail et al. [15] constructed the NCSs based on SVNSs [16] and IVNSs[17].

Definition 1 [15]. The NCSs are constructed:

$$A=\left\{\left(\theta,\left(I_A(\theta),S_A(\theta)\right)\right)\middle|\theta\in X\right\} \tag{1}$$

where $I_A(\theta)$ is IVNSs and $S_A(\theta)$ is SVNSs in X. And

$$I_A(\theta)=\left(\left[TL_A(\theta),TR_A(\theta)\right],\left[IL_A(\theta),IR_A(\theta)\right],\left[FL_A(\theta),FR_A(\theta)\right]\right) \quad (2)$$

$$S_A(\theta)=\left(TT_A(\theta),II_A(\theta),FF_A(\theta)\right) \quad (3)$$

$$\left[TL_A(\theta),TR_A(\theta)\right],\left[IL_A(\theta),IR_A(\theta)\right],\left[FL_A(\theta),FR_A(\theta)\right]\subseteq[0,1] \quad (4)$$

$$TT_A(\theta),II_A(\theta),FF_A(\theta)\in[0,1] \quad (5)$$

$$0\le TR_A(\theta)+IR_A(\theta)+FR_A(\theta)\le 3,\ \ 0\le TT_A(\theta)+II_A(\theta)+FF_A(\theta)\le 3 \quad (6)$$

The neutrosophic cubic number (NCN) is constructed:

$A=\left\{\left(\left[TL_A,TR_A\right],\left[IL_A,IR_A\right],\left[FL_A,FR_A\right]\right),\left(TT_A,II_A,FF_A\right)\right\}$.

Definition 2[15]. Let $A=\left\{\left(\left[TL_A,TR_A\right],\left[IL_A,IR_A\right],\left[FL_A,FR_A\right]\right),\left(TT_A,II_A,FF_A\right)\right\}$ and $B=\left\{\left(\left[TL_B,TR_B\right],\left[IL_B,IR_B\right],\left[FL_B,FR_B\right]\right),\left(TT_B,II_B,FF_B\right)\right\}$, the laws of the operation be constructed:

$$(1)\ A\oplus B=\left\{\begin{pmatrix}\left(TL_A+TL_B-TL_ATL_B,TR_A+TR_B-TR_ATR_B\right),\\ \left[IL_AIL_B,IR_AIR_B\right],\\ \left[FL_AFL_B,FR_AFR_B\right]\end{pmatrix},\begin{pmatrix}TT_A+TT_B-TT_ATT_B,\\ II_AII_B,\\ FF_AFF_B\end{pmatrix}\right\};$$

$$(2)\ A\otimes B=\left\{\begin{pmatrix}\left[TL_ATL_B,TR_ATR_B\right],\\ \left[IL_A+IL_B-IL_AIL_B,IR_A+IR_B-IR_AIR_B\right],\\ \left[FL_A+FL_B-FL_AFL_B,FR_A+FR_B-FR_AFR_B\right]\end{pmatrix},\begin{pmatrix}TT_ATT_B,\\ II_A+II_B-II_AII_B,\\ FF_A+FF_B-FF_AFF_B\end{pmatrix}\right\};$$

$$(3)\ \lambda A=\left\{\begin{pmatrix}\left[1-\left(1-TL_A\right)^{\lambda},1-\left(1-TR_A\right)^{\lambda}\right],\\ \left[\left(IL_A\right)^{\lambda},\left(IR_A\right)^{\lambda}\right],\\ \left[\left(FL_A\right)^{\lambda},\left(FR_A\right)^{\lambda}\right]\end{pmatrix},\begin{pmatrix}1-\left(1-TT_A\right)^{\lambda},\\ \left(II_A\right)^{\lambda},\\ \left(FF_A\right)^{\lambda}\end{pmatrix}\right\},\lambda>0;$$

$$(4)\ \left(A\right)^{\lambda}=\left\{\begin{pmatrix}\left[\left(TL_A\right)^{\lambda},\left(TR_A\right)^{\lambda}\right],\\ \left[1-\left(1-IL_A\right)^{\lambda},1-\left(1-IR_A\right)^{\lambda}\right],\\ \left[1-\left(1-FL_A\right)^{\lambda},1-\left(1-FR_A\right)^{\lambda}\right]\end{pmatrix},\begin{pmatrix}\left(TT_A\right)^{\lambda},\\ 1-\left(1-II_A\right)^{\lambda},\\ 1-\left(1-FF_A\right)^{\lambda}\end{pmatrix}\right\},\lambda>0.$$

Definition 3[18]. Let $A=\left\{\left(\left[TL_A,TR_A\right],\left[IL_A,IR_A\right],\left[FL_A,FR_A\right]\right),\left(TT_A,II_A,FF_A\right)\right\}$, the score, accuracy, certainty values are constructed:

$$NSV\left(A\right)=\frac{\left(6+TL_A+TR_A+TT_A-IL_A-IR_A-FL_A-FR_A-II_A-FF_A\right)}{9} \tag{7}$$

$$NAV\left(A\right)=\frac{\left(TL_A+TR_A+TT_A-IL_A-IR_A-FF_A\right)}{3} \tag{8}$$

$$NCV\left(A\right)=\frac{\left(TL_A+TR_A+TT_A\right)}{3} \tag{9}$$

Definition 4[18]. Let $A=\left\{\left(\left[TL_A,TR_A\right],\left[IL_A,IR_A\right],\left[FL_A,FR_A\right]\right),\left(TT_A,II_A,FF_A\right)\right\}$ and $B=\left\{\left(\left[TL_B,TR_B\right],\left[IL_B,IR_B\right],\left[FL_B,FR_B\right]\right),\left(TT_B,II_B,FF_B\right)\right\}$, Then the comparison laws for NCNs be constructed:

(1) if $NSV\left(A\right)<NSV\left(B\right)$, then, $A<B$;

(2)if $NSV\left(A\right)=NSV\left(B\right)$, and $NAV\left(A\right)<NAV\left(B\right)$, we have $A<B$;

(3) if $NSV\left(A\right)=NSV\left(B\right)$ and $NAV\left(A\right)=NAV\left(B\right)$, and $NCV\left(A\right)<NCV\left(B\right)$, we have $A<B$;

(4) if $NSV\left(A\right)=NSV\left(B\right)$ and $NAV\left(A\right)=NAV\left(B\right)$, and $NCV\left(A\right)=NCV\left(B\right)$, we have $A=B$.

Definition 5[19]. Let $A=\left\{\left(\left[TL_A,TR_A\right],\left[IL_A,IR_A\right],\left[FL_A,FR_A\right]\right),\left(TT_A,II_A,FF_A\right)\right\}$ and $B=\left\{\left(\left[TL_B,TR_B\right],\left[IL_B,IR_B\right],\left[FL_B,FR_B\right]\right),\left(TT_B,II_B,FF_B\right)\right\}$, the NCN Euclidean distance (NCNED) and NCN Hamming distance (NCNHD) be constructed:

$$NCNED\left(A,B\right)=\sqrt{\frac{1}{9}\left(\begin{array}{l}\left|TL_A-TL_B\right|^2+\left|TR_A-TR_B\right|^2+\left|IL_A-IL_B\right|^2+\\\left|IR_A-IR_B\right|^2+\left|FL_A-FL_B\right|^2+\left|FR_A-FR_B\right|^2+\\\left|TT_A-TT_B\right|^2+\left|II_A-II_B\right|^2+\left|FF_A-FF_B\right|^2\end{array}\right)} \tag{10-a}$$

$$NCNHD\left(A,B\right)=\frac{1}{9}\left(\begin{array}{l}\left|TL_A-TL_B\right|+\left|TR_A-TR_B\right|+\left|IL_A-IL_B\right|+\\\left|IR_A-IR_B\right|+\left|FL_A-FL_B\right|+\left|FR_A-FR_B\right|+\\\left|TT_A-TT_B\right|+\left|II_A-II_B\right|+\left|FF_A-FF_B\right|\end{array}\right) \tag{10-b}$$

3. Methodology

This section shows the steps of the NCN-MOORA approach. In 2004, Brauers presented the Multi-Objective Optimization Ratio Analysis (MOORA) method, which is regarded as an objective (non-subjective) approach. Additionally, while ranking options, both desirable and unpleasant criteria are applied simultaneously to choose a higher or better option. There are numerous uses for this approach. The following characteristics of the MOORA approach are present: It falls under the category of compensating techniques. Qualitative qualities are transformed into quantitative attributes; attributes are independent.

Step 1. Create a decision matrix.

$$R = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{pmatrix}; i = 1, \dots, m; j = 1, \dots., n \tag{11}$$

Step 2. Normalize the decision matrix

We normalize the decision matrix between the criteria and the alternatives.

$$n_{ij} = \frac{r_{ij}}{\sqrt{\sum_{i=1}^{m} r_{ij}^2}} \tag{12}$$

Step 3. Determine the reference points.

We compute the reference points for the positive criteria by maximizing the normalized values and the negative criteria by minimizing the normalized values.

Step 4. Compute assessment values.

$$U_j = \sum_{j=1}^{g} n_{ij} w_j - \sum_{j=g+1}^{n} n_{ij} w_j \tag{13}$$

Where g refers to the number of positive criteria and n-g refers to the number of negative criteria.

Step 5. Final rank of alternatives.

We ranked alternatives based on the highest value of the assessment values.

4. Results

This section shows the results of the proposed method. The mobile advertising users' propagation ability evaluation is a MADM issue. Eight mobile advertising companies are assessed with 10 attributes.

Step 1. We build the decision matrix using the neutrosophic numbers as shown in Table 1. Then we obtained the score function. Then we combined these values into one matrix. Then we compute the criteria weights by the average method.

C1= 0.098935017,C2= 0.0997919,C3= 0.099730694,C4= 0.100176623, C5= 0.100832401,
C6= 0.099678232,C7= 0.100342753,C8= 0.099573307,C9= 0.100561346, C10= 0.100377728.

Step 2. Then we used Eq. (12) to normalize the decision matrix.

$$n_{11} = \left(\frac{0.51667}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \right) 0.34866$$

$$n_{12} = \left(\frac{0.52481}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \right) 0.35416$$

$$n_{13} = \left(\frac{0.53037}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \right) 0.35791$$

$$n_{14} = \left(\frac{0.52889}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \right) 0.35691$$

$$n_{15} = \left(\frac{0.52111}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \right) 0.35166$$

$$n_{16} = \left(\frac{0.50926}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \frac{0.50926}{0.34367} \right)$$

$$n_{17} = \left(\frac{0.50926}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \frac{0.50926}{0.35141} \right)$$

$$n_{18} = \left(\frac{0.53889}{\sqrt{(0.51667 + 0.52481 + 0.53037 + 0.52889 + 0.52111 + 0.50926 + 0.52074 + 0.53889)}} = \frac{0.53889}{0.36366} \right)$$

Step 3. Then we compute the reference points for all positive criteria.

Step 4. Eq. (13) was used to compute assessment values.

Step 5. Final rank of alternatives. A7>A3>A4>A6>A5>A8>A1>A2

Table 1. The information of neutrosophic numbers.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
a_1	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))
a_2	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))
a_3	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
a_4	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))
a_5	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))
a_6	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.046, 0.41, 0.50))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))
a_7	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
a_8	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.66, 0.41, 0.50))
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
a_1	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))
a_2	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))
a_3	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
a_4	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))
a_5	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))
a_6	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.046, 0.41, 0.50))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))
a_7	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
a_8	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	([0.12, 0.40], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.46, 0.41, 0.50))	([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	([0.42, 0.69], [0.31, 0.55], [0.26, 0.40]), (0.66, 0.41, 0.50))

	⟨0.30⟩, (0.43, 0.26, 0.34))	⟨0.40⟩, (0.46, 0.30, 0.39))	⟨0.39⟩, (0.59, 0.32, 0.44))	⟨0.40⟩, (0.66, 0.41, 0.50))	⟨0.61⟩, (0.70, 0.39, 0.47))	⟨0.40⟩, (0.46, 0.30, 0.39))	⟨0.30⟩, (0.43, 0.26, 0.34))	⟨0.40⟩, (0.46, 0.30, 0.39))	⟨0.30⟩, (0.43, 0.26, 0.34))	⟨0.30⟩, (0.43, 0.26, 0.34))
A_3	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.42⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.42⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.42⟩, (0.37, 0.28, 0.42))
A_4	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))
A_5	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))
A_6	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))
A_7	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))
A_8	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))
A_2	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.42⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))
A_3	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))
A_4	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))
A_5	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.39, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))
A_6	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))
A_7	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))
A_8	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.23, 0.57], [0.17, 0.36], [0.11, 0.45⟩, (0.48, 0.21, 0.37))	⟨(0.12, 0.48], [0.21, 0.39], [0.15, 0.43⟩, (0.37, 0.28, 0.42))	⟨(0.19, 0.44], [0.15, 0.38], [0.13, 0.30⟩, (0.43, 0.26, 0.34))	⟨(0.22, 0.49], [0.19, 0.37], [0.15, 0.40⟩, (0.46, 0.30, 0.39))	⟨(0.13, 0.40], [0.14, 0.33], [0.22, 0.50⟩, (0.38, 0.28, 0.35))	⟨(0.38, 0.71], [0.29, 0.49], [0.26, 0.61⟩, (0.70, 0.39, 0.47))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))	⟨(0.35, 0.62], [0.28, 0.47], [0.19, 0.39⟩, (0.59, 0.32, 0.44))	⟨(0.42, 0.69], [0.31, 0.55], [0.26, 0.48⟩, (0.66, 0.41, 0.50))

4.1 Discussion of the results

The evaluation of mobile advertising users' propagation ability is a critical aspect of understanding how effectively information spreads within a digital ecosystem. In the context of mobile advertising, propagation ability refers to a user's capacity to influence others and disseminate content, which is vital for boosting the reach and impact of advertising campaigns. With the rapid growth of mobile platforms and the increasing reliance on user-driven content sharing, evaluating this ability has become a fundamental task for advertisers and marketers aiming to optimize campaign performance. Mobile advertising operates within a networked digital environment where users play a dual role as

consumers and distributors of content. Propagation ability highlights the extent to which users facilitate the spread of promotional messages, which is influenced by factors such as their social connections, activity levels, and influence within their networks. By identifying users with strong propagation abilities, advertisers can strategically target these individuals to amplify the visibility and effectiveness of their campaigns. One of the key challenges in evaluating propagation ability lies in the inherent uncertainty and complexity of user behavior. Mobile advertising operates in a dynamic environment where factors such as user preferences, engagement levels, and network structures constantly evolve. To address these challenges, advanced decision-making methods are often employed to effectively analyze and evaluate propagation ability while accounting for uncertainties. Techniques such as MADM approaches, including TODIM and VIKOR, are widely used to navigate the complexity of these evaluations. These methods enable a structured assessment of users by considering multiple attributes and identifying the most influential individuals within a network. Another consideration in evaluating propagation ability is the role of data-driven techniques and tools to process and analyze large volumes of information. Mobile platforms generate vast amounts of data, including user interactions, sharing patterns, and network connections. This data serves as the foundation for evaluating propagation ability, enabling marketers to uncover insights into user behavior and identify key influencers within the system. Advanced modeling techniques, such as neutrosophic sets, have been introduced to manage the uncertainty inherent in such data, providing a more comprehensive and accurate evaluation. Ultimately, the evaluation of mobile advertising users' propagation ability serves as a strategic tool for optimizing advertising efforts. By understanding how users disseminate content and identifying those with the greatest influence, advertisers can improve targeting strategies, enhance campaign reach, and maximize engagement. This evaluation process is essential for leveraging the full potential of mobile platforms in modern advertising. The results show alternative 7 is the best and alternative 2 is the worst.

4.2 Comparative analysis

The NCN-MOORA approach is compared with the NC-VIKOR approach [20], NC-Cross Entropy[21], and NCN-GRA approach [22]. The comparative results are constructed in Table 2.

Table 2. Order of different approaches

Methods	*Order*
NC-VIKOR approach [20]	A7>A3>A6>A5>A4>A8>A1>A2
NC-Cross Entropy[22]	A7>A3>A5>A6>A4>A8>A1>A2
NCN-GRA approach [23]	A7>A3>A4>A6>A5>A8>A1>A2
Our model	A7>A3>A4>A6>A5>A8>A1>A2

From the detailed analysis above, it is evident that all five models lead to the same optimal choice, and the ranking order of these approaches remains consistent. This consistency validates the reasonableness and effectiveness of the proposed NCN-MOORA method. By demonstrating alignment across different models, the reliability of the NCN-MOORA approach is further confirmed, solidifying its utility in addressing MADM problems. Additionally, the proposed NCN-MOORA approach offers a more robust and precise decision-making framework compared to other existing methods. Its ability to handle complex decision environments with uncertain and inconsistent information sets it apart as a superior technique. This method integrates the strengths of both MOORA under the framework of neutrosophic cubic sets (NCSs), making it a versatile and dependable tool for decision-making scenarios.

One of the key features of the NCN-MOORA approach is its consideration of the bounded rationality of decision-makers (DMs). Unlike traditional methods that assume fully rational behavior, the NCN-MOORA approach acknowledges that DMs often operate under constraints such as limited information, time pressures, and cognitive biases. By incorporating this realistic perspective, the proposed method provides results that are better aligned with real-world decision-making processes. Furthermore, the NCN-MOORA approach effectively minimizes the impact of abnormal or extreme

evaluation values. In MADM problems, outlier values can significantly distort results, leading to unreliable conclusions. The proposed method addresses this challenge by integrating the neutrosophic cubic set framework so that the influence of such abnormal values is reduced. This leads to more balanced and accurate decision outcomes. In conclusion, the NCN-MOORA approach is a powerful and effective approach for solving MADM problems. Its ability to accommodate bounded rationality, mitigate the effects of extreme values, and deliver consistent and reliable results makes it a valuable tool for decision analysis. By surpassing the precision and reliability of other methods, this approach contributes significantly to the advancement of decision-making methodologies in uncertain environments.

4.3 Managerial Implications of This Study

By effectively evaluating mobile advertising users' propagation ability, managers can strategically enhance their advertising impact, foster meaningful user relationships, and drive sustainable brand growth.

1. Identifying High-Value Users

Implication: Managers can pinpoint users with the highest propagation potential, such as influencers or opinion leaders, to amplify advertising reach. Action: Focus resources on collaborating with these high-value users for targeted campaigns to maximize ROI.

2. Optimizing Content Strategy

Implication: Insights into user engagement and sharing behaviors help managers tailor content to align with users' preferences and interests. Action: Create engaging, shareable, and relevant content that resonates with users and motivates them to propagate the message.

3. Enhancing Campaign Targeting

Implication: Propagation ability evaluation allows managers to identify specific demographics and platforms where users have the greatest influence. Action: Allocate advertising budgets to platforms and user segments that offer the highest propagation potential.

4. Improving Engagement Metrics

Implication: Understanding engagement patterns enables managers to refine strategies for fostering deeper connections between users and the brand. Action: Focus on increasing likes, shares, and comments by designing interactive and emotionally resonant campaigns.

5. Encouraging Peer-to-Peer Advocacy

Implication: Users with high propagation ability act as brand advocates, fostering organic growth through peer recommendations. Action: Develop referral programs, gamified sharing incentives, or exclusive offers to encourage users to share content widely.

5. Conclusion

The evaluation of mobile advertising users' propagation ability is essential for understanding how effectively users can spread advertising content within their networks. In the digital advertising ecosystem, users act not only as content consumers but also as distributors, playing a critical role in amplifying the reach and influence of campaigns. Propagation ability reflects a user's capacity to engage with and share advertisements, influencing how well messages are disseminated across social platforms or mobile networks. This evaluation is particularly important as advertisers aim to identify key influencers who can enhance the visibility and impact of their campaigns. By understanding the dynamics of content sharing and user influence, businesses can better target individuals who maximize reach and engagement. However, the evaluation process is complex due

to the uncertainties in user behavior and the dynamic nature of mobile platforms. Advanced decision-making models, such as those incorporating neutrosophic cubic sets, are often used to address these challenges. These methods allow advertisers to account for uncertainty, incomplete data, and behavioral variability, offering a more robust framework for evaluating users' propagation capabilities. Ultimately, assessing propagation ability helps optimize advertising strategies, ensure more effective resource allocation, and improve overall campaign performance. It is a crucial step in leveraging the power of mobile advertising.

The evaluation of mobile advertising users' propagation ability is a MADM problem. This type of evaluation is crucial for understanding how effectively users disseminate advertising content within their networks, which directly impacts the success of advertising campaigns. Recently, the MOORA approach has been widely applied to solve MADM problems due to its ability to handle conflicting criteria and provide a balanced solution. However, in environments with uncertain and incomplete information, conventional methods may struggle to deliver accurate and reliable results. To address this, NCSs have been introduced as a powerful tool for characterizing and managing uncertainty during the evaluation of mobile advertising users' propagation ability. In this manuscript, we propose the NCN-MOORA model, a novel integrated approach designed to solve MADM problems under the framework of NCSs. This model combines the strengths of MOORA while leveraging the flexibility of NCSs to represent uncertain and imprecise information. The NCN-MOORA approach is specifically applied to evaluate mobile advertising users' propagation ability, providing a robust and comprehensive framework for decision analysis in uncertain environments.

To validate the effectiveness of the proposed approach, a numerical case study is conducted. The case study demonstrates how the NCN-MOORA model performs in real-world scenarios and

highlights its ability to deliver consistent and reliable results. Additionally, a comparative analysis with existing methods is provided to further confirm its practicality and superiority.

The key contributions of this research are summarized:

- The NCN-MOORA model is introduced to address MADM problems under the framework of NCSs, offering a novel approach for handling uncertainty and complexity.
- The NCN-MOORA model is specifically designed for evaluating mobile advertising users' propagation ability and is compared with other existing approaches.
- The comparative analysis reveals that the proposed NCN-MOORA approach is highly effective and reliable for evaluating propagation ability in uncertain environments.

This research provides a valuable contribution to MADM methodologies and offers a practical tool for optimizing mobile advertising strategies.

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Numerical and Semi Analytical Scheme for developing a solution to Falkner Skan equation in Neutrosophic environments

B. Amudha[1], M. Shanmugapriya[2], R. Sundareswaran[3*], Said Broumi[4]

[1, 2,3]Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Chennai - 603 110, India

[1]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco
Email: shanmugapriyam@ssn.edu.in, sundareswaranr@ssn.edu.in, s.broumi@flbenmsik.ma,
**Corresponding author detail: sundareswaranr@ssn.edu.in*

Abstract: This study presents a detailed analysis of the Falkner-Skan equation under uncertainty by employing neutrosophic numbers to represent the uncertainty in no slip conditions and the parameter $\tilde{\lambda}$. By applying appropriate similarity transformation, partial differential equation (PDE) are converted into ordinary differential equation (ODE). Both the Shooting Method and the Homotopy Perturbation Method (HPM) are utilized to solve ODE's. These ODE's are then transmuted into neutrosophic differential equation (NDE) by employing $(\tilde{A}, \tilde{B}, \tilde{\Gamma})_{cut}$ approach. The parameter $\tilde{\lambda}$ and no slip conditions are taken as triangular and trapezoidal neutrosophic numbers. The results are presented graphically to illustrate the comparative effectiveness of these methods. The analysis reveals that use of trapezoidal neutrosophic numbers and triangular neutrosophic numbers in Falkner-Skan equation gives strong neutrosophic solution. A 3D error analysis is conducted to compare the performance of triangular and trapezoidal neutrosophic numbers, highlighting their relative accuracy in solving the Falkner-Skan equation under varying degrees of uncertainty.

Keywords: Falkner-Skan Equation; Neutrosophic set; Triangular Neutrosophic Number; Trapezoidal Neutrosophic Number; Homotopy Perturbation Method.

1. Introduction

Classical set theory is widely applied in various real-world contexts to organize, categorize, and analyze information. A classical set incorporates elements that satisfy factual membership properties. Fuzzy set theory is an extension of classical set theory popularized by Zadeh [1] in 1965 that deals with uncertainty and imprecision. It is highly effective in controlling ambiguities emerging from an element's vagueness in a set, although it is incapable to simulate all forms of uncertainty encountered in various real-world physical problems with incomplete information. The abstraction of fuzzy set is recognized as Intuitionistic fuzzy sets invented in 1986 by Atanassov [2–5]. Each element in Intuitionistic fuzzy set has membership and non-membership values where their sum is greater than or equal to one. Later, in 1998, Smarandache [6–8] introduced Neutrosophic set theory, further generalizing classical, fuzzy, and intuitionistic fuzzy set theories. It is designed to handle various types

of uncertainties, including incomplete, indeterminate, and inconsistent information in real world. They are characterized by three membership functions: truth, falsity, and indeterminacy, each of which can take values in the range $]0^-,1^+[$.

Differentiation is crucial in science and engineering for modeling and solving various problems involving rates of change. Many real-world problems come with uncertain or imprecise parameters, which traditional differential equations struggle to handle. Fuzzy differential equations (FDE) handle uncertainty by representing variables with fuzzy sets. Bede et al. [9] investigated first order linear FDE's under various interpretations and demonstrated that the behavior of the solutions varies depending on the interpretation used. Vasavi et al. [10] applied fuzzy logic to Newton's law of cooling using different interpretations of fuzzy differential equations. They utilize a specific mathematical method (variation of constants) to find solutions that can better reflect the actual behavior of systems under uncertainty. Further, to improve upon fuzzy differential equations, intuitionistic fuzzy differential equations were developed. Ettoussi et al. [11] explored how intuitionistic fuzzy differential equations can be solved uniquely and exist using method of successive approximation and discusses the continuity of these solutions in the context of fuzzy set theory. In engineering, science, economics, and other fields, many systems are influenced by factors that cannot be precisely measured or predicted. Neutrosophic differential equations allow for a more accurate and realistic modeling of these systems by capturing all facets of uncertainty. Moi et al. [12] has examined second order boundary value problem through neutrosophic differential equation. Many problems involving multicriteria decision making problems [13], medical diagnostics [14], pattern recognition [15] and mine safety [16] have all been solved extensively with neutrosophic differential equation.

A $\mathcal{N}_{number}$ is an extension of classical numbers and fuzzy numbers designed to handle uncertainty, indeterminacy, and vagueness more comprehensively. Parikh et al. [17] investigated logistic growth model that incorporates neutrosophic logic to better predict population dynamics, particularly when dealing with uncertainties and potential deviations. Bhaumik et al. [18] introduces a new ranking approach for handling uncertainty in game theory using single-valued triangular neutrosophic numbers by applying bi-matrix games. He demonstrated how neutrosophic logic can enhance decision-making processes in uncertain environments. Shanmugapriya et al. [19] explores a neutrosophic method to solve a system of first-order differential equations using Trapezoidal Neutrosophic Numbers. They incorporated $\left(\tilde{A},\tilde{B},\tilde{\Gamma}\right)_{cut}$, the approach accounts for varying levels of truth, indeterminacy, and falsity.

An important tool for understanding fluid flow dynamics under pressure gradients is the Falkner-Skan equation, which forms the basis of boundary layer theory. Its relevance and adaptability in the field of fluid dynamics are demonstrated by its applications in engineering design, theoretical analysis, numerical validation, and larger flow issues. Han [20] applied finite difference method for

finding numerical solution of Falkner-Skan wedge flow equation. Parand et al. [21] studied magnetohydrodynamics (MHD) Falkner-Skan wedge flow equation by leveraging the orthogonal Sinc functions to approximate solutions on semi-infinite domains with high accuracy. Elnady et al. [22] examined the solution of the Falkner–Skan equation by addressing the challenge of its semi-infinite domain. He tackled this by truncating the domain to a finite interval and then used a Chebyshev series to approximate the solution. Bararnia et al. [23] applied Homotopy Analysis Method (HAM) to solve the momentum and energy equation of Falkner-Skan wedge flow equation in incompressible fluid by using trial and error and Padé approximation. Shanmugapriya et al. [24, 25] and Gopi Krishna et al. [26] have studied fluid flow in wedge using various techniques.

Zulqarnain et al [27] employed a triangular fuzzy number to capture uncertainties and uses numerical methods to investigate how various parameters affect flow and heat transfer. He studied tri-hybrid nanofluid model incorporating Al_2O_3, Cu, TiO_2 in engine oil over a Riga wedge. Siddique et al [28] investigated the thermal properties of a second-grade hybrid nanofluid ($Al_2O_3 + Cu/EO$) over a Riga wedge, considering factors like heat source, stagnation point, and nonlinear thermal radiation in fuzzy environment. Shanmugapriya et al [29] explores the effects of endothermic/exothermic chemical reactions, thermal radiation, thermophoresis, and Brownian diffusion on the flow and heat transfer of a Casson hybrid nanofluid over a moving wedge under fuzzy environment.

1.1 Research gap

In the literature survey mentioned above, researchers have developed various methods to solve the Falkner-Skan equation, including similarity transformations, series solutions, and numerical methods like the finite difference method. However, but no research has addressed solutions using triangular and trapezoidal neutrosophic numbers. This research seeks to address this gap by examining neutrosophic solutions to the Falkner-Skan Boundary Layer Wedge flow equation through the application of both the shooting method and the Homotopy Perturbation Method (HPM).

1.2 Objective

The aim of this study is to investigate Falkner-Skan equation using $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. The ODE's are transformed into NDE's with the help of $\left(\widetilde{\mathrm{A}}, \widetilde{\mathrm{B}}, \widetilde{\Gamma}\right)_{cut}$. The $\mathcal{N}_{number}$ are applied in no slip conditions and wedge angle parameter. The shooting method via MATLAB and HPM via MAPLE were utilized to handle NDE's. The outcomes are validated through available literature, 2D and 3D plots.

1.3 Novelty

The novelty of the present study is

i. The behavior of Falkner-Skan wedge flow equation in neutrosophic environment has not been addressed previously.

ii. The wedge angle parameter $\tilde{\lambda}$ within the Falkner-Skan equation and no slip conditions are taken as $TriN_{number}$ and $TrapN_{number}$ through $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$ approach covering a range of possible behavior.

iii. A comparative analysis of mean values and error profiles for $TriN_{number}$ and $TrapN_{number}$ are determined which highlights the performance in controlling uncertainty.

1.4 Applications

The use of trapezoidal and triangular neutrosophic numbers into the Falkner-Skan equation provides noteworthy benefits across several engineering domains through efficient uncertainty management:

i. Aerodynamics: By incorporating neutrosophic numbers into the design process, designers of automobiles and aircraft may better account for uncertainties in pressure gradients and flow conditions, resulting in more resilient and efficient designs that minimize drag and improve performance in ever changing situations.

ii. Industrial Heat Transfer: The use of neutrosophic numbers in heat exchangers and cooling systems makes designs more dependable and maintains efficiency in the face of temperature and flow rate variations, which is essential for energy-intensive businesses.

The above application shows how the Falkner-Skan equation's use of neutrosophic numbers improves fluid dynamics designs' performance and dependability by taking into account the inherent uncertainties of real-world scenarios.

The basic concepts of N_{set}, $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$ of a N_{set}, N_{number}, $TriN_{number}$, $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$ of $TriN_{number}$, $TrapN_{number}$, $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$ of $TrapN_{number}$, and the $strong_{solution}$ of NDE are introduced in § 2.The mathematical formulation of Falkner-Skan equation is derived in § 3. In § 4, the equation is transmuted into neutrosophic equation and application of shooting method and HPM for neutrosophic governing equation. Finally, § 5 contains profile graphs, tables, neutrosophic analysis and comparative analysis of shooting method and HPM. The conclusion is given in § 6.

Abbreviation	**Description**
N_{set}	Neutrosophic Set
N_{number}	Neutrosophic Number
$(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$	$(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})$ cut
$TriN_{number}$	Triangular Neutrosophic Number
$(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$ of a $TriN_{number}$	$(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})$ cut of Triangular Neutrosophic Number

$TrapN_{number}$	Trapezoidal Neutrosophic Number
$(\tilde{A}, \tilde{B}, \tilde{\Gamma})_{cut}$ of a $TrapN_{number}$	$(\tilde{A}, \tilde{B}, \tilde{\Gamma})$ cut of Trapezoidal Neutrosophic Number
$strong_{solution}$	Strong Solution
$weak_{solution}$	Weak Solution
$truth_{function}$	Truth function
$indeterministic_{function}$	Indeterministic function
$falsity_{function}$	Falsity function
$TriN$ velocity	Triangular Neutrosophic Velocity
$TrapN$ velocity	Trapezoidal Neutrosophic Velocity

2. Preliminaries

In this § 2, we recall a few preliminary concepts of neutrosophic set theory and some notations for better understanding.

Definition 2.1.

The universal set is typically denoted by X and is defined as the set of all objects or elements relevant to a discussion or problem.

Definition 2.2.[30]

Let X be a universe set. A $\mathcal{N}_{set}$ $(\widetilde{\mathcal{N}_s})$ on X is defined as $\widetilde{\mathcal{N}_s} = \{\langle \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x), \mathcal{I}_{\widetilde{\mathcal{N}_s}}(x), \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x)\rangle : x \in X\}$, where $\mathcal{T}_{\widetilde{\mathcal{N}_s}}(x), \mathcal{I}_{\widetilde{\mathcal{N}_s}}(x), \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x): X \to {}^{-}]0,1[^{+}$ represents the degree of certainty, degree of hesitation and degree of falseness respectively of the element $x \in X$ such that ${}^{-}0 \leq \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x), +\mathcal{I}_{\widetilde{\mathcal{N}_s}}(x) + \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x) \leq 3^{+}$.

Definition 2.3.[31]

The $(\tilde{A}, \tilde{B}, \tilde{\Gamma})_{cut}$ of $\mathcal{N}_{set}$ is defined as $\widetilde{\mathcal{N}_s}_{(\tilde{A},\tilde{B},\tilde{\Gamma})} = \{\langle \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x), \mathcal{I}_{\widetilde{\mathcal{N}_s}}(x), \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x)\rangle : x \in X, \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x) \geq \tilde{A}, \mathcal{I}_{\widetilde{\mathcal{N}_s}}(x) \leq \tilde{B}, \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x) \leq \tilde{\Gamma}\}$, where $\tilde{A}, \tilde{B}, \tilde{\Gamma} \in [0,1]$, such that $0 \leq \tilde{A} + \tilde{B} + \tilde{\Gamma} \leq 3$.

Definition 2.4.[31]

A $\mathcal{N}_{set}$ defined on the set of real numbers $\mathbb{R}$ is a $\mathcal{N}_{number}$ if it satisfies the following properties:

i. $\widetilde{\mathcal{N}_s}$ is normal if $\exists\, x_0 \in \mathbb{R}: \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x_0) = 1. \left(\mathcal{I}_{\widetilde{\mathcal{N}_s}}(x_0) = \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x_0) = 0\right)$.

ii. $\widetilde{\mathcal{N}_s}$ is convex for the $truth_{function}$ $\mathcal{T}_{\widetilde{\mathcal{N}_s}}(x)$. $(ie)\mathcal{T}_{\widetilde{\mathcal{N}_s}}(\varsigma x_1 + (1-\varsigma)x_2) \geq \min\left(\mathcal{T}_{\widetilde{\mathcal{N}_s}}(x_1), \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ and $\varsigma \in [0,1]$.

iii. $\widetilde{\mathcal{N}_s}$ is concave set for the $indeterministic_{function}$ $\mathcal{I}_{\widetilde{\mathcal{N}_s}}(x)$ and $falsity_{function}$ $\mathcal{F}_{\widetilde{\mathcal{N}_s}}(x)$. $(ie)\ \mathcal{I}_{\widetilde{\mathcal{N}_s}}(\varsigma x_1 + (1-\varsigma)x_2) \geq \max\left(\mathcal{I}_{\widetilde{\mathcal{N}_s}}(x_1), \mathcal{I}_{\widetilde{\mathcal{N}_s}}(x_2)\right)$ and $(ie)\mathcal{F}_{\widetilde{\mathcal{N}_s}}(\varsigma x_1 + (1-\varsigma)x_2) \geq \max\left(\mathcal{F}_{\widetilde{\mathcal{N}_s}}(x_1), \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ and $\varsigma \in [0,1]$.

Definition 2.5.[31]

A $Tri\mathcal{N}_{number}$ is a subset of $\widetilde{\mathcal{N}_s}$ in $\mathbb{R}$ with the following $truth_{function}$, $indeterministic_{function}$, $falsity_{function}$ is defined as

$$\mathcal{T}_{\widetilde{\mathcal{N}_s}}(x) = \begin{cases} \left(\frac{x-a_1}{a_2-a_1}\right)\omega_{\widetilde{\mathcal{N}_s}} & if\ a_1 \le x \le a_2 \\ \omega_{\widetilde{\mathcal{N}_s}} & if\ x = a_2 \\ \left(\frac{a_3-x}{a_3-a_2}\right) & if\ a_2 \le x \le a_3 \\ 0 & otherwise \end{cases}$$

$$\mathcal{I}_{\widetilde{\mathcal{N}_s}}(x) = \begin{cases} \left(\frac{a_2-x}{a_2-a_1}\right)\delta_{\widetilde{\mathcal{N}_s}} & if\ a_1 \le x \le a_2 \\ \delta_{\widetilde{\mathcal{N}_s}} & if\ x = a_2 \\ \left(\frac{x-a_3}{a_3-a_2}\right) & if\ a_2 \le x \le a_3 \\ 1 & otherwise \end{cases} \tag{2.1}$$

$$\mathcal{F}_{\widetilde{\mathcal{N}_s}}(x) = \begin{cases} \left(\frac{a_2-x}{a_2-a_1}\right)\varepsilon_{\widetilde{\mathcal{N}_s}} & if\ a_1 \le x \le a_2 \\ \varepsilon_{\widetilde{\mathcal{N}_s}} & if\ x = a_2 \\ \left(\frac{x-a_3}{a_3-a_2}\right) & if\ a_2 \le x \le a_3 \\ 1 & otherwise \end{cases}$$

where $^-0 \le \mathcal{T}_{\widetilde{\mathcal{N}_s}}(x) + \mathcal{I}_{\widetilde{\mathcal{N}_s}}(x) + \mathcal{F}_{\widetilde{\mathcal{N}_s}}(x) \le 3^+, x \in \widetilde{\mathcal{N}_s}$.

Definition 2.6.[31]

The $\left(\widetilde{\mathrm{A}}, \widetilde{\mathrm{B}}, \widetilde{\Gamma}\right)_{cut}$ of a $Tri\mathcal{N}_{number} = \langle (a_1, a_2, a_3); \omega_{\widetilde{\mathcal{N}_s}}, \delta_{\widetilde{\mathcal{N}_s}}, \varepsilon_{\widetilde{\mathcal{N}_s}} \rangle$ is defined as follows:

$\widetilde{\mathcal{N}_s}_{(\widetilde{\mathrm{A}},\widetilde{\mathrm{B}},\widetilde{\Gamma})} = \left[\mathcal{T}_{\widetilde{\mathcal{N}_s}^1}(\widetilde{\mathrm{A}}), \mathcal{T}_{\widetilde{\mathcal{N}_s}^2}(\widetilde{\mathrm{A}}); \mathcal{I}_{\widetilde{\mathcal{N}_s}^1}(\widetilde{\mathrm{B}}), \mathcal{I}_{\widetilde{\mathcal{N}_s}^2}(\widetilde{\mathrm{B}}); \mathcal{F}_{\widetilde{\mathcal{N}_s}^1}(\widetilde{\Gamma}), \mathcal{F}_{\widetilde{\mathcal{N}_s}^2}(\widetilde{\Gamma})\right]$, where

$$\begin{aligned} &\mathcal{T}_{\widetilde{\mathcal{N}_s}^1}(\widetilde{\mathrm{A}}) = \left[a_1 + \widetilde{\mathrm{A}}(a_2 - a_1)\right]\omega_{\widetilde{\mathcal{N}_s}}, \mathcal{T}_{\widetilde{\mathcal{N}_s}^2}(\widetilde{\mathrm{A}}) = \left[a_3 - \widetilde{\mathrm{A}}(a_3 - a_2)\right]\omega_{\widetilde{\mathcal{N}_s}} \\ &\mathcal{I}_{\widetilde{\mathcal{N}_s}^1}(\widetilde{\mathrm{A}}) = \left[a_2 - \widetilde{\mathrm{B}}(a_2 - a_1)\right]\delta_{\widetilde{\mathcal{N}_s}}, \mathcal{I}_{\widetilde{\mathcal{N}_s}^2}(\widetilde{\mathrm{A}}) = \left[a_2 + \widetilde{\mathrm{B}}(a_3 - a_2)\right]\delta_{\widetilde{\mathcal{N}_s}} \\ &\mathcal{F}_{\widetilde{\mathcal{N}_s}^1}(\widetilde{\mathrm{A}}) = \left[a_2 - \widetilde{\Gamma}(a_2 - a_1)\right]\varepsilon_{\widetilde{\mathcal{N}_s}}, \mathcal{F}_{\widetilde{\mathcal{N}_s}^2}(\widetilde{\mathrm{A}}) = \left[a_2 + \widetilde{\Gamma}(a_3 - a_2)\right]\varepsilon_{\widetilde{\mathcal{N}_s}} \end{aligned} \tag{2.2}$$

Here $0 \le \widetilde{\mathrm{A}} \le 1, 0 \le \widetilde{\mathrm{B}} \le 1, 0 \le \widetilde{\Gamma} \le 1$ and $^-0 \le \widetilde{\mathrm{A}} + \widetilde{\mathrm{B}} + \widetilde{\Gamma} \le 3^+$ and is displayed in Figure 1.

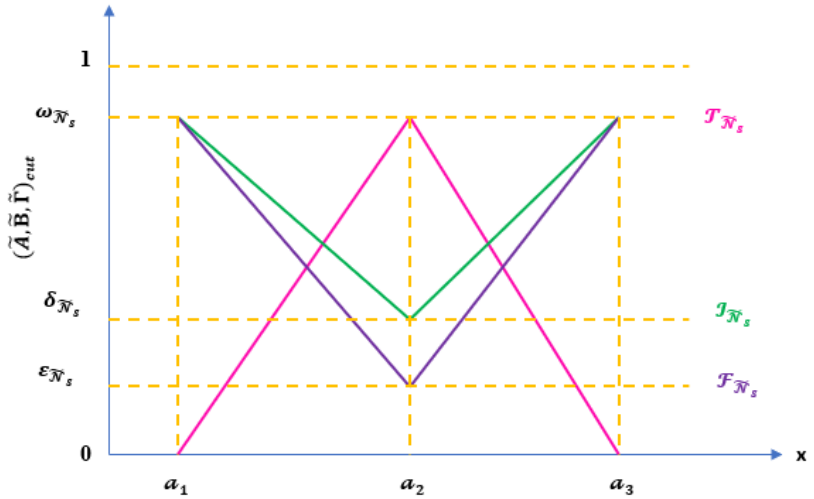

Figure 1. Membership function of $TriN_{number}$

Definition 2.7.[32]

A $TrapN_{number}$ is a subset of $\widetilde{N_s}$ in $\mathbb{R}$ with the following $truth_{function}$, $indeterministic_{function}$, $falsity_{function}$ is defined as

$$\mathcal{T}_{\widetilde{N_s}}(x) = \begin{cases} \left(\frac{x-a_1}{a_2-a_1}\right)\omega_{\widetilde{N_s}} & if\ a_1 \le x \le a_2 \\ \omega_{\widetilde{N_s}} & if\ a_2 \le x \le a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right) & if\ a_3 \le x \le a_4 \\ 0 & otherwise \end{cases}$$

$$\mathcal{I}_{\widetilde{N_s}}(x) = \begin{cases} \left(\frac{a_2-x}{a_2-a_1}\right)\delta_{\widetilde{N_s}} & if\ a_1 \le x \le a_2 \\ \delta_{\widetilde{N_s}} & if\ a_2 \le x \le a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right) & if\ a_3 \le x \le a_4 \\ 1 & otherwise \end{cases} \quad (2.3)$$

$$\mathcal{F}_{\widetilde{N_s}}(x) = \begin{cases} \left(\frac{a_2-x}{a_2-a_1}\right)\varepsilon_{\widetilde{N_s}} & if\ a_1 \le x \le a_2 \\ \varepsilon_{\widetilde{N_s}} & if\ a_2 \le x \le a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right) & if\ a_3 \le x \le a_4 \\ 1 & otherwise \end{cases}$$

where $^-0 \le \mathcal{T}_{\widetilde{N_s}}(x) + \mathcal{I}_{\widetilde{N_s}}(x) + \mathcal{F}_{\widetilde{N_s}}(x) \le 3^+, x \in \widetilde{N_s}$.

Definition 2.8.[32]

The $\left(\widetilde{\mathrm{A}}, \widetilde{\mathrm{B}}, \widetilde{\Gamma}\right)_{cut}$ of a $TrapN_{number} = \langle (a_1, a_2, a_3, a_4); \omega_{\widetilde{N_s}}, \delta_{\widetilde{N_s}}, \varepsilon_{\widetilde{N_s}} \rangle$ is defined as follows:

$\widetilde{N_s}_{(\widetilde{\mathrm{A}},\widetilde{\mathrm{B}},\widetilde{\Gamma})} = \left[\mathcal{T}_{\widetilde{N_{s1}}}(\widetilde{\mathrm{A}}), \mathcal{T}_{\widetilde{N_{s2}}}(\widetilde{\mathrm{A}}); \mathcal{I}_{\widetilde{N_{s1}}}(\widetilde{\mathrm{B}}), \mathcal{I}_{\widetilde{N_{s2}}}(\widetilde{\mathrm{B}}); \mathcal{F}_{\widetilde{N_{s1}}}(\widetilde{\Gamma}), \mathcal{F}_{\widetilde{N_{s2}}}(\widetilde{\Gamma})\right]$, where

$$\begin{aligned} &\mathcal{T}_{\widetilde{N_{s1}}}(\widetilde{\mathrm{A}}) = \left[a_1 + \widetilde{\mathrm{A}}(a_2 - a_1)\right]\omega_{\widetilde{N_s}}, \mathcal{T}_{\widetilde{N_{s2}}}(\widetilde{\mathrm{A}}) = \left[a_4 - \widetilde{\mathrm{A}}(a_4 - a_3)\right]\omega_{\widetilde{N_s}} \\ &\mathcal{I}_{\widetilde{N_{s1}}}(\widetilde{\mathrm{A}}) = \left[a_2 - \widetilde{\mathrm{B}}(a_2 - a_1)\right]\delta_{\widetilde{N_s}}, \mathcal{I}_{\widetilde{N_{s2}}}(\widetilde{\mathrm{A}}) = \left[a_3 + \widetilde{\mathrm{B}}(a_4 - a_3)\right]\delta_{\widetilde{N_s}} \\ &\mathcal{F}_{\widetilde{N_{s1}}}(\widetilde{\mathrm{A}}) = \left[a_2 - \widetilde{\Gamma}(a_2 - a_1)\right]\varepsilon_{\widetilde{N_s}}, \mathcal{F}_{\widetilde{N_{s2}}}(\widetilde{\mathrm{A}}) = \left[a_3 + \widetilde{\Gamma}(a_4 - a_3)\right]\varepsilon_{\widetilde{N_s}} \end{aligned} \quad (2.4)$$

Here $0 \le \widetilde{\mathrm{A}} \le 1, 0 \le \widetilde{\mathrm{B}} \le 1, 0 \le \widetilde{\Gamma} \le 1$ and $^-0 \le \widetilde{\mathrm{A}} + \widetilde{\mathrm{B}} + \widetilde{\Gamma} \le 3^+$ and is displayed in Figure 2.

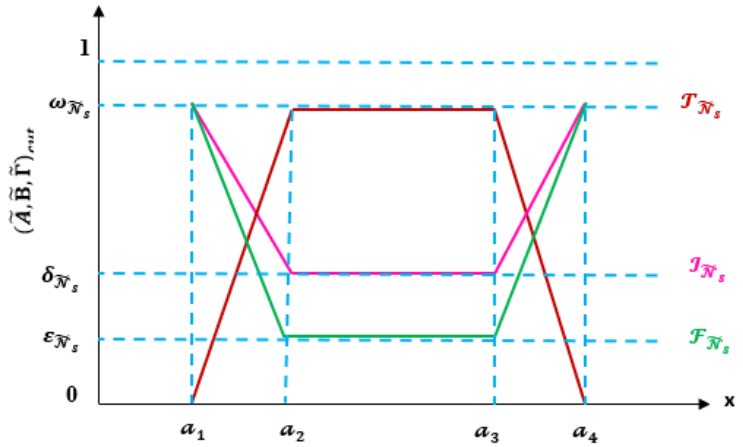

Figure 2. Membership function of $Trap\mathcal{N}_{number}$

Definition 2.9.[32] Let the solution of the neutrosophic differential equation be $y(x)$ and its $(\tilde{A},\tilde{B},\tilde{\Gamma})_{cut}$ be $[y(x,\tilde{A},\tilde{B},\tilde{\Gamma})] = \left[\left(y_1(x,\tilde{A}), y_2(x,\tilde{A}), y_1'(x,\tilde{B}), y_2'(x,\tilde{B}), y_1''(x,\tilde{\Gamma}), y_2''(x,\tilde{\Gamma})\right)\right]$. The solution is a $strong_{solution}$

$$\begin{aligned}
&i. \quad \frac{dy_1(x,\tilde{A})}{d\tilde{A}} > 0, \frac{dy_2(x,\tilde{A})}{d\tilde{A}} < 0, \forall\, \tilde{A} \in [0,1], y_1(x,1) \le y_2(x,1).\\
&ii. \quad \frac{dy_1'(x,\tilde{B})}{d\tilde{B}} < 0, \frac{dy_2'(x,\tilde{B})}{d\tilde{B}} > 0, \forall\, \tilde{B} \in [0,1], y_1'(x,0) \le y_2'(x,0). \qquad (2.5)\\
&iii. \quad \frac{dy_1''(x,\tilde{\Gamma})}{d\tilde{\Gamma}} < 0, \frac{dy_2''(x,\tilde{\Gamma})}{d\tilde{\Gamma}} > 0, \forall\, \tilde{\Gamma} \in [0,1], y_1''(x,0) \le y_2''(x,0).
\end{aligned}$$

Otherwise the solution is $week_{solution}$.

3. Falkner-Skan Equation

The Falkner-Skan equation is a third-order nonlinear ordinary differential equation (ODE) that arises in the study of boundary layer flows over a wedge. Named after V.M. Falkner and S.W. Skan, this equation generalizes the Blasius equation for boundary layer flows over a flat plate to include the effects of a pressure gradient. The governing equation is derived from the boundary layer equations, which are simplifications of the Navier-Stokes equations for high Reynolds number flows where viscous effects are confined to thin regions near solid boundaries. The solution to the Falkner–Skan equation is significant in fluid dynamics as it describes the behavior of the laminar boundary layer over an infinite wedge. The wedge has a vertex angle of $\tilde{\lambda}\pi$, where $0 \le \tilde{\lambda} \le 2$.The parameter $\tilde{\lambda}$ controls the shape and flow characteristics of the boundary layer. The range $0 \le \tilde{\lambda} \le 2$ indicates different flow regimes: 1) $\tilde{\lambda} = 0$ represents the boundary layer over a flat plate (Blasius solution). 2) $0 < \tilde{\lambda} < 1$ describes accelerating flows, where the flow speed increases as it moves along the surface. 3) $\tilde{\lambda} = 1$:

Represents a stagnation point flow, where the fluid approaches the surface perpendicularly. 4) $1 < \tilde{\lambda} \leq 2$: Describes decelerating flows, where the flow speed decreases along the surface.

For a steady, incompressible, two-dimensional boundary layer flow over a wedge, the boundary layer equations in Cartesian coordinates (x, y) are: [33]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} + \varpi\frac{\partial^2 u}{\partial y^2} \tag{3.2}$$

with boundary conditions

$$u = 0, v = 0 \; at \; y = 0 \text{ and } u(x, y) = U(x) \; as \; y \to \infty. \tag{3.3}$$

Here $u$and v are the velocity components in the x and y directions, respectively. U is the external velocity at the edge of the boundary layer. ϖ is the kinematic viscosity of the fluid.

3.1 Similarity Transformation

To simplify these equations, we introduce a similarity variable ζ and a stream function χ defined as: [34]

$$\zeta = y\sqrt{\frac{c(m+1)}{2\varpi_f}x^{m-1}}\;, \chi(x, y) = \left[\frac{2c\varpi_f}{m+1}x^{m+1}\right]^{0.5} f(\zeta) \tag{3.4}$$

where $f(\zeta)$ is the dimensionless stream function. The velocity components can be expressed in terms of $f(\zeta)$ as:

$$u = \frac{\partial \chi}{\partial y} = cx^m f'(\zeta), v = \frac{-\partial \chi}{\partial x} = \left[\frac{-\varpi_f c x^{m-1}(m+1)}{2}\right]^{0.5}\left[\zeta f'(\zeta)\frac{(m-1)}{(m+1)} + f(\zeta)\right] \tag{3.5}$$

Substituting these into the boundary layer equations and simplifying, we obtain the Falkner-Skan equation:

$$f''' + ff' + \tilde{\lambda}[1 - (f')^2] = 0 \tag{3.6}$$

where $\tilde{\lambda}$ is the Falkner-Skan parameter related to the pressure gradient and is defines as: $\tilde{\lambda} = \frac{2m}{m+1}$.

Here, m is related to the external velocity $U(x)$ as $U(x) \propto x^m$.

The boundary conditions for the Falkner-Skan equation are:

$$\left.\begin{matrix} f(\zeta) = 0,\; f'(\zeta) = 0 & as\; \zeta \to 0 \\ f'(\zeta) \to 1 & as\; \zeta \to \infty \end{matrix}\right\} \tag{3.7}$$

These conditions correspond to the no-slip condition at the wall and the matching of the velocity to the external flow far from the wall.

The Falkner-Skan equation is used to analyze the effects of pressure gradients on boundary layer flows. Positive $\tilde{\lambda}$ (or $m > 0$) corresponds to a favorable pressure gradient (accelerating flow), while negative $\tilde{\lambda}$ (or $m < 0$) corresponds to an adverse pressure gradient (decelerating flow).

4. Methodology

4.1 Application of numerical technique

We can use the shooting method to solve boundary value problem (BVP) by converting them into initial value problem (IVP). By transforming the third-order ODE into a first-order ODE and solved using MATLAB's numerical solvers. The shooting method involves guessing the initial conditions for the IVP at $\zeta = 0$, integrating the first-order ODE, and adjusting the guessed initial conditions to satisfy the boundary conditions at the other endpoint (as $\zeta \to \infty$). The tolerance level in the case of the present problem is 10^{-6}. It is highly sensitive to initial guesses for the unknown initial conditions, and poor choices can be to diverges or incorrect solutions. This method may not always converge particularly for highly nonlinear coupled equations. These challenges make this method to less reliable in certain situation.

Let $f = i_1, f' = i_2, f'' = i_3$.

Then the system of equations becomes

$$\begin{bmatrix} i_1' \\ i_2' \\ i_3' \end{bmatrix} = \begin{bmatrix} i_2 \\ i_3 \\ -i_1 i_3 - \tilde{\lambda}[1 - (i_2)^2] \end{bmatrix} \tag{4.1}$$

along with boundary conditions

$$i_1(\zeta) = i_2(\zeta) = 0, \ \ as\ \zeta \to\ 0 \text{ and } i_2(\zeta) = 1\ as\ \zeta \to \infty \tag{4.2}$$

4.1.1 Formulation of Falkner-Skan Equation in Neutrosophic Environment

In the neutrosophic context, the function $f(\zeta)$ and its derivatives will have three components each: $\widetilde{\mathrm{A}}$ (truth), $\widetilde{\mathrm{B}}$ (indeterminacy), and $\tilde{\Gamma}$ (falsity). Denote these as:

$$f(\zeta) = \left(f(\zeta, \widetilde{\mathrm{A}}), f(\zeta, \widetilde{\mathrm{B}}), f(\zeta, \tilde{\Gamma})\right) \tag{4.3}$$

This means that for the function $f(\zeta)$ its derivatives, and the constants, we will consider three corresponding components. The neutrosophic form of the Falkner-Skan equation can then be written as a set of three coupled differential equations:

$$\left.\begin{aligned} f'''(\zeta, \widetilde{\mathrm{A}}) + f(\zeta, \widetilde{\mathrm{A}}) f'(\zeta, \widetilde{\mathrm{A}}) + \tilde{\lambda}^{\widetilde{\mathrm{A}}}\left[1 - \left(f'(\zeta, \widetilde{\mathrm{A}})\right)^2\right] = 0 \\ f'''(\zeta, \widetilde{\mathrm{B}}) + f(\zeta, \widetilde{\mathrm{B}}) f'(\zeta, \widetilde{\mathrm{B}}) + \tilde{\lambda}^{\widetilde{\mathrm{B}}}\left[1 - \left(f'(\zeta, \widetilde{\mathrm{B}})\right)^2\right] = 0 \\ f'''(\zeta, \tilde{\Gamma}) + f(\zeta, \tilde{\Gamma}) f'(\zeta, \tilde{\Gamma}) + \tilde{\lambda}^{\tilde{\Gamma}}\left[1 - \left(f'(\zeta, \tilde{\Gamma})\right)^2\right] = 0 \end{aligned}\right\} \tag{4.4}$$

with the transformed boundary condition

$$\left.\begin{array}{l} f(\zeta,\widetilde{\mathrm{A}})=\tilde{0},\ f'(\zeta,\widetilde{\mathrm{A}})=\tilde{0}\ as\ \zeta\to\ 0\ ,\ f'(\zeta,\widetilde{\mathrm{A}})\to 1\ as\ \zeta\to\infty \\ f(\zeta,\widetilde{\mathrm{B}})=\tilde{0},\ f'(\zeta,\widetilde{\mathrm{B}})=\tilde{0} as\ \zeta\to\ 0\ ,\ f'(\zeta,\widetilde{\mathrm{B}})\to 1\ as\ \zeta\to\infty \\ f(\zeta,\tilde{\Gamma})=\tilde{0},\ f'(\zeta,\tilde{\Gamma})=\tilde{0}\ as\ \zeta\to\ 0\ ,\ f'(\zeta,\tilde{\Gamma})\to 1\ as\ \zeta\to\infty \end{array}\right\} \quad (4.5)$$

where the neutrosophic velocity profile $f'(\zeta,\widetilde{\mathrm{A}})=\left[\underline{\mathrm{f}'}(\zeta,\widetilde{\mathrm{A}}),\overline{\mathrm{f}'}(\zeta,\widetilde{\mathrm{A}})\right], 0\le\widetilde{\mathrm{A}}\le 1$. The term $\underline{\mathrm{f}'}(\zeta,\widetilde{\mathrm{A}})$ and $\overline{\mathrm{f}'}(\zeta,\widetilde{\mathrm{A}})$ represents the lower and upper bounds of the neutrosophic velocity, respectively for $\widetilde{\mathrm{A}}$ (truth). Similarly, $f'(\zeta,\widetilde{\mathrm{B}})=\left[\underline{\mathrm{f}'}(\zeta,\widetilde{\mathrm{B}}),\overline{\mathrm{f}'}(\zeta,\widetilde{\mathrm{B}})\right]$, $f'(\zeta,\tilde{\Gamma})=\left[\underline{\mathrm{f}'}(\zeta,\tilde{\Gamma}),\overline{\mathrm{f}'}(\zeta,\tilde{\Gamma})\right], 0\le\widetilde{\mathrm{B}}\le 1, 0\le\tilde{\Gamma}\le 1$ for $\widetilde{\mathrm{B}}$ (indeterminacy) and $\tilde{\Gamma}$ (falsity), respectively and $^{-}0\le\widetilde{\mathrm{A}}+\widetilde{\mathrm{B}}+\tilde{\Gamma}\le 3^{+}$.

4.2 Application of semi-analytic technique

The basic concept of the HPM to a non-linear differential equation is

$$R(C)-s(\widetilde{\Lambda})=0,\qquad \widetilde{\Lambda}\in\Omega \quad (4.6)$$

The corresponding boundary conditions are

$$Z\left(C,\frac{\partial C}{\partial n}\right)=0\ ,\widetilde{\Lambda}\in\Gamma \quad (4.7)$$

where R is a general non-linear differential operator, B is a boundary operator, $s(\widetilde{\Lambda})$ a known analytical function and $\widetilde{\Lambda}$ the boundary of domain Ω.

The operator R can be divided into two namely L and N, where L is the linear operator, while N is a non-linear operator of Eq. (4.6) which can be written as:

$$L(C)+N(C)-s(\widetilde{\Lambda})=0,\widetilde{\Lambda}\in\Omega \quad (4.8)$$

Define a homotopy as follows:

$$\mathcal{H}(f,\xi)=(1-\xi)[L(f)]+\xi[N(f)]=0, \quad (4.9)$$

where $\xi=[0,1]$ is an embedding parameter.

Its accuracy depends on the initial approximation, and poor choices may lead to inaccurate solutions. This method may also produce dual solutions, depending on how the homotopy id constructed, creating ambiguity in selecting the correct one. Moreover, it doesn't guarantee convergence highly nonlinear coupled equations.

For Falkner-Skan equation, $L(f)=f'''$ and $N(f)=f'''+ff'+\tilde{\lambda}[1-(f')^2]$. Now construct the homotopy as:

$$(1-\xi)f'''+\xi\left[f'''+ff'+\tilde{\lambda}[1-(f')^2]\right]=0 \quad (4.10)$$

Assume a solution in the form of a power series in ξ:

$$f=f_0+\xi f_1+\xi^2 f_2+\cdots=\sum_{j=1}^{n}\xi^j f_j \quad (4.11)$$

Substituting Eq. (4.11) into Eq. (4.10), we get the co-efficient of different powers of ξ:

$$\xi^0: f_0'''=0 \quad (4.12)$$

$$f_0(0)=f_0'(0)=0\ ,f_0'(\infty)=1$$

$$\xi^1: f_1'''=-f_0f_0''-\tilde{\lambda}[1-(f_0')^2] \quad (4.13)$$

$$f_1(0) = f_1'(0) = 0\,, f_1'(\infty) = 0$$
$$\xi^2 : f_2''' = -f_0 f_1'' - f_0'' f_1 + 2\tilde{\lambda} f_0' f_1' \tag{4.14}$$
$$f_2(0) = f_2'(0) = 0\,, f_2'(\infty) = 0$$

Solving (4.12), (4.13) and (4.14), we have

$$\left.\begin{aligned} f_0(\zeta) &= \frac{\zeta^2}{2\zeta_\infty} \\ f_1(\zeta) &= -\frac{k}{60}\zeta^5 - \frac{\tilde{\lambda}}{6}\zeta^3 \\ f_2(\zeta) &= \frac{\left[\left(\frac{(1-\tilde{\lambda})k + (k/10)}{1764}\right)\zeta^8 + \left(\frac{(2\tilde{\lambda}/3) - \tilde{\lambda}^2}{120}\right)\zeta^6\right]}{\zeta_\infty} \end{aligned}\right\} \tag{4.15}$$

4.2.1 Formulation of Falkner-Skan Equation in Neutrosophic Environment

We introduce a neutrosophic approach to solve uncertainty parameter $\tilde{\lambda}$ and boundary condition, using $TriN_{number}$ and $TrapN_{number}$. Our methodology involves applying $\left(\widetilde{A}, \widetilde{B}, \tilde{\Gamma}\right)_{cut}$ to represent various degrees of truth, indeterminacy, and falsity. We developed a solution procedure that integrates these neutrosophic numbers into HPM.

$$\left.\begin{aligned} (1-\zeta)\, f'''\left(\zeta, \widetilde{A}\right) + \zeta\left[f'''\left(\zeta, \widetilde{A}\right) + f\left(\zeta, \widetilde{A}\right) f'\left(\zeta, \widetilde{A}\right) + \tilde{\lambda}^{\widetilde{A}}\left[1 - \left(f'\left(\zeta, \widetilde{A}\right)\right)^2\right]\right] = 0 \\ (1-\zeta) f'''\left(\zeta, \widetilde{B}\right) + \zeta\left[f'''\left(\zeta, \widetilde{B}\right) + f\left(\zeta, \widetilde{B}\right) f'\left(\zeta, \widetilde{B}\right) + \tilde{\lambda}^{\widetilde{B}}\left[1 - \left(f'\left(\zeta, \widetilde{B}\right)\right)^2\right]\right] = 0 \\ (1-\zeta) f'''\left(\zeta, \tilde{\Gamma}\right) + \zeta\left[f'''\left(\zeta, \tilde{\Gamma}\right) + f\left(\zeta, \tilde{\Gamma}\right) f'\left(\zeta, \tilde{\Gamma}\right) + \tilde{\lambda}^{\tilde{\Gamma}}\left[1 - \left(f'\left(\zeta, \tilde{\Gamma}\right)\right)^2\right]\right] = 0 \end{aligned}\right\} \tag{4.16}$$

with the transformed boundary condition

$$\left.\begin{aligned} f\left(\zeta, \widetilde{A}\right) = \tilde{0},\ f'\left(\zeta, \widetilde{A}\right) = \tilde{0}\ as\ \zeta \to 0\ ,\ f'\left(\zeta, \widetilde{A}\right) \to 1\ as\ \zeta \to \infty \\ f\left(\zeta, \widetilde{B}\right) = \tilde{0},\ f'\left(\zeta, \widetilde{B}\right) = \tilde{0}\ as\ \zeta \to 0\ ,\ f'\left(\zeta, \widetilde{B}\right) \to 1\ as\ \zeta \to \infty \\ f\left(\zeta, \tilde{\Gamma}\right) = \tilde{0},\ f'\left(\zeta, \tilde{\Gamma}\right) = \tilde{0}\ as\ \zeta \to 0\ ,\ f'\left(\zeta, \tilde{\Gamma}\right) \to 1\ as\ \zeta \to \infty \end{aligned}\right\} \tag{4.17}$$

where the neutrosophic velocity profile $f'\left(\zeta, \widetilde{A}\right) = \left[\underline{f'}\left(\zeta, \widetilde{A}\right), \overline{f'}\left(\zeta, \widetilde{A}\right)\right], 0 \le \widetilde{A} \le 1$. The term $\underline{f'}\left(\zeta, \widetilde{A}\right)$ and $\overline{f'}\left(\zeta, \widetilde{A}\right)$ represents the lower and upper bounds of the neutrosophic velocity, respectively for $\widetilde{A}$ (truth). Similarly, $f'\left(\zeta, \widetilde{B}\right) = \left[\underline{f'}\left(\zeta, \widetilde{B}\right), \overline{f'}\left(\zeta, \widetilde{B}\right)\right]$, $f'\left(\zeta, \tilde{\Gamma}\right) = \left[\underline{f'}\left(\zeta, \tilde{\Gamma}\right), \overline{f'}\left(\zeta, \tilde{\Gamma}\right)\right], 0 \le \widetilde{B} \le 1, 0 \le \tilde{\Gamma} \le 1$ for $\widetilde{B}$ (indeterminacy) and $\tilde{\Gamma}$ (falsity), respectively and $^{-}0 \le \widetilde{A} + \widetilde{B} + \tilde{\Gamma} \le 3^{+}$.

The $TriN_{number}$ and $TriN_{number}$ are transformed using $\left(\widetilde{A}, \widetilde{B}, \tilde{\Gamma}\right)_{cut}$ technique shown in Table 1 and 2.

	Crisp value	$\mathcal{N}_{number}$	**Alpha cut approach**	**Beta cut approach**	**Gamma cut approach**
$\boldsymbol{TriN_{number}}$	[0.01-0.4]	[0.2,0.4,0.6;0.7,0.3,0.5]	[(0.2+0.2$\widetilde{A}$)0.7, (0.6 – 0.2$\widetilde{A}$)0.7]	[(0.4-0.2$\widetilde{B}$)0.3, (0.4 + 0.2$\widetilde{B}$)0.3]	[(0.4-0.2$\tilde{\Gamma}$)0.5, (0.4 + 0.2$\tilde{\Gamma}$)0.5]

$\boldsymbol{TrapN_{number}}$	[0.01-0.4]	[0.2,0.4,0.6,0.8;0.7,0.3,0.5]	[(0.2+0.2$\widetilde{A}$)0.7, (0.8 – 0.2$\widetilde{A}$)0.7]	[(0.4-0.2$\widetilde{B}$)0.3, (0.6 + 0.2$\widetilde{B}$)0.3]	[(0.4-0.2$\widetilde{\Gamma}$)0.5, (0.6 + 0.2$\widetilde{\Gamma}$)0.5]

Table 1. Parametric form of $TriN_{number}$ and $TrapN_{number}$ for wedge angle parameter $\tilde{\lambda}$

Table 2. Parametric form of $TriN_{number}$ and $TrapN_{number}$ for no slip conditions

	No slip Conditions	**Crisp value**	$\boldsymbol{N_{number}}$	**Alpha cut approach**	**Beta cut approach**	**Gamma cut approach**
$\boldsymbol{TriN_{number}}$	$f(\zeta) = \tilde{0}$,	[0.01-0.4]	[0.1,0.3,0.5; 0.6,0.4,0.2]	[(0.1+0.2$\widetilde{A}$)0.6, (0.5 – 0.2$\widetilde{A}$)0.6]	[(0.3-0.2$\widetilde{B}$)0.4, (0.3 + 0.2$\widetilde{B}$)0.4]	[(0.3-0.2$\widetilde{\Gamma}$)0.2, (0.3 + 0.2$\widetilde{\Gamma}$)0.2]
	$f'(\zeta) = \tilde{0}$	[0.01-0.4]	[0.2,0.4,0.6; 0.7,0.3,0.5]	[(0.2+0.2$\widetilde{A}$)0.7, (0.6 – 0.2$\widetilde{A}$)0.7]	[(0.4-0.2$\widetilde{B}$)0.3, (0.4 + 0.2$\widetilde{B}$)0.3]	[(0.4-0.2$\widetilde{\Gamma}$)0.5, (0.4 + 0.2$\widetilde{\Gamma}$)0.5]
$\boldsymbol{TrapN_{number}}$	$f(\zeta) = \tilde{0}$	[0.01-0.4]	[0.1,0.3,0.5,0.7; 0.6,0.4,0.2]	[(0.1+0.2$\widetilde{A}$)0.6, (0.7 – 0.2$\widetilde{A}$)0.6]	[(0.3-0.2$\widetilde{B}$)0.4, (0.5 + 0.2$\widetilde{B}$)0.4]	[(0.3-0.2$\widetilde{\Gamma}$)0.2, (0.5+ 0.2$\widetilde{\Gamma}$)0.2]
	$f'(\zeta) = \tilde{0}$	[0.01-0.4]	[0.2,0.4,0.6,0.8; 0.7,0.3,0.5]	[(0.2+0.2$\widetilde{A}$)0.7, (0.8 – 0.2$\widetilde{A}$)0.7]	[(0.4-0.2$\widetilde{B}$)0.3, (0.6 + 0.2$\widetilde{B}$)0.3]	[(0.4-0.2$\widetilde{\Gamma}$)0.5, (0.6 + 0.2$\widetilde{\Gamma}$)0.5]

4.3 Validation

To check the accuracy of the numerical and semi-analytic scheme, the results of $f''(0)$ of Falkner-Skan equation was compared with Zhang et al. [35] and Salama [36].

Table 3. Comparison of $f''(0)$ for various values of $\tilde{\lambda}$

$\tilde{\lambda}$	**Zhang et al. [35]**	**Salama [36]**	**Present (Shooting method)**	**Present (HPM)**
-0.15	0.216362	0.216362	0.216362	0.288217
-0.1	0.319270	0.319270	0.319269	0.359690
0	0.469600	0.469600	0.469599	0.493155
0.5	0.927680	0.927680	0.927680	0.970833
1	1.232587	1.232588	1.232587	1.132440

5. Result and discussion

In this section, we concentrated on the neutrosophic analysis of Falkner-Skan equation. The uncertainty of wedge angle parameter $\tilde{\lambda}$ and no slip conditions are considered as $TriN_{number}$ and $TrapN_{number}$. The governing equation is transformed as a NDE's, are numerically solved using the shooting method and HPM. The solution process involves application of the $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma})_{cut}$ techniques.

5.1 Profile graph of Falkner-Skan Equation via Shooting Method

Figure 3 illustrates that the velocity distribution within the boundary layer increases as $\tilde{\lambda}$ is augmented. The observed trends show that controlling the wedge angle parameter $\tilde{\lambda}$ is an effective

strategy to modulate boundary layer characteristics, particularly in applications requiring specific flow acceleration.

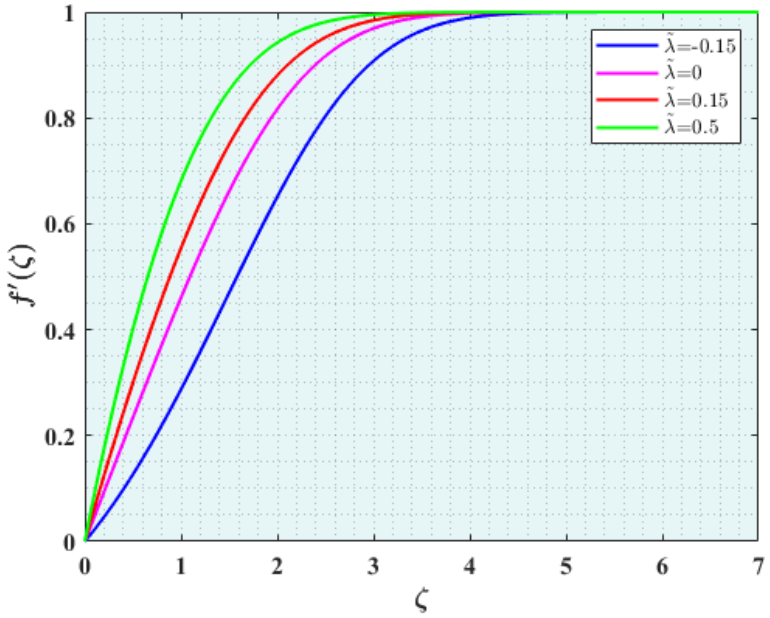

Figure 3. Behavior of $f'(\zeta)$ against $\tilde{\lambda}$

5.1.1 Neutrosophic Analysis using shooting method

Case 1: No slip condition as $\mathcal{N}_{number}$

The no slip conditions are taken as $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. From Figure 4 and 5, we see that $\underline{f'}(\zeta, \tilde{A})$ is increasing and $\overline{f'}(\zeta, \tilde{A})$ is decreasing for all $\tilde{A} \in [0,1]$, $\underline{f'}(\zeta, \tilde{A}) \leq \overline{f'}(\zeta, \tilde{A})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$.

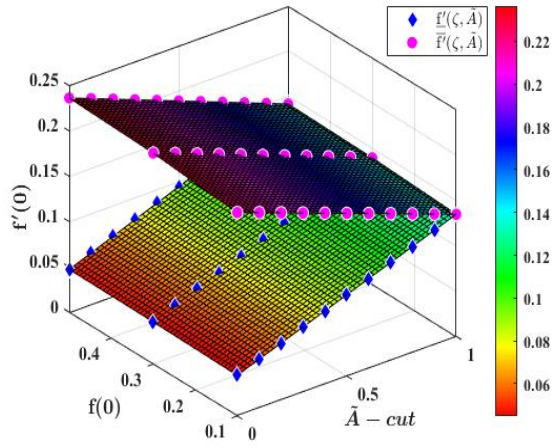

Figure 4. $Tri\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

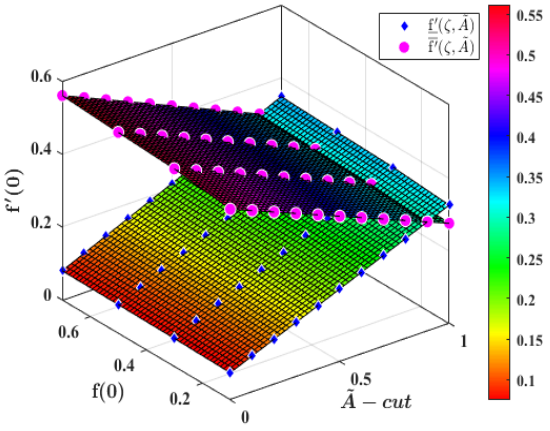

Figure 5. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

From Figure 6 and 7, we see that $\underline{f'}(\zeta, \widetilde{B})$ is decreasing and $\overline{f'}(\zeta, \widetilde{B})$ is increasing for all $\widetilde{B} \in [0,1]$, $\underline{f'}(\zeta, \widetilde{B}) \leq \overline{f'}(\zeta, \widetilde{B})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ which implies that neutrosophic solution exist.

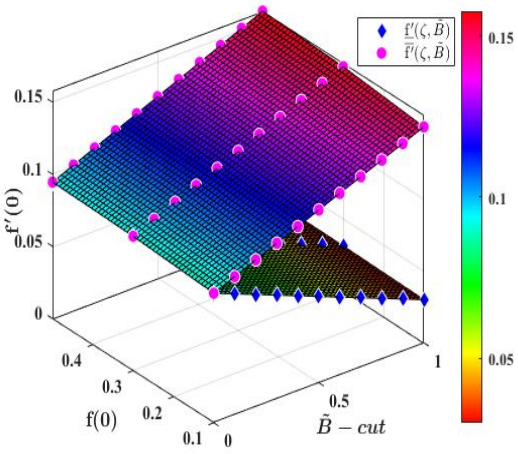

Figure 6. $Tri\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

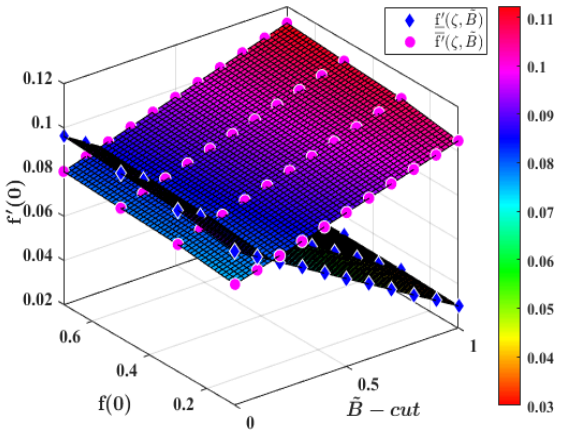

Figure 7. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

From Figure 8 and 9, it shows that $\underline{f'}(\zeta, \tilde{\Gamma})$ is decreasing and $\overline{f'}(\zeta, \tilde{\Gamma})$ is increasing for all $\tilde{\Gamma} \in [0,1]$, $\underline{f'}(\zeta, \tilde{\Gamma}) \leq \overline{f'}(\zeta, \tilde{\Gamma})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. Hence by Definition 2.9, the solution of (3.6), $f'(\zeta, \tilde{A}, \widetilde{B}, \tilde{\Gamma})$ is a strong neutrosophic solution.

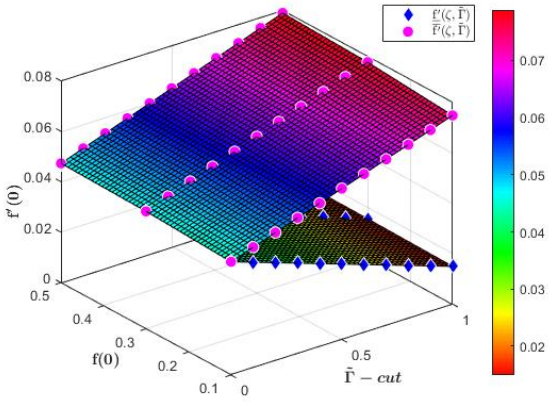

Figure 8. $Tri\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

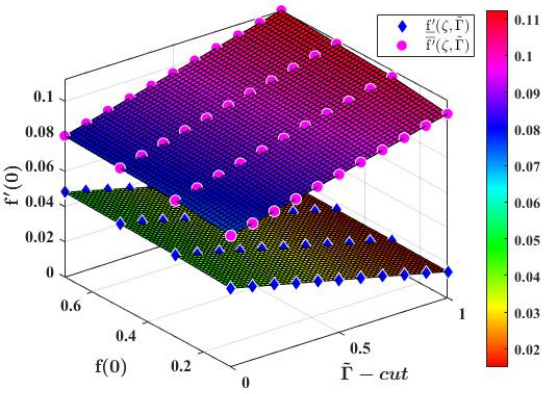

Figure 9. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

From Figure 10 and 11, we see that $\underline{f'}(\zeta, \widetilde{A})$ is increasing and $\overline{f'}(\zeta, \widetilde{A})$ is decreasing for all $\widetilde{A} \in [0,1]$, $\underline{f'}(\zeta, \widetilde{A}) \leq \overline{f'}(\zeta, \widetilde{A})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$.

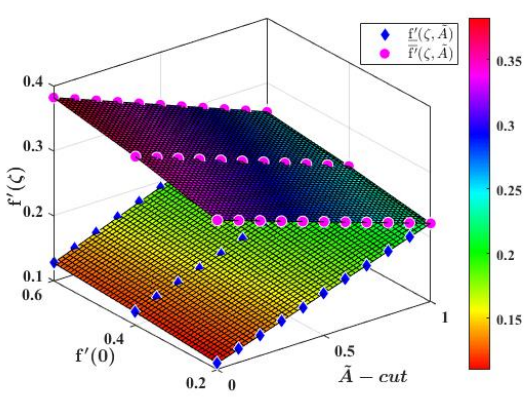

Figure 10. $Tri\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

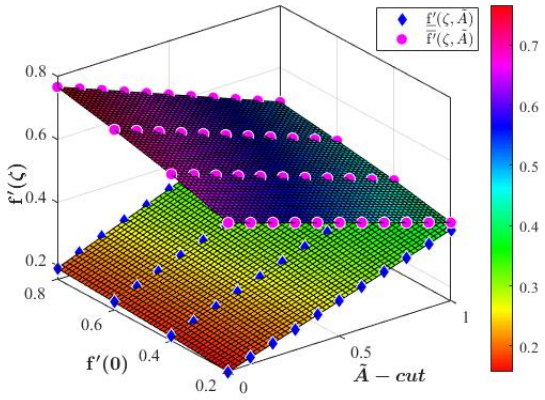

Figure 11. $Trap\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

From Figure 12 and 13, we see that $\underline{f}'(\zeta,\tilde{B})$ is decreasing and $\overline{f}'(\zeta,\tilde{B})$ is increasing for all $\tilde{B} \in [0,1]$, $\underline{f}'(\zeta,\tilde{B}) \leq \overline{f}'(\zeta,\tilde{B})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ which implies that neutrosophic solution exist.

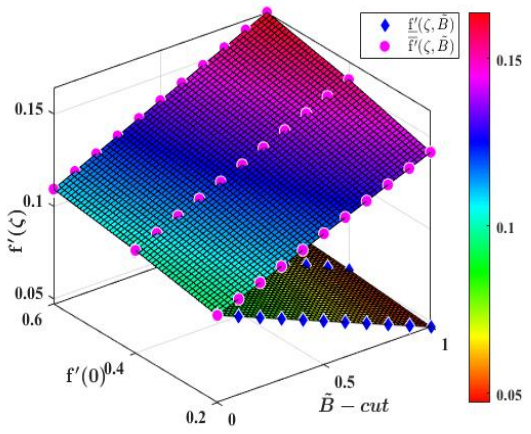

Figure 12. $Tri\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

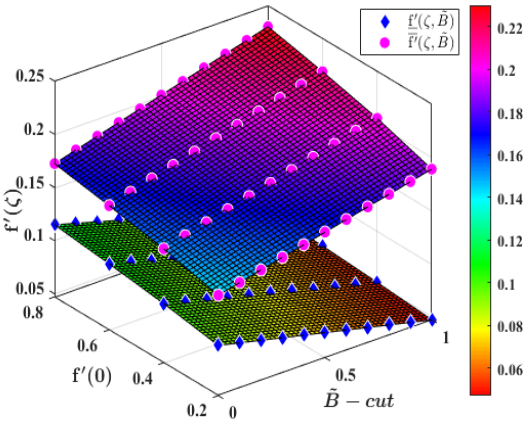

Figure 13. $Trap\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

From Figure 14 and 15, it shows that $\underline{f}'(\zeta,\tilde{\Gamma})$ is decreasing and $\overline{f}'(\zeta,\tilde{\Gamma})$ is increasing for all $\tilde{\Gamma} \in [0,1]$, $\underline{f}'(\zeta,\tilde{\Gamma}) \leq \overline{f}'(\zeta,\tilde{\Gamma})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. Hence by Definition 2.9, the solution of (3.6), $f'(\zeta,\tilde{A},\tilde{B},\tilde{\Gamma})$ is a strong neutrosophic solution.

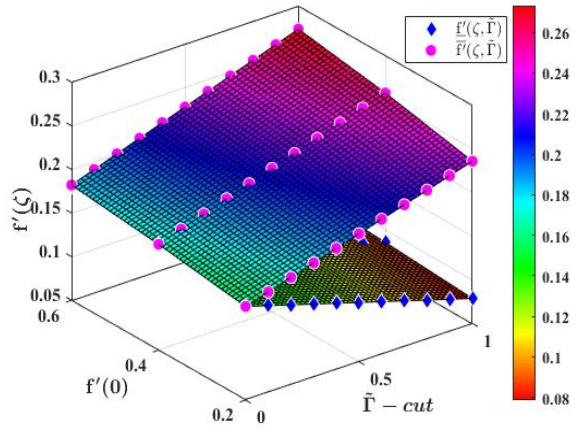

Figure 14. $Tri\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

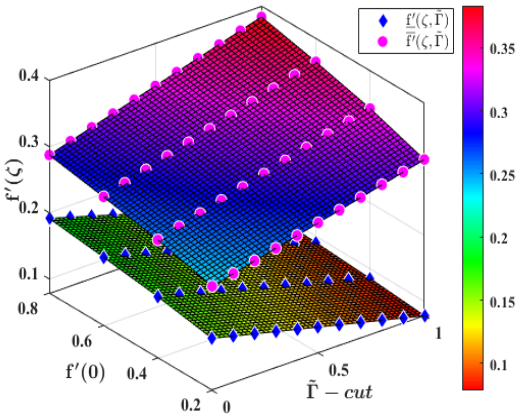

Figure 15. $Trap\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

Case 2: Wedge angle parameter $\tilde{\lambda}$ as $\mathcal{N}_{number}$

Figure 16 and 17 depicts the neutrosophic velocity profiles $f'(\zeta, \tilde{A})$ corresponding to the truth component of the $\mathcal{N}_{number}$. The wedge angle parameter $\tilde{\lambda}$ is taken as $Tri\mathcal{N}_{number}$ [0.2, 0.4, 0.6;0.7, 0.3, 0.5] and $Trap\mathcal{N}_{number}$ [0.2, 0.4, 0.6, 0.8;0.7, 0.3, 0.5] in the Falkner-Skan equation at fixed $\zeta = 3$. The velocity profiles are plotted along the x-axis represents the various cuts $\tilde{A}, \tilde{B}, \tilde{\Gamma}$, the y-axis corresponds to the uncertainty parameter ζ, and the z-axis represents $f'(\zeta, \tilde{A})$. The figures shows that $\underline{f'}(\zeta, \tilde{A})$ is strictly increasing and $\overline{f'}(\zeta, \tilde{A})$ is strictly decreasing for all $\tilde{A} \in [0,1]$, $\underline{f'}(\zeta, \tilde{A}) \leq \overline{f'}(\zeta, \tilde{A})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$.

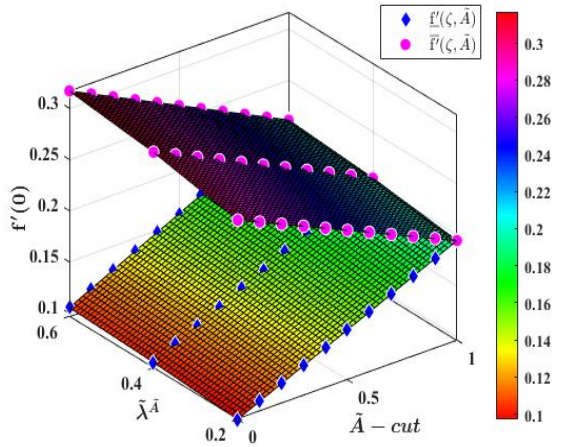

Figure 16. $Tri\mathcal{N}$ velocity for $truth_{function}$ at $\zeta = 3$

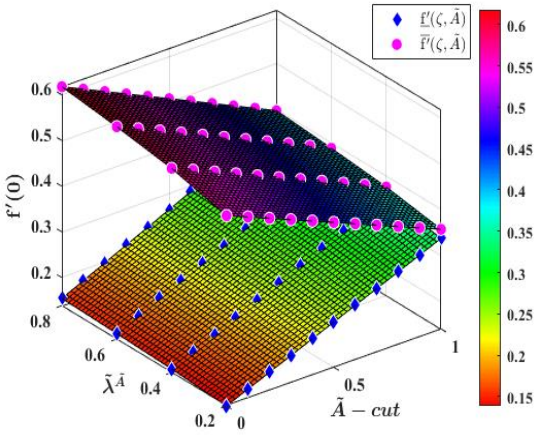

Figure 17. $Trap\mathcal{N}$ velocity for $truth_{function}$ at $\zeta = 3$

From Figure 18 and 19 illustrates that the neutrosophic velocity profile focusing on the indeterminacy. We see that $\underline{f'}(\zeta, \widetilde{B})$ is strictly decreasing and $\overline{f'}(\zeta, \widetilde{B})$ is strictly increasing for all $\widetilde{B} \in [0,1]$, $\underline{f'}(\zeta, \widetilde{B}) \leq \overline{f'}(\zeta, \widetilde{B})$ in both $TriN_{number}$ and $TrapN_{number}$.

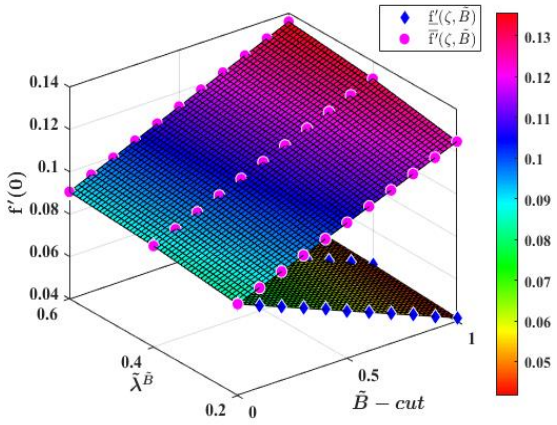

Figure 18. $TriN$ velocity for $indeterministic_{function}$ at $\zeta = 3$

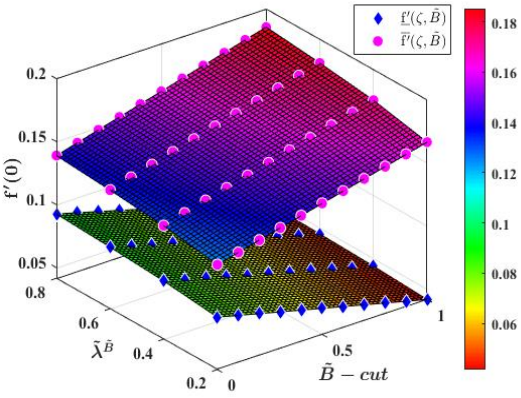

Figure 19. $TrapN$ velocity for $indeterministic_{function}$ at $\zeta = 3$

From Figure 20 and 21 displays $f'(\zeta, \tilde{\Gamma})$ associated with the falsity. It shows that $\underline{f'}(\zeta, \tilde{\Gamma})$ is strictly decreasing and $\overline{f'}(\zeta, \tilde{\Gamma})$ is strictly increasing for all $\tilde{\Gamma} \in [0,1]$, $\underline{f'}(\zeta, \tilde{\Gamma}) \leq \overline{f'}(\zeta, \tilde{\Gamma})$ in both $TriN_{number}$ and $TrapN_{number}$. Hence by Definition 2.9, the solution of (3.6), $f'(\zeta, \tilde{A}, \widetilde{B}, \tilde{\Gamma})$ is a strong neutrosophic solution.

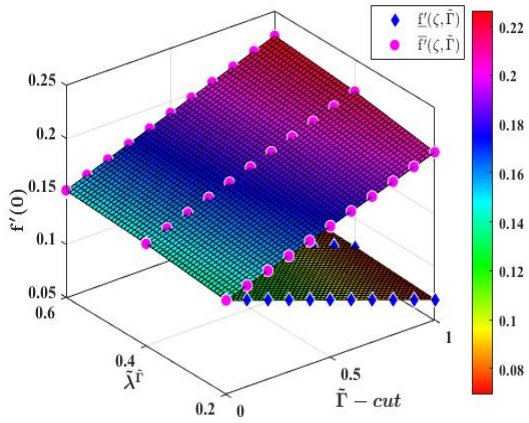

Figure 20. $TriN$ velocity for $falsity_{function}$ at $\zeta = 3$

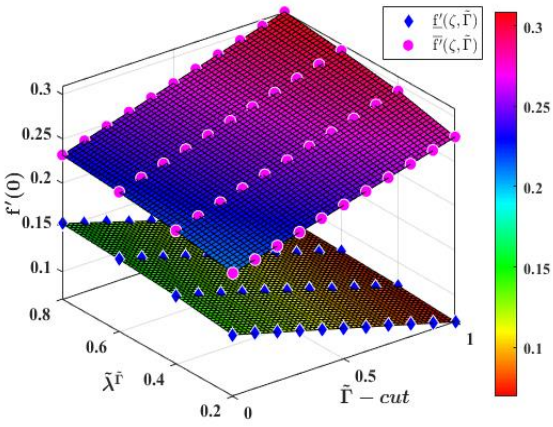

Figure 21. $TrapN$ velocity for $falsity_{function}$ at $\zeta = 3$

5.2 Profile graph of Falkner-Skan Equation via HPM

The velocity profile of $\tilde{\lambda}$ is displayed in Figure 22. The velocity profiles steepen near the wall, indicating that the flow accelerates more rapidly with higher $\tilde{\lambda}$ values. This behavior is attributed to

the enhanced favorable pressure gradient associated with larger $\tilde{\lambda}$ values, which effectively accelerates the fluid particles adjacent to the surface.

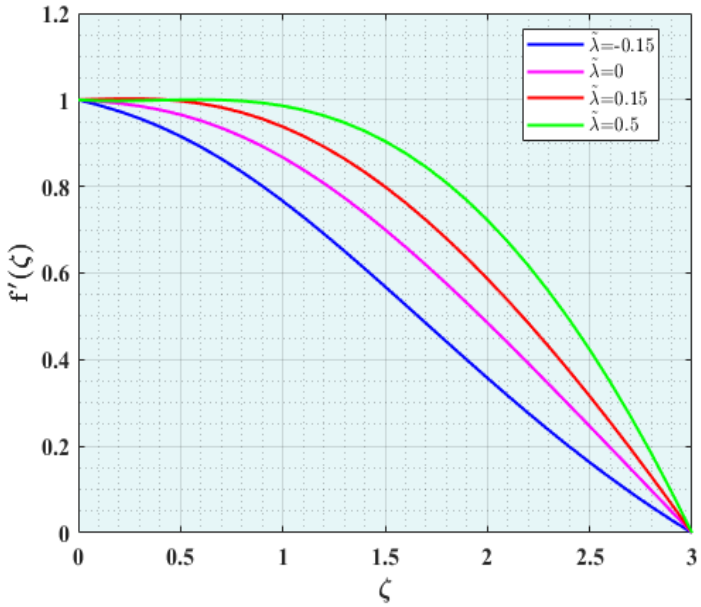

Figure 22. Behavior of $f'(\zeta)$ against $\tilde{\lambda}$

5.2.1 Neutrosophic Analysis using HPM

Case 1: No slip condition as $\mathcal{N}_{number}$

The no slip conditions are taken as $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. From Figure 23 and 24, we see that $\underline{f'}(\zeta, \tilde{A})$ is increasing and $\overline{f'}(\zeta, \tilde{A})$ is decreasing for all $\tilde{A} \in [0,1]$, $\underline{f'}(\zeta, \tilde{A}) \leq \overline{f'}(\zeta, \tilde{A})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$.

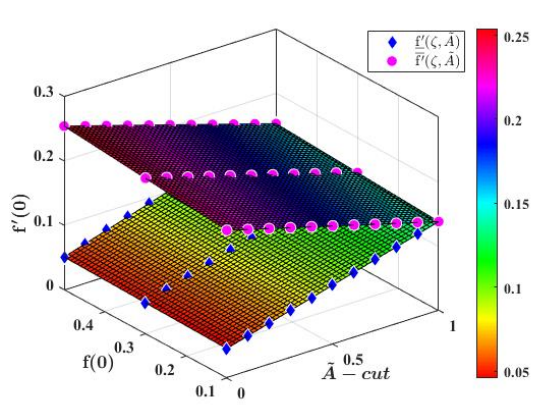

Figure 23. $Tri\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

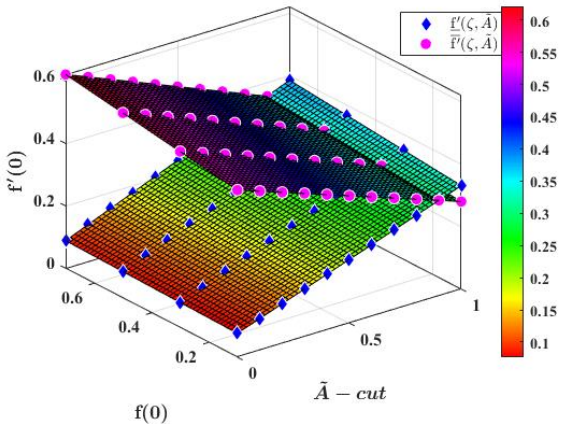

Figure 24. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

From Figure 25 and 26, we see that $\underline{f'}(\zeta, \widetilde{B})$ is decreasing and $\overline{f'}(\zeta, \widetilde{B})$ is increasing for all $\widetilde{B} \in [0,1]$, $\underline{f'}(\zeta, \widetilde{B}) \leq \overline{f'}(\zeta, \widetilde{B})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ which implies that neutrosophic solution exist.

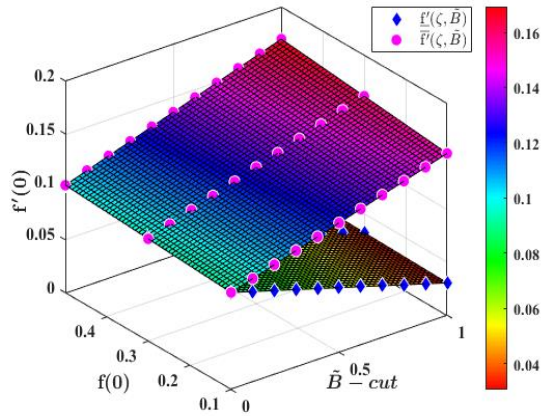

Figure 25. $Tri\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

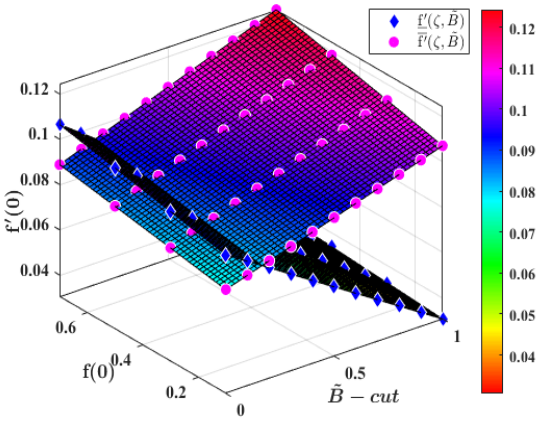

Figure 26. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

From Figure 27 and 28, it shows that $\underline{f'}(\zeta, \tilde{\Gamma})$ is decreasing and $\overline{f'}(\zeta, \tilde{\Gamma})$ is increasing for all $\tilde{\Gamma} \in [0,1]$, $\underline{f'}(\zeta, \tilde{\Gamma}) \leq \overline{f'}(\zeta, \tilde{\Gamma})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. Hence by Definition 2.9, the solution of (3.6), $f'(\zeta, \tilde{A}, \widetilde{B}, \tilde{\Gamma})$ is a strong neutrosophic solution.

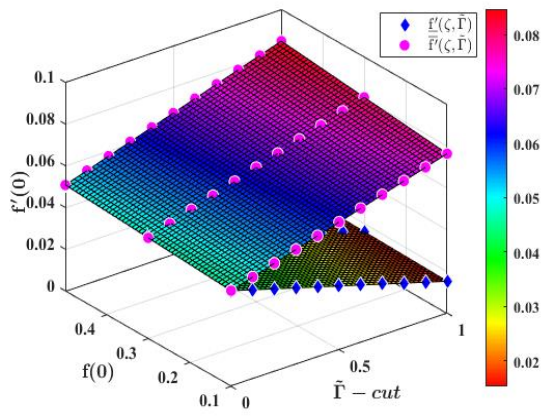

Figure 27. $Tri\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

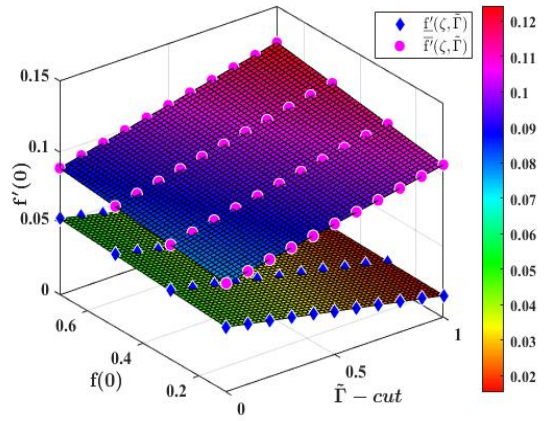

Figure 28. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta =$

From Figure 29 and 30, we see that $\underline{f'}(\zeta, \widetilde{A})$ is increasing and $\overline{f'}(\zeta, \widetilde{A})$ is decreasing for all $\widetilde{A} \in [0,1]$, $\underline{f'}(\zeta, \widetilde{A}) \leq \overline{f'}(\zeta, \widetilde{A})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$.

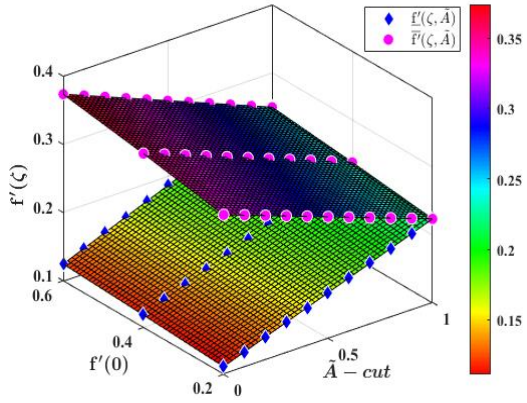

Figure 29. $Tri\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

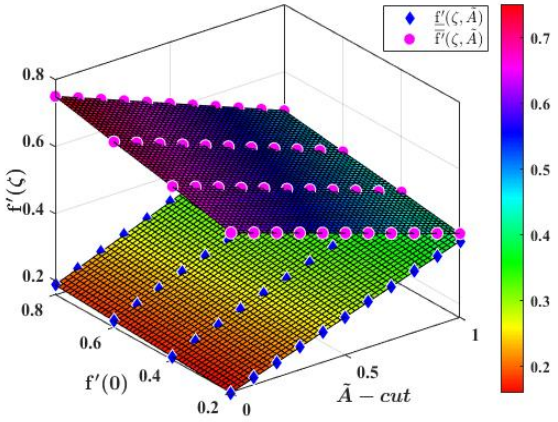

Figure 30. $Trap\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

From Figure 31 and 32, we see that $\underline{f}'(\zeta,\tilde{B})$ is decreasing and $\overline{f}'(\zeta,\tilde{B})$ is increasing for all $\tilde{B} \in [0,1]$, $\underline{f}'(\zeta,\tilde{B}) \leq \overline{f}'(\zeta,\tilde{B})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ which implies that neutrosophic solution exist.

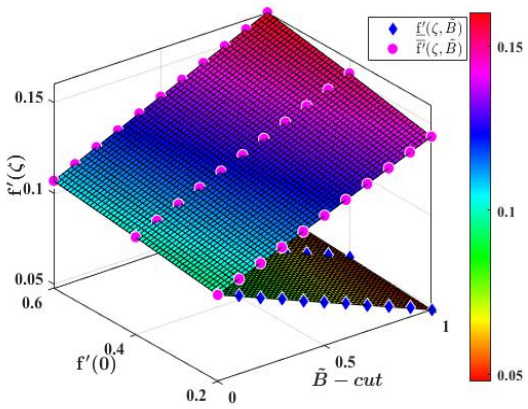

Figure 31. $Tri\mathcal{N}$ velocity for $f'(0)$ at $\zeta = 3$

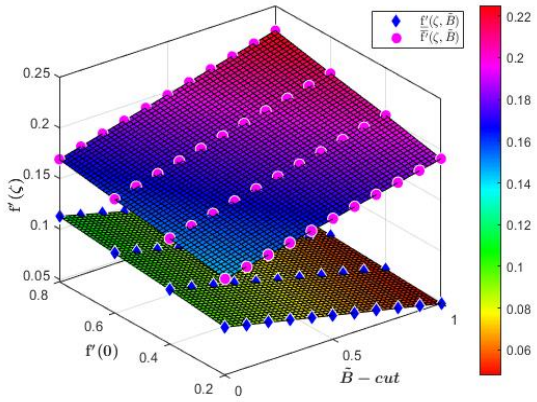

Figure 32. $Trap\mathcal{N}$ velocity for $f(0)$ at $\zeta = 3$

From Figure 33 and 34, it shows that $\underline{f}'(\zeta,\tilde{\Gamma})$ is decreasing and $\overline{f}'(\zeta,\tilde{\Gamma})$ is increasing for all $\tilde{\Gamma} \in [0,1]$, $\underline{f}'(\zeta,\tilde{\Gamma}) \leq \overline{f}'(\zeta,\tilde{\Gamma})$ in both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. Hence by Definition 2.9, the solution of (3.6), $f'(\zeta,\tilde{A},\tilde{B},\tilde{\Gamma})$ is a strong neutrosophic solution.

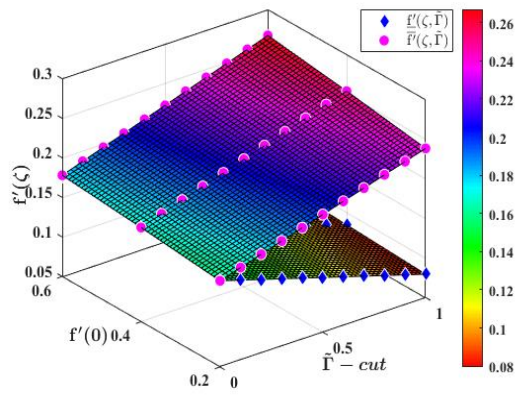

Figure 33. $TriN$ velocity for $f(0)$ at $\zeta = 3$

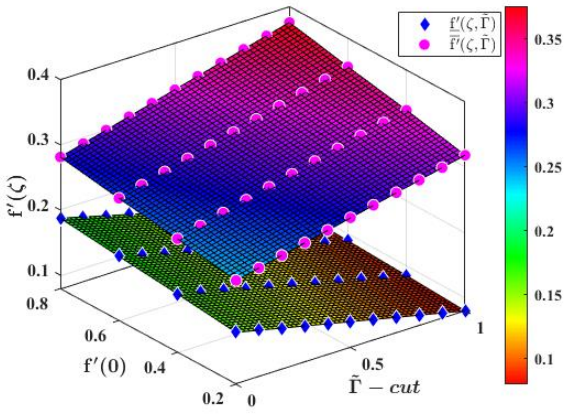

Figure 34. $TrapN$ velocity for $f(0)$ at $\zeta = 3$

Case 2: Wedge angle parameter $\tilde{\lambda}$ as $\mathcal{N}_{number}$

Figure 35 and 36 demonstrates $f'(\zeta,\tilde{A})$ derived from the truth aspect of the $\mathcal{N}_{number}$. The analysis reveals that both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$, $\underline{f}'(\zeta,\tilde{A})$ is strictly increasing and $\overline{f}'(\zeta,\tilde{A})$ is strictly decreasing for all $\tilde{A} \in [0,1]$, $\underline{f}'(\zeta,\tilde{A}) \leq \overline{f}'(\zeta,\tilde{A})$.

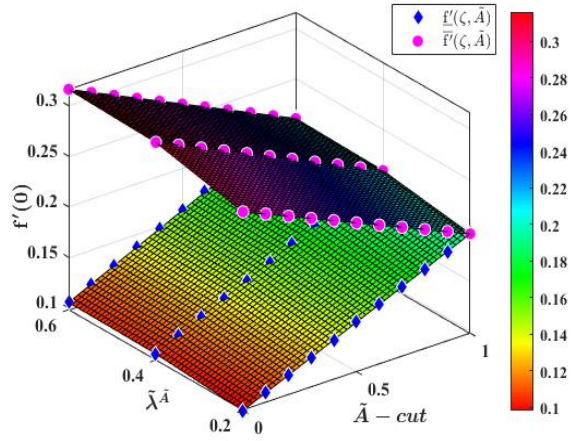

Figure 35. $f'(\zeta,\tilde{A})$ at $\zeta = 3$ using $Tri\mathcal{N}_{number}$

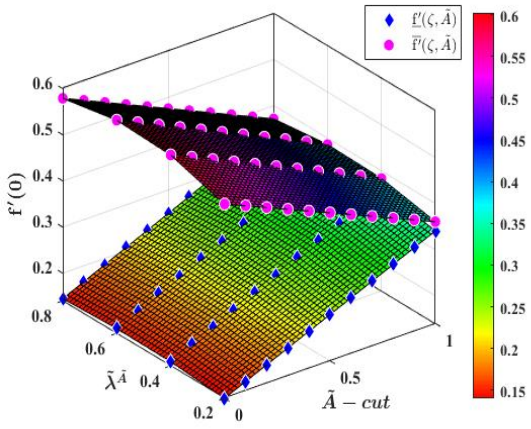

Figure 36. $f'(\zeta,\tilde{A})$ at $\zeta = 3$ using $Trap\mathcal{N}_{number}$

From Figure 37 and 38 showcases the neutrosophic velocity profile corresponding to the indeterminacy aspect. In both $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ we see that $\underline{f}'(\zeta,\tilde{B})$ is strictly decreasing and $\overline{f}'(\zeta,\tilde{B})$ is strictly increasing for all $\tilde{B} \in [0,1]$, $\underline{f}'(\zeta,\tilde{B}) \leq \overline{f}'(\zeta,\tilde{B})$.

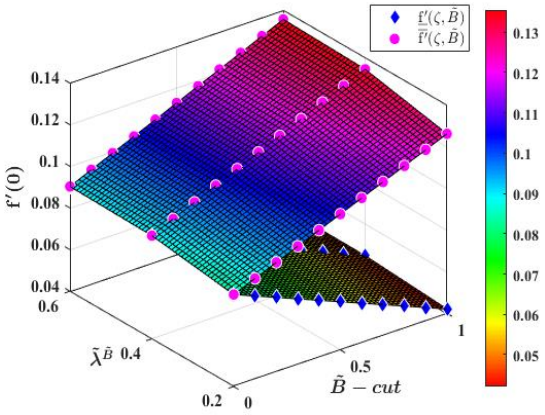

Figure 37. $f'(\zeta,\widetilde{\mathrm{B}})$ at $\zeta=3$ using $Tri\mathcal{N}_{number}$

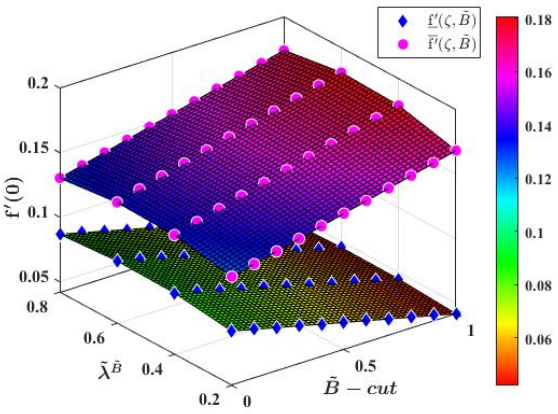

Figure 38. $f'(\zeta,\widetilde{\mathrm{B}})$ at $\zeta=3$ using $Trap\mathcal{N}_{number}$

From Figure 39 and 40 presents the neutrosophic velocity profile related to the falsity aspect. Similar to the observations made for indeterminacy, the $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ reveals that $\underline{f}'(\zeta,\tilde{\Gamma})$ is strictly decreasing and $\overline{f}'(\zeta,\tilde{\Gamma})$ is strictly increasing for all $\tilde{\Gamma}\in[0,1]$, $\underline{f}'(\zeta,\tilde{\Gamma})\le\overline{f}'(\zeta,\tilde{\Gamma})$. Hence by Definition 2.9, the solution of (3.6), $f'(\zeta,\tilde{A},\widetilde{\mathrm{B}},\tilde{\Gamma})$ is a strong neutrosophic solution.

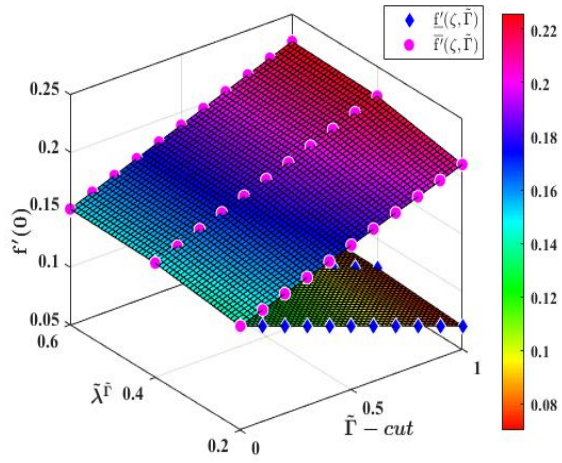

Figure 39. $f'(\zeta,\tilde{\Gamma})$ at $\zeta=3$ using $Tri\mathcal{N}_{number}$

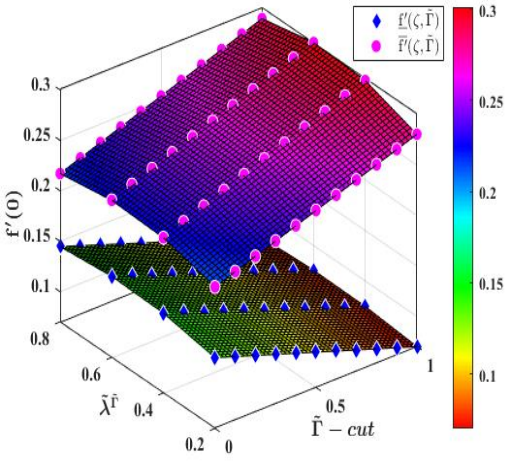

Figure 40. $f'(\zeta,\tilde{\Gamma})$ at $\zeta=3$ using $Trap\mathcal{N}_{number}$

5.3 Comparison of Shooting method and HPM

We computed the absolute error between the values obtained from the Shooting method and the HPM for the $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$. The absolute errors were computed separately for each $(\widetilde{\mathrm{A}},\widetilde{\mathrm{B}},\tilde{\Gamma})_{cut}$. These errors were then visualized in a 3D plot, with the x-axis represents the cuts $\tilde{A},\widetilde{\mathrm{B}},\tilde{\Gamma}$, the y-axis corresponds to the uncertainty parameter $\tilde{\lambda}$, and the z-axis represents the absolute error. The analysis demonstrates that the choice between $Tri\mathcal{N}_{number}$and $Trap\mathcal{N}_{number}$ impacts the

resulting velocity profiles significantly. Based on the results, $TriN_{number}$ and $TrapN_{number}$ exhibits lower errors at higher cuts.

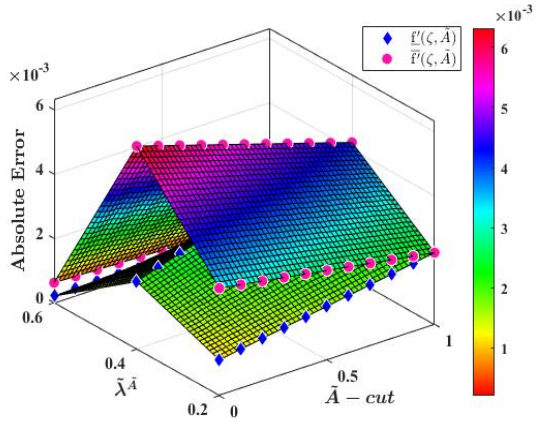

Figure 41. Absolute Error of $f'(\zeta,\tilde{A})$ using $TriN_{number}$

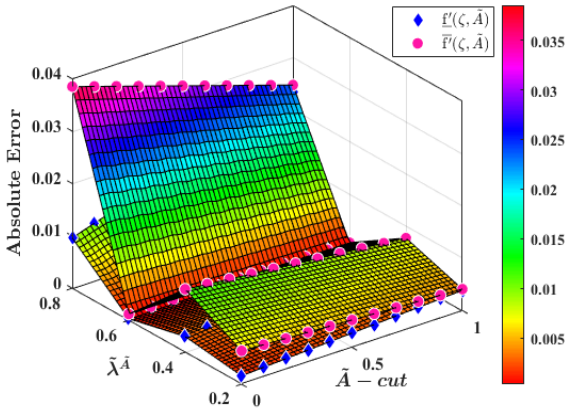

Figure 42. Absolute Error of $f'(\zeta,\tilde{A})$ using $TrapN_{number}$

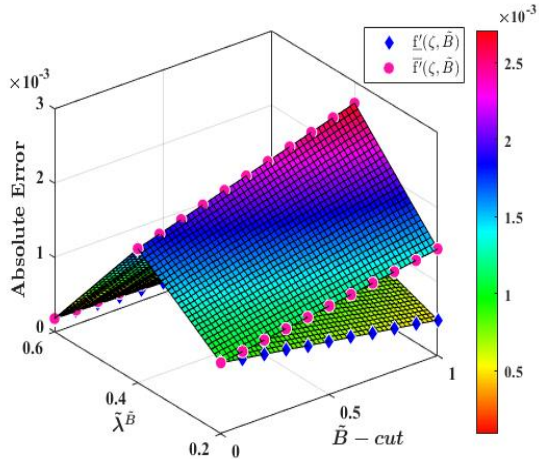

Figure 43. Absolute Error of $f'(\zeta,\tilde{B})$ using $TriN_{number}$

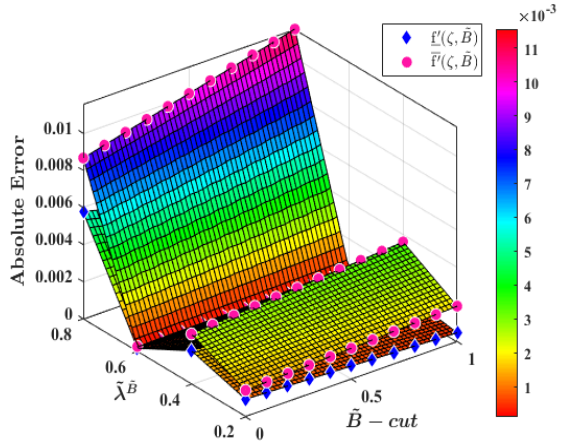

Figure 44. Absolute Error of $f'(\zeta,\tilde{B})$ using $TrapN_{number}$

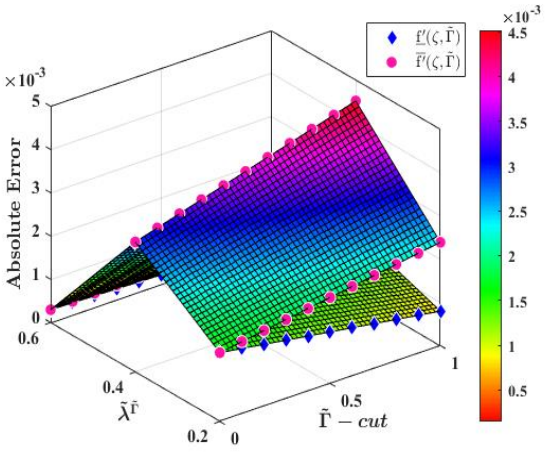

Figure 45. Absolute Error of $f'(\zeta,\tilde{\Gamma})$ using $Tri\mathcal{N}_{number}$

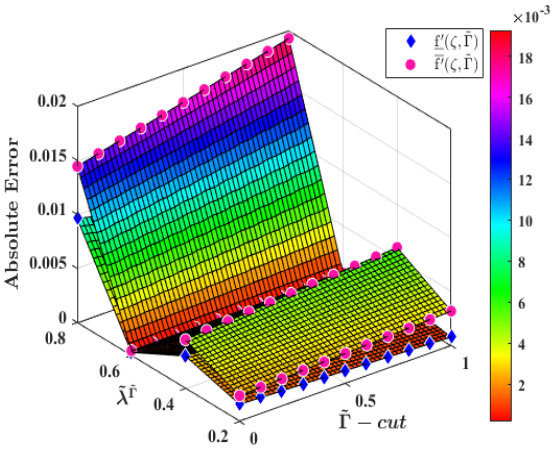

Figure 46. Absolute Error of $f'(\zeta,\tilde{\Gamma})$ using $Trap\mathcal{N}_{number}$

The results, as shown in Table 4, indicate that the $Trap\mathcal{N}_{number}$ exhibits higher truth mean values for both shooting method and HPM. However, the $Tri\mathcal{N}_{number}$consistently shows lower mean indeterminacy and falsity values. $Tri\mathcal{N}_{number}$ gives a better performance in minimizing uncertainty and falsity. The Shooting Method slightly outperforms HPM in achieving a higher truth value for both types of $\mathcal{N}_{number}$. The Shooting Method shows slightly lower indeterminacy for both types, indicates better performance in minimizing uncertainty. The Shooting Method again results in slightly lower falsity values, it is more effective in minimizing errors.

Table 4. Comparison of Mean Values of Neutrosophic Velocity

	Shooting Method		**HPM**	
	$\boldsymbol{Tri\mathcal{N}_{number}}$	$\boldsymbol{Trap\mathcal{N}_{number}}$	$\boldsymbol{Tri\mathcal{N}_{number}}$	$\boldsymbol{Trap\mathcal{N}_{number}}$
$\widetilde{\mathrm{A}}_{cut}$	0.203434	0.368977	0.205443	0.365649
$\widetilde{\mathrm{B}}_{cut}$	0.087186	0.110693	0.088047	0.109695
$\tilde{\Gamma}_{cut}$	0.14531	0.184488	0.146745	0.182824

6. Conclusion

This study focused on neutrosophic analysis of Falkner-Skan boundary layer equation over wedge. The impact of wedge angle parameter $\tilde{\lambda}$ on velocity profile has been studied by employing shooting method and HPM, the NDEs are solved numerically. The wedge parameter $\tilde{\lambda}$ and no slip conditions are considered as $Tri\mathcal{N}_{number}$ and $Trap\mathcal{N}_{number}$ with the help of $(\widetilde{\mathrm{A}},\widetilde{\mathrm{B}},\tilde{\Gamma})_{cut}$ which control uncertainty. The key outcomes of this study are as follows:

- The velocity profile curve $f'(\zeta)$ upsurge for $\tilde{\lambda}$ in both shooting method and HPM.

- The Shooting Method is considered superior for solving the Falkner-Skan equation under uncertainty with $\tilde{\lambda}$.
- Mean values of $Tri\mathcal{N}_{number}$ and $Tri\mathcal{N}_{number}$ are compared, which show triangular membership function performs better than trapezoidal membership function in reducing uncertainty.
- For both $Tri\mathcal{N}_{number}$ and $Tri\mathcal{N}_{number}$, the error was found to vary significantly across different cuts.
- The neutrosophic concept has been applied to Falkner-Skan equation whose solution follows the conditions of strong neutrosophic solution in both methods.

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Conflicts of Interest

The authors declare no conflicts of interest.

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The PAMSSEM Approach for Multi-Attribute Group Decision-Making Using Single-Valued Neutrosophic Sets: Applications to Evaluating the Effectiveness of College Students' Psychological Education

Xiaolan Liang
Guangxi Vocational Normal University, Nanning, 530007, Guangxi, China
*Corresponding author, E-mail: lxl202405@126.com

Abstract: The evaluation of the effectiveness of college students' psychological education involves using scientific approaches and tools to assess the actual impact of psychological education on improving students' mental health, fostering personality development, and enhancing psychological adjustment abilities. The evaluation covers aspects such as students' mental health status, the achievement of educational goals, the effectiveness of interventions, and student satisfaction. Through such evaluations, psychological education programs can be optimized to improve quality, help students better cope with psychological challenges in academics, life, and relationships, and promote their holistic development and healthy growth. The results also provide a scientific basis for continuous improvement of psychological education. The effectiveness evaluation of college students' psychological education is MADM. In this paper, the single-valued neutrosophic sets (SVNSs) and the average are employed to determine the attribute weights within decision-making processes. To address multi-attribute group decision-making (MADM) problems under SVNSs, the single-valued neutrosophic numbers PAMSSEM (SVNN-PAMSSEM) approach is proposed and systematically structured. This approach integrates the advantages of SVNSs, which effectively handle uncertainty, imprecision, and inconsistency in decision-making scenarios, with the PAMSSEM approach, known for its robustness in evaluating alternatives based on multiple attributes. To demonstrate the practicality and effectiveness of the proposed SVNN- PAMSSEM approach, a case study on the effectiveness evaluation of college students' psychological education is presented. This example illustrates how the approach can be applied to assess the outcomes of psychological education programs, highlighting its applicability in real-world scenarios. Additionally, comparative decision analyses with other existing approaches are conducted to validate the SVNN- PAMSSEM approach further. The results confirm its superior performance in handling complex decision-making problems involving linguistic and fuzzy information, proving its value as a robust tool for MADM under SVNSs.

Keywords: MADM; single-valued neutrosophic sets (SVNSs); PAMSSEM approach; effectiveness evaluation; college students' psychological education.

1. Introduction

The evaluation of the effectiveness of psychological education for college students is aimed at scientifically assessing the impact of psychological education on students' mental health and overall quality development. As social pressures increase and mental health issues among college students become more

prominent, psychological education plays a critical role as an intervention and support measure. Effectiveness evaluation helps educators understand the outcomes of psychological education, identify its strengths and weaknesses, and provide a scientific basis for subsequent improvements. This evaluation process not only focuses on the role of psychological education in addressing mental health problems but also examines its impact on students' emotional regulation, stress-coping skills, and personality development. Through systematic and comprehensive analysis, the evaluation reveals the actual effects of psychological education among college students, helping to optimize educational content and methods while enhancing its relevance and effectiveness. Furthermore, effectiveness evaluation is a vital part of ensuring educational quality, offering directions for continuous improvement in psychological education. In addition, evaluating the effectiveness of psychological education for college students carries significant social value. As college students are the future backbone of society, their mental health directly influences social stability and development. Scientific evaluation not only supports students' healthy growth during their academic years but also lays a solid foundation for their psychological adaptation after entering society. A well-conducted evaluation enables psychological education to better meet students' needs and promotes their comprehensive development. In summary, the evaluation of psychological education effectiveness is a process that integrates systematic, scientific, and practical approaches. It is not merely a summary of educational efforts but also a crucial tool for driving the continuous advancement of psychological education. Effective evaluation can greatly improve the quality of psychological education, safeguard the mental health of college students, and contribute to societal harmony and progress. Research on the psychological education of college students has deepened over the past two decades, expanding its focus and improving the educational framework. The following is a chronological summary of the research content in nine papers. The earliest study by Yuan [1] analyzed the characteristics of psychological education and the factors affecting its effectiveness. He emphasized that psychological education differs significantly from other forms of education and highlighted the importance of addressing specific issues based on its unique characteristics to improve its outcomes. This research underlined the theoretical understanding of psychological education. Later, Jing et al. [2, 3] conducted a two-year longitudinal study on the mental health education of Henan Polytechnic University's 2008 cohort. They found that students who received psychological education showed significant improvements in their mental health, with psychological lectures being particularly effective. The study also emphasized that psychological education is critical for

fostering student growth and cultivating innovative talents. Zhan and Lian [4] focused on students in agricultural disciplines at Fujian Agriculture and Forestry University. Through experimental comparisons using the SCL-90 scale, they found significant differences in students' mental health levels before and after receiving education. They proposed building a systematic education framework and implementing need-oriented education strategies. He et al. [5] investigated the effects of crisis education on college students. Their study revealed that crisis education significantly improved students' knowledge of psychological crises, enabling them to adopt effective coping strategies and seek help proactively, thus supporting crisis prevention. Yang and Ouyang [6] tracked college students with abnormal psychological survey results from 2016 to 2018. They found that psychological education effectively improved students' mental health and reduced SCL-90 scores. The study highlighted the importance of addressing specific psychological issues among college students to support their development. Xie and Yao [7] examined the impact of mental health on the effectiveness of online education among agricultural students. They found that mental health significantly influenced students' performance in online learning, particularly in interpersonal relationships and emotional aspects. Qiu et al. [8] explored the application of diary-based psychological education for students with bipolar disorder. Their results showed that the intervention significantly improved mental states, increased positive emotions, reduced negative emotions, optimized stress coping strategies, and enhanced satisfaction with psychological interventions. Pei [9] analyzed the current state of psychological education in art colleges and proposed innovative methods that integrate professional characteristics. She emphasized a student-centered approach to fully realize the educational potential of psychological education and contribute to the cultivation of artistic talents. In conclusion, research on college students' psychological education has shifted from theoretical explorations and general studies to more specific intervention methods and educational practices in recent years. These studies provide valuable theoretical foundations and practical references for improving the effectiveness of psychological education.

The effectiveness evaluation of college students' psychological education is a MADM problem under an SVNN environment. However, there are three main shortcomings in addressing such MADM problems, which form the motivations for this study:

- Complexity of Existing Decision Approaches: Current decision-making approaches often involve complex computational processes [10-21]. Simplifying these approaches while maintaining their

effectiveness is a critical and emerging research focus. Therefore, the first motivation of this study is to develop new decision-making approaches with relatively simpler computational requirements, ensuring they are practical and accessible for real-world applications.

- Limitations of Existing Weighting Approaches: Most existing weighting approaches assume completely known weight information [22-27]. However, in many practical scenarios, such as the evaluation of psychological education, the weight information may be completely unknown. Exploring approaches to handle this uncertainty is both challenging and significant. Thus, the second motivation of this paper is to propose a novel weight determination approach that can effectively address situations with completely unknown weight information.
- Lack of Research on Psychological Education Evaluation: The effectiveness evaluation of college students' psychological education in China remains largely unexplored. This gap highlights the need for exploratory research to advance the understanding of this field and enrich its theoretical and practical content. Recognizing the importance of psychological education in fostering students' mental well-being, this study aims to address the lack of systematic decision-making frameworks for its evaluation. Therefore, the third motivation is to design new MADM approaches tailored specifically for the effective evaluation of college students' psychological education, contributing to both theory and practice in this domain.

By addressing these three challenges, this study seeks to contribute innovative approaches to the field of MADM, particularly in the context of evaluating the effectiveness of college students' psychological education, which holds significant theoretical and practical importance.

On this basis, combined with the specific characteristics of the effectiveness evaluation of college students' psychological education, a new MADM approach in the SVNNs environment is proposed. The specific research contributions are summarized as follows:

- Development of the SVNN-PAMSSEM Approach: This paper introduces the SVNN-PAMSSEM approach, which is constructed by integrating the PAMSSEM [28] with SVNNs under conditions of completely unknown weight information. The SVNN-PAMSSEM approach incorporates distance-based measures to improve decision-making processes.
- Weight Determination Using the Average Approach: To derive the completely unknown attribute weights, an optimization model is established using the average method This model determines the

attribute weights. By combining the traditional PAMSSEM model with SVNN information, the SVNN- PAMSSEM approach is established, and a systematic computational process for MADM is provided.

- Advantages of the SVNN- PAMSSEM Approach: The SVNN- PAMSSEM approach offers several benefits: it provides stable computational results, utilizes simple calculation equations, accounts for the latent values of gains and losses, and can be flexibly combined with other decision-making approaches. As a result, the SVNN-PAMSSEM approach is a robust and effective tool for obtaining reasonable decision-making outcomes.
- Application to Psychological Education Evaluation: A numerical example is presented to demonstrate the practical application of the SVNN- PAMSSEM approach in the effective evaluation of college students' psychological education. Comparative analyses are conducted to highlight the advantages of the proposed approach. This paper primarily offers approach logical guidance and technical support for applying the SVNN- PAMSSEM approach.

This research provides significant insights into decision-making in the context of college students' psychological education. It also has broader implications for public-sector decision-making, infrastructure development, and the promotion of national security and stability. The proposed approach contributes to the theoretical and technical advancements in MADM under SVNN environments.

2. Preliminaries

This section introduces the fundamental concepts and definitions necessary for understanding the proposed methodology, including an overview of single-valued neutrosophic sets (SVNSs) and their role in multi-attribute decision-making.

2.1 SVNSs and Their Role in Multi-Attribute Decision-Making

Single-Valued Neutrosophic Sets (SVNSs) are an advanced mathematical tool used to handle uncertainty, vagueness, and imprecision in decision-making processes. Unlike traditional methods that rely on precise values, SVNSs provide a flexible framework by representing information through three independent membership functions: truth (T), indeterminacy (I), and falsity (F). Each function takes a value between 0 and 1, allowing for a nuanced representation of uncertainty.

In multi-attribute decision-making (MADM), where alternatives are evaluated based on multiple criteria, SVNSs offer a powerful way to incorporate imprecise and inconsistent data. Decision-makers can use SVNSs to express their preferences and judgments in situations where exact values are difficult to obtain, such as subjective evaluations or linguistic assessments. By capturing the complexity of real-world scenarios, SVNSs improve the accuracy and reliability of decision-making outcomes.

Overall, SVNSs enhance the ability to analyze and prioritize alternatives in MADM, making them a valuable tool for tackling complex problems with uncertain or incomplete information.

2.1 Definitions

This subsection provides key definitions related to SVNSs, including their structure, operations, and core concepts, which form the foundation for their application in multi-attribute decision-making. These definitions are essential for understanding how SVNSs handle uncertainty and imprecision effectively.

Definition 1 [29]. The SVNSs are articulated:

$$PA=\left\{\left(y,PT\left(y\right),PI\left(y\right),PF\left(y\right)\right)\middle|y\in Y\right\} \tag{1}$$

with truth-membership $PT\left(y\right)$, indeterminacy-membership $PI\left(y\right)$ and falsity-membership $PF\left(y\right)$, $PT\left(y\right),PI\left(y\right),PF\left(y\right)\in\left[0,1\right],\ 0\le PT\left(y\right)+PI\left(y\right)+PF\left(y\right)\le 3.$

The SVNN is articulated as $PA=\left(PT_A,PI_A,PF_A\right)$, $PT_A,PI_A,PF_A\in\left[0,1\right]$, $0\le PT_A+PI_A+PF_A\le 3.$

Definition 2 [30]. Let $PA=\left(PT_A,PI_A,PF_A\right)$ be SVNN, the score value is articulated:

$$SV\left(PA\right)=\frac{\left(2+PT_A-PI_A-PF_A\right)}{3},\ SV\left(PA\right)\in\left[0,1\right]. \tag{2}$$

Definition 3 [30]**.** Let $PA=\left(PT_A,PI_A,PF_A\right)$ be the SVNN, the accuracy value is articulated:

$$HV\left(PA\right)=PT_A-PF_A,\ HV\left(PA\right)\in\left[-1,1\right]. \tag{3}$$

Peng et al. [30] suggested an order relation for SVNNs to facilitate their comparison in decision-making scenarios. This order relation evaluates SVNNs based on their truth, indeterminacy, and falsity memberships, providing a systematic way to rank alternatives even under uncertain or imprecise conditions.

Definition 4 [30]**.** Let $PA=\left(PT_A,PI_A,PF_A\right)$ and $PB=\left(PT_B,PI_B,PF_B\right)$ be SVNNs, let $SV\left(PA\right)=\frac{\left(2+PT_A-PI_A-PF_A\right)}{3}$ and $SV\left(PB\right)=\frac{\left(2+PT_B-PI_B-PF_B\right)}{3}$, and let $HV\left(PA\right)=PT_A-PF_A$ and $HV\left(PB\right)=PT_B-PF_B$, if $SV\left(PA\right)<SV\left(PB\right)$, $PA<PB$; if $SV\left(PA\right)=SV\left(PB\right)$, then (1)if $HV\left(PA\right)=HV\left(PB\right)$, $PA=PB$; (2) if $HV\left(PA\right)<HV\left(PB\right)$, $PA<PB$.

Definition 5 [31]**.** Let $PA=\left(PT_A,PI_A,PF_A\right)$ and $PB=\left(PT_B,PI_B,PF_B\right)$ be SVNNs, the basic operations are articulated:

(1) $PA\oplus PB=\left(PT_A+PT_B-PT_A\cdot PT_B,PI_A\cdot PI_B,PF_A\cdot PF_B\right)$;

(2) $PA\otimes PB=\left(PT_A\cdot PT_B,PI_A+PI_B-PI_A\cdot PI_B,PF_A+PF_B-PF_A\cdot PF_B\right)$;

(3) $\lambda PA=\left(1-\left(1-PT_A\right)^{\lambda},\left(PI_A\right)^{\lambda},\left(PF_A\right)^{\lambda}\right),\lambda>0$;

(4) $\left(PA\right)^{\lambda}=\left(\left(PT_A\right)^{\lambda},\left(PI_A\right)^{\lambda},1-\left(1-PF_A\right)^{\lambda}\right),\lambda>0.$

Definition 6 [32]. Let $PA=\left(PT_A,PI_A,PF_A\right)$ and $PB=\left(PT_B,PI_B,PF_B\right)$, the Hamming distance between $PA=\left(PT_A,PI_A,PF_A\right)$ and $PB=\left(PT_B,PI_B,PF_B\right)$ is articulated:

$$HD\left(PA,PB\right)=\frac{\left|PT_A-PT_B\right|+\left|PI_A-PI_B\right|+\left|PF_A-PF_B\right|}{3} \tag{4}$$

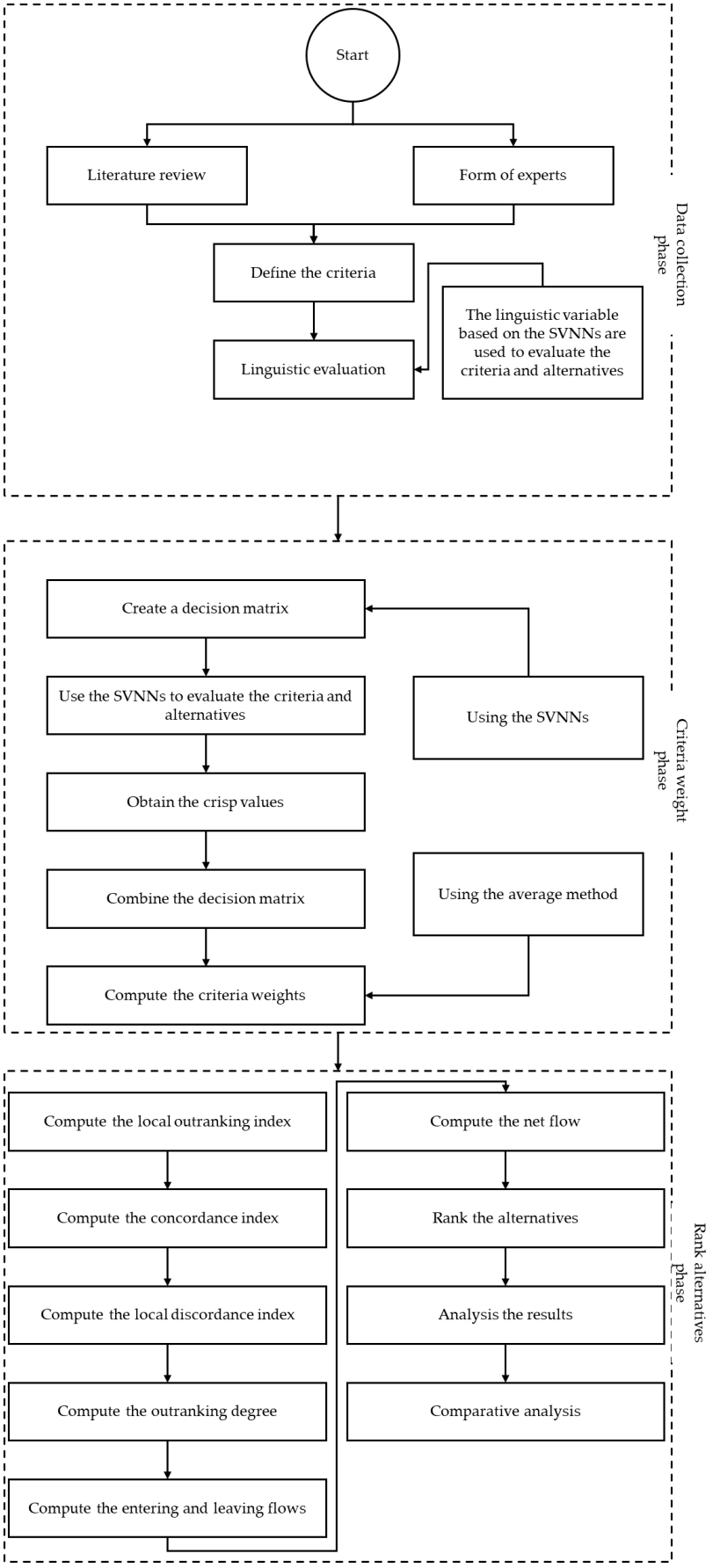
Start
Literature review
Form of experts
Define the criteria
Linguistic evaluation
The linguistic variable based on the SVNNs are used to evaluate the criteria and alternatives
Data collection phase
Create a decision matrix
Use the SVNNs to evaluate the criteria and alternatives
Obtain the crisp values
Combine the decision matrix
Compute the criteria weights
Using the SVNNs
Using the average method
Criteria weight phase
Compute the local outranking index
Compute the concordance index
Compute the local discordance index
Compute the outranking degree
Compute the entering and leaving flows
Compute the net flow
Rank the alternatives
Analysis the results
Comparative analysis
Rank alternatives phase

Figure 1. The flowchart of the proposed method.

3. Methodology

3.1 Research Basis

The research framework of this study is presented in Figure 1 which shows the steps of the proposed methodology under the neutrosophic sets.

3.2 Phase 1 Data Collection

Data collection is a process of gathering different data on criteria and alternatives to answer various questions, test hypotheses, and evaluate results. A survey was conducted to gather data on the criteria and alternatives and analysis of different criteria. Experts are decision-making and are required to obtain their opinions to evaluate the criteria and alternatives. The collected data is primary data. The decision-makers create the decision matrix based on the criteria and alternatives. We replaced their opinions by using single-valued neutrosophic numbers (SVNNs).

3.3. Phase 2 Data Evaluation

This phase refers to applying the proposed methodology to evaluate the data by obtaining the criteria weights and ranking the alternatives.

Step 1. Define the criteria of this study based on the literature.

Step 2. Define a set of alternatives

Step 3. Create a decision matrix

Step 4. Obtain the crisp values.

Step 5. Combined the decision matrix.

Step 6. Compute the criteria weights.

The criteria weights are computed using the average method.

Step 7. Compute the local outranking index.

$$R(A_i, A_{i'}) = \sum_{A_{i'}} \left(\sum_{A_i} R_j^-(A_i, A_{i'}) f_j(A_i) \right) f_j(A_{i'}) \tag{5}$$

Where $f_j(A_i)\ and\ f_j(A_{i'})$ refers to the probability density function and is assumed to be equal to one

$$R_j^-(A_i, A_{i'}) = \begin{cases} 0 & if\ \Delta_j < -p_j \\ \dfrac{\Delta_j - p_j}{p_j - q_j} & if - p_j < \Delta_j - q_j \\ 1 & if\ \Delta_j \geq -q_j \end{cases} \tag{6}$$

$$\Delta_j = k_j(A_i) - k_j(A_{i'}) \tag{7}$$

For the cardinal criteria

$$R_j(A_i, A_{i'}) = \begin{cases} 0 & if\ \Delta_j < -1 \\ 0.5 & if - 1 < \Delta_j \leq 0 \\ 1 & if\ \Delta_j \geq 0 \end{cases} \tag{8}$$

Step 8. Compute the concordance index.

$$C(A_i, A_{i'}) = \sum_{j=1}^{n} R_j(A_i, A_{i'}) w_j \tag{9}$$

Step 9. Compute the local discordance index.

$$D(A_i, A_{i'}) = \sum_{A_i} \left(\sum_{A_{i'}} D_j^-(A_i, A_{i'})\ f_j(A_i) \right) f_j(A_{i'}) \tag{10}$$

$$D_j^-(A_i, A_{i'}) = \begin{cases} 0 \quad if\ \Delta_j < -v_j \\ -\left(\dfrac{\Delta_j + p_j}{p_j - p_j}\right) \quad if - v_j < \Delta_j - p_j \\ 1 \quad if\ \Delta_j \geq -p_j \end{cases} \tag{11}$$

For the cardinal criteria

$$R_j(A_i, A_{i'}) = \begin{cases} \min\left\{1, t(w_j)\Delta_j + \dfrac{e_j + 1}{2}\right\} \quad if\ \Delta_j < -\left[\dfrac{e_j + 1}{2}\right] \\ 0 \quad if \Delta_j \geq -\left[\dfrac{e_j + 1}{2}\right] \end{cases} \tag{12}$$

.

Where $e_j > 3$ and

$$t(w_j) = 0.2\left(1 + \frac{w_j}{2}\right) \tag{13}$$

Step 9. Compute the outranking degree

$$N(A_i, A_{i'}) = C(A_i, A_{i'}) \prod_{j=1}^{n} \left[1 - D_j^3(A_i, A_{i'})\right] \tag{14}$$

Step 10. Compute the entering and leaving flows.

$$N^+(A_i) = \sum_{A_i \in A} N(A_i, A_{i'}) \tag{15}$$

$$N^-(A_i) = \sum_{A_i \in A} N(A_{i'}, A_i) \tag{16}$$

Step 11. Compute the net flow

$$N(A_i) = N^+(A_i) - N^-(A_i) \tag{17}$$

Step 12. Rank the alternatives.

Step 13. End

4. Case Study

4.1 Criteria weights.

The effective evaluation of college students' psychological education is the MADM. Five local colleges are evaluated with 8 attributes. We define eight criteria in this study and five alternatives.

a. Awareness and Literacy

Assesses the extent to which students are informed about mental health concepts, psychological well-being, and coping strategies. Programs should aim to destigmatize mental health issues and improve overall mental health literacy.

b. Accessibility of Psychological Counseling Services

Evaluates the availability and accessibility of counseling centers, mental health professionals, and helplines for students. This includes factors like the number of counselors, ease of appointments, and online/remote options.

c. Effectiveness of Intervention Programs

Measures the success of targeted intervention programs such as stress management workshops, cognitive-behavioral therapy sessions, and resilience training. The effectiveness is determined through feedback, outcomes, and measurable improvements.

d. Early Detection and Prevention Mechanisms

Assesses systems for identifying students at risk of mental health issues early. This includes tools like screenings, peer monitoring, faculty training for early warning signs, and prevention programs.

e. Student Engagement in Psychological Programs

Analyzes how actively students participate in mental health education activities such as workshops, seminars, and awareness campaigns. Higher engagement indicates greater program success.

f. Integration with Academic Curriculum

Examines whether psychological education is incorporated into the broader curriculum. For example, life skills courses, well-being seminars, and topics in personal development enhance students' mental resilience.

g. Faculty and Staff Training for Psychological Support

Evaluate whether teachers, mentors, and non-teaching staff are trained to support students' psychological well-being. This includes their ability to identify concerns and refer students for professional help.

h. Peer Support Systems

Measures the effectiveness of peer counseling and mentoring programs where students support each other under the guidance of professionals. Peer networks create trust and reduce stigma.

We invited three experts to evaluate the criteria and alternatives. Then we obtain the crisp values. Then we combined these values into a single decision matrix. Then we compute the criteria weights as shown in Figure 2.

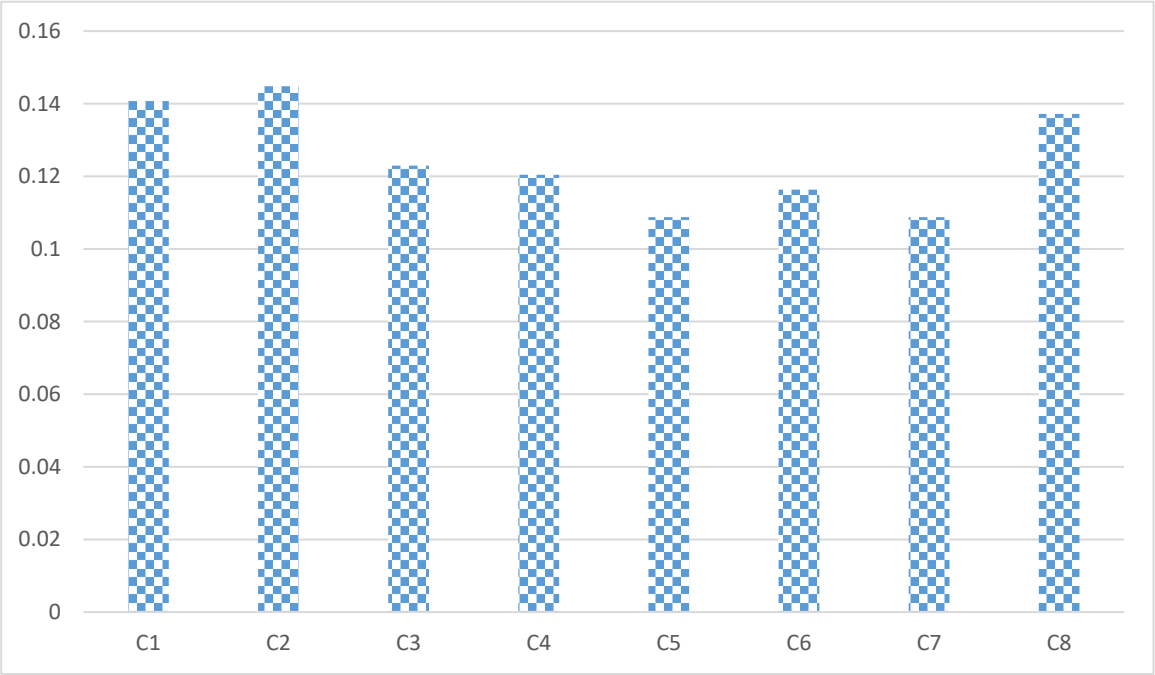

Figure 2. The criteria weights.

4.2 Rank of alternatives by the proposed method

In this part, we rank the alternatives by the proposed method.

Eq. (5) was used to compute the local outranking index. Eq. (9) was used to compute the concordance index. Eq. (10) was used to compute the local discordance index. Eq. (14) was used to compute the outranking degree Eqs. (15 and 16) were used to compute the entering and leaving flows. Then Eq. (17) was used to compute the net flow. Finally, we Ranked the alternatives.

A4>A2>A3>A1>A5

The results show alternative 4 is the best and alternative 5 is the worst.

The opinions of the first expert

(0.9,0.1,0.2) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9)

(0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.9,0.1,0.2)

(0.5,0.5,0.5) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.5,0.5,0.5) (0.8,0.2,0.3)

(0.6,0.4,0.5) (0.7,0.3,0.4) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.2,0.7,0.8) (0.3,0.6,0.7) (0.7,0.3,0.4)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.2,0.7,0.8) (0.3,0.6,0.7) (0.5,0.5,0.5) (0.6,0.4,0.5)

The opinions of the second expert

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.7,0.3,0.4) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.7,0.3,0.4)

(0.9,0.1,0.2) (0.7,0.3,0.4) (0.1,0.8,0.9) (0.7,0.3,0.4) (0.9,0.1,0.2) (0.7,0.3,0.4) (0.7,0.3,0.4) (0.9,0.1,0.2)

(0.2,0.7,0.8) (0.9,0.1,0.2) (0.7,0.3,0.4) (0.9,0.1,0.2) (0.2,0.7,0.8) (0.9,0.1,0.2) (0.9,0.1,0.2) (0.2,0.7,0.8)

(0.6,0.4,0.5) (0.2,0.7,0.8) (0.9,0.1,0.2) (0.2,0.7,0.8) (0.9,0.1,0.2) (0.2,0.7,0.8) (0.2,0.7,0.8) (0.7,0.3,0.4)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.2,0.7,0.8) (0.3,0.6,0.7) (0.5,0.5,0.5) (0.6,0.4,0.5)

The opinions of the third expert

(0.9,0.1,0.2) (0.1,0.8,0.9) (0.7,0.3,0.4) (0.1,0.8,0.9) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9)

(0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.7,0.3,0.4) (0.1,0.8,0.9) (0.9,0.1,0.2)

(0.9,0.1,0.2) (0.8,0.2,0.3) (0.1,0.8,0.9) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.8,0.2,0.3)

(0.8,0.2,0.3) (0.7,0.3,0.4) (0.9,0.1,0.2) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.7,0.3,0.4)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.8,0.2,0.3) (0.8,0.2,0.3) (0.2,0.7,0.8) (0.8,0.2,0.3) (0.5,0.5,0.5) (0.6,0.4,0.5)

4.3 Results and Discussion

The evaluation of the effectiveness of college students' psychological education is a crucial process for assessing the actual impact of psychological education practices on university students. As society develops rapidly and the pressures of academic and everyday life increase, the mental health issues of college students have gradually become a focus of public concern. Psychological education, as an essential means of promoting the development of students' psychological quality and improving their mental health, directly affects their holistic development and the long-term interests of society. Therefore, a scientific evaluation of its effectiveness is particularly important. The core of psychological education effectiveness evaluation lies in assessing whether the educational activities have achieved their intended goals, improved students' mental health, enhanced their psychological adjustment abilities and facilitated their personal growth and social adaptability. Through evaluation, educators can understand the outcomes of psychological education, optimize its content and approaches, and improve its quality. Meanwhile, such evaluations can identify common psychological issues and potential crises among students, providing a scientific basis for subsequent interventions and improvements in education practices. This evaluation process typically involves monitoring the dynamic changes in students' psychological states and analyzing the impact of psychological education on their behaviors, cognition, and emotions. By conducting in-depth analyses of educational practices, it is possible to uncover strengths and weaknesses, clarify future directions for improvement, and guide more effective psychological education. The evaluation is not merely a summary of educational activities but also serves as a guiding tool, offering theoretical and practical references for continuous improvement in psychological education for college students.

Moreover, the evaluation of psychological education effectiveness holds significant social value. As college students are the backbone of society's future, their mental health directly affects social stability and development. Through scientific and rational evaluations, the psychological education system can be improved, ensuring the cultivation of healthier and more positive psychological qualities among students. This not only enhances their mental resilience during their time at university but also lays a solid psychological foundation for their future integration into society. In conclusion, the evaluation of the effectiveness of college students' psychological education is a process that integrates systematic, scientific, and practical approaches. It is not only a means of assessing psychological education practices but also an essential pathway for driving continuous advancements in psychological education. Effective evaluations can improve the relevance and efficacy of psychological education while contributing positively to the holistic development of students and the harmonious progress of society.

4.4 Compare analysis

The SVNN-PAMSSEM approach is compared with SVNN-CODAS approach and SVNN-EDAS approach, SVNN-TOPSIS, SVNN-VIKOR. The comparative results of different approaches are in Table 1.

Table 1. Results for different given approaches

Approaches	Order
SVNN-CODAS approach	A4>A3>A2>A1>A5
SVNN-EDAS approach	A4>A2>A5>A1>A3
SVNN-TOPSIS approach	A4>A2>A3>A1>A5
SVNN-VIKOR approach	A4>A2>A3>A1>A5
SVNN-PAMSSEM approach	A4>A2>A3>A1>A5

From Table 1, the order of these approaches has a light difference and the best local college and the worst local college are the same. The five given models each have distinct advantages that make them valuable in different decision-making scenarios:

- SVNN-CODAS Approach: This approach evaluates the overall performance of alternatives using both Euclidean distance and Hamming distance from negative ideal points. The Euclidean distance serves as the primary measure for comparison. If the Euclidean distances between two alternatives are very close,

the Hamming distance is used as a secondary measure to differentiate between them. The threshold parameter is employed to determine the closeness of the Euclidean distance, enhancing the precision of the evaluation. This dual-measure approach ensures a robust and detailed assessment of alternatives.

- SVNN-EDAS Approach: A key advantage of this approach is its computational simplicity, requiring fewer calculations while achieving the same ranking of alternatives. Unlike methods such as TOPSIS and VIKOR, the EDAS approach evaluates alternatives based on their distance from average solutions for each criterion. This distinction makes EDAS a highly efficient and effective method for decision-making, particularly in contexts where computational resources or time are limited.
- SVNN- PAMSSEM Approach: This model offers several notable features, including stable computation results, simple calculating equations, and the ability to account for latent gains and losses. Additionally, it can be easily integrated with other decision-making models, increasing its flexibility and applicability. These characteristics make the SVNN-PAMSSEM approach an excellent tool for deriving reasonable and reliable decision-making outcomes.

In summary, each of these models has unique strengths tailored to specific decision-making needs. Whether emphasizing group or individual influences, offering computational efficiency, or providing robust evaluation measures, these approaches collectively address a wide range of decision-making challenges in multi-criteria environments.

5. Conclusion

The evaluation of the effectiveness of psychological education for college students is a crucial tool for assessing whether the intended goals of psychological education have been achieved. It aims to understand the actual impact of psychological education on students' mental health, psychological quality, and adaptability. As social competition intensifies and life pressures increase, the mental health issues of college students have become more prominent, making psychological education an essential solution to these challenges. Effectiveness evaluation helps educators grasp the current state of psychological education, identify its strengths and weaknesses, and provide a scientific basis for optimizing educational content and methods. Through systematic analysis of the outcomes of psychological education, the evaluation can assess its effectiveness in alleviating psychological problems, improving emotional regulation, and fostering personal growth. It serves not only as a summary of educational activities but also as guidance for the future development of psychological education, offering practical support and theoretical references for mental health education. The significance of this evaluation also lies in advancing the improvement of the psychological education system, helping students better cope with challenges in their studies and lives, and laying a solid psychological foundation for their future development. This process holds great value for both the personal growth of students and the harmonious

development of society. The effectiveness evaluation of college students' psychological education is a MADM problem. In this paper, the PAMSSEM approach is developed under the framework of SVNN-MADM. First, the average method is employed to determine the decision weights. Based on this, the SVNN- PAMSSEM approach is constructed to handle MADM problems under SVNSs. Finally, an example focused on evaluating the effectiveness of college students' psychological education is provided, along with comparative decision analyses to validate the proposed SVNN- PAMSSEM method.

The main contributions of this research are as follows:

- Attribute Weight Determination: The attribute weights are objectively derived using the average approach.
- Development of the SVNN- PAMSSEM Approach: This paper constructs a novel SVNN- PAMSSEM approach specifically tailored for evaluating the effectiveness of college students' psychological education.
- Case Study Application: The proposed approach is demonstrated through a practical case study, effectively showcasing its applicability and relevance in real-world scenarios.
- Comparative Analysis: Comparative decision-making analyses are conducted to verify the rationality and advantages of the SVNN- PAMSSEM method, highlighting its robustness and effectiveness in addressing MADM problems.

In summary, this work provides a systematic and innovative methodology for evaluating the effectiveness of psychological education for college students. Leveraging the SVNN-PAMSSEM approach, not only ensures a comprehensive evaluation process but also offers a reliable tool for decision-makers to assess and improve psychological education strategies. The findings confirm the practicality and superiority of the proposed method in tackling complex MADM problems.

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Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems

M.Anandhkumar[1], S. Prathap[2], R. Ambrose Prabhu[3], , P.Tharaniya[4], K. Thirumalai[5], B. Kanimozhi[6]

[1]Department of Mathematics, IFET College of Engineering (Autonomous), Villupuram, Tamilnadu, India.
anandhkumarmm@gmail.com

[2] Department of Mathematics, Panimalar Engineering College, Chennai, Tamil Nadu, India.

prathapmaths@gmailcom

[3]Department of Mathematics, Rajalakshmi Institute of Technology, Poonamalle, Chennai, Tamilnadu, India
bennyamb0457@gmail.com

[4]Department of Mathematics, Rajalakshmi Institute of Technology, Poonamalle, Chennai, Tamilnadu, India
tharamanisharmi@gmail.com

[5]Department Of Mathematics, Saveetha Engineering College, Chennai, Tamilnadu, India.
thirumalaikmaths@gmail.com

[6]Department of Mathematics, Sri Manakula Vinayagar Engineering College (Autonomous), Madagadipet Puducherry, India.
kanimozhimaths@smvec.ac.in

Abstract: We explore the determinant theory for Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs), investigating their properties. In this study, we establish that $\det\left(Padj\left(P\right)\right)=\det\left(P\right)=\det\left(adj\left(P\right)P\right)$. Additionally, we propose a refined method to compute the determinant of matrices with a higher number of rows and columns. Furthermore, an algorithm is developed to address decision-making problems based on QPNFMs. An illustrative example is provided to demonstrate the effectiveness of the proposed method.

Keywords: Quadri Partitioned Neutrosophic fuzzy sets , Adjoint, Determinant, Decision-Making.

1. Introduction

The introduction of fuzzy sets by Zadeh [1] provided a foundational framework to handle uncertainty by representing degrees of membership. This framework has since evolved into a crucial tool for managing imprecise information. Zadeh later advanced this concept by introducing linguistic variables to facilitate approximate reasoning [3]. Pawlak [2] further contributed to the field with rough set theory, offering an alternative approach for representing uncertainty through boundary

regions. Atanassov [4] extended fuzzy set theory by proposing IFSs, incorporating an additional parameter to capture the degree of hesitation, which is not present in traditional fuzzy sets. He explored various theoretical aspects of IFS, including applications [5] and implications, such as fuzzy modus ponens and types of negations [6, 10]. Anandhkumar,et al [7,8,9] have studied Pseudo Similarity of NFM, On various Inverse of NFM, Reverse Sharp and Left-T Right-T Partial Ordering on NFM. These advancements have paved the way for a deeper understanding of uncertainty in decision-making contexts. Research has also applied fuzzy and intuitionistic fuzzy theories to matrices. Kim and Roush [14] explored generalized fuzzy matrices, and Thomason [15] investigated convergence properties in fuzzy matrices. Later, Xu and Yager [13] developed geometric operators based on intuitionistic fuzzy sets, which have been useful in multi-criteria decision-making scenarios, enhancing the applicability of IFS in complex systems.

In addition to foundational works on fuzzy sets and IFSs , significant advancements have been made in fuzzy matrix theory. Kim [16] examined idempotents and inverses in fuzzy matrices, which led to further exploration of matrix properties such as transitivity and sub-inverses in fuzzy matrices, as studied by Mishref and Eman [17]. The determinant and adjoint concepts for square fuzzy matrices were later developed by Ragab and Eman [18], adding depth to the algebraic study of fuzzy matrices.Further research by Kim [19, 20] addressed T-type fuzzy idempotent matrices and established a determinant theory specific to square fuzzy matrices. Punithavalli [21] has discussed Kernel and K-Kernel Symmetric IFM. Anandhkumar et al [22, 23] have present Partial orderings, Characterizations and Generalization of k-idempotent NFM, Reverse Tilde (T) and Minus Partial Ordering on IFM. Lun [24] expanded this by exploring determinant theory for D01 lattice matrices, adding a novel perspective within the fuzzy matrix domain.

For IFM, Atanassov [25, 26] proposed the concept of generalized index matrices and applied it to represent intuitionistic fuzzy graphs. This was followed by work by Pal [27], who introduced an intuitionistic fuzzy determinant, and by Im, Lee, and Park [28], who analyzed determinants in square intuitionistic fuzzy matrices. Pal, Khan, and Shyamal [29] further contributed by exploring general properties of intuitionistic fuzzy matrices, expanding the theoretical framework for applications in multi-criteria decision-making.The body of work surrounding intuitionistic fuzzy matrices has continued to grow, contributing to both theoretical foundations and practical applications. Pal, Khan, and Shyamal [30] elaborated on the characteristics of intuitionistic fuzzy matrices, enhancing the understanding of their structures in various mathematical contexts. Meenakshi and Gandhimathi [31] investigated intuitionistic fuzzy relational equations, highlighting their applications in fuzzy logic and decision-making. Sriram and Murugadas focused on the concept of sub-inverses of IFM [32], expanding the algebraic framework, while also examining semi-rings associated with these matrices [33]. Meenakshi's work on Fuzzy Matrix Theory and Applications [34] provided a comprehensive resource for understanding both the theoretical and applied aspects of fuzzy matrices.

Research on distances between intuitionistic fuzzy matrices was conducted by Shyamal and Pal [35], offering new metrics for comparison. Bhowmik and Pal [36, 37] explored various properties of

intuitionistic fuzzy matrices, including generalized forms and circulant structures, which are important for complex systems in fuzzy logic. Im [38] analyzed determinants of square intuitionistic fuzzy matrices, which is crucial for many mathematical applications. Atanassov [39] provided insights into index matrices and their relevance to augmented matrix calculus, further contributing to the theoretical landscape. Anandhkumar et al [40] have present Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices. Punithavalli [41] has studied Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices. Adak and colleagues [42, 43] also focused on generalized IFM, investigating their applications in multi-criteria decision-making and exploring properties of generalized fuzzy nilpotent matrices, reinforcing the practical significance of these concepts in real-world scenarios.

The research on intuitionistic fuzzy matrices continues to expand with various contributions, focusing on their structures, properties, and applications.Lee and Jeong [44] examined the canonical form of transitive intuitionistic fuzzy matrices, contributing to a deeper understanding of the properties that define these matrices. Mondal and Pal's studies [45, 46] explored similarity relations, invertibility, and eigenvalues of intuitionistic fuzzy matrices, along with their determinants, providing essential insights for both theoretical exploration and practical implementation in fuzzy systems. Radhika [47,48] et al have presented On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices and Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making. Prathab [49] et al have studied Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses. Pradhan and Pal [50] contributed significantly to the understanding of intuitionistic fuzzy matrices by exploring convergence properties of different arithmetic means of intuitionistic fuzzy matrices [51], linear transformations [52]. The study by Im, Lee, and Park [53] on the determinant of square IFMs is significant in understanding the mathematical properties and applications of intuitionistic fuzzy sets. Smarandache [56] has studied Neutrosophic set, a generalization of the IFs. Murugadas and Padder [54] and Uma [55] their research focuses on defining and calculating the determinants of these matrices, which are essential for various applications in decision-making, optimization, and fuzzy logic systems. The authors explore the characteristics of determinants in the context of intuitionistic fuzzy matrices, examining how the unique features of intuitionistic fuzzy sets—such as membership and non-membership degrees—affect the determinant's computation and interpretation. Their findings contribute to the broader field of fuzzy mathematics by providing foundational knowledge that can be applied in both theoretical research and practical applications where uncertainty and vagueness are present.

In recent years, advancements in neutrosophic fuzzy matrices have facilitated deeper analysis in fields involving uncertainty and indeterminate information. Symmetric NSMs, as explored by Anandhkumar et al. [57], have proven valuable in complex decision-making and data classification. Further work by Anandhkumar et al. [58] introduced secondary k-column symmetric NFM, while Anandhkumar et al. [59] expanded this framework with interval-valued secondary k-range

symmetric neutrosophic fuzzy matrices, enhancing models with interval-based evaluations. Additionally, Anandhkumar et al. [60] extended these concepts to generalized symmetric Fermatean neutrosophic fuzzy matrices, emphasizing their applicability in systems requiring nuanced symmetrical handling. The development of quadripartitioned neutrosophic soft sets has enriched the analysis of complex data sets under uncertainty. Mary [61] introduced foundational concepts in this area, providing a framework that allows for a more nuanced approach to handling uncertain information. Chatterjee, Majumdar, and Samanta [62] furthered this research by exploring similarity measures and entropy within quadripartitioned single-valued neutrosophic sets, which facilitate enhanced classification and decision-making. Subsequent analysis by Smith and Doe [63] examined the properties and applications of quadri-partitioned neutrosophic soft sets, contributing to their growing utility in handling intricate data structures.

1.1 Abbrivations

IFM: Intuitionistic Fuzzy Matrices

IFSs: Intuitionistic Fuzzy Sets

NFM: Neutrosophic fuzzy matrices.
QPNFM: Quadri Partitioned Neutrosophic fuzzy matrices.
MCDMP: Multi-Criteria Decision- Making Problem.

2. The structure of this article is arranged as follows: In Section 3 presents the objectives of the present work, laying the foundation for the study. Section 4 delves into the motivation behind this research, where a comparative analysis of the Quadri Partitioned Neutrosophic Fuzzy Matrix (QPNFM) model with existing soft models is provided to showcase its advantages. Section 5 identifies the research gap, emphasizing the limitations of current approaches and the need for the proposed model. Section 6 highlights the novelty of the QPNFM model, underscoring its unique contributions to the field. Section 7 introduces the preliminaries necessary for understanding the model, followed by Section 8, which formally defines Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs) and their mathematical structure. In Section 9 present Properties of the Quadri-Partitioned Neutrosophic Fuzzy Matrices In Section 10, relevant theorems and results are presented to establish the theoretical foundations of the model. Section 11 describes an algorithm based on QPNSS specifically designed for decision-making problems, while Section 12 demonstrates a practical application of this algorithm through a detailed example. This structure facilitates a systematic exploration of the model, providing insights into its theoretical background, methodological rigor, and real-world applicability in complex decision-making scenarios.

3. The objectives of the present work are given:

- **To develop determinant theory for Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs):** Establish fundamental principles and properties of determinants specific to QPNFMs, focusing on unique characteristics within the neutrosophic fuzzy domain.

- **To investigate determinant relationships in QPNFMs:** Prove key determinant relationships, such as $\det\left(Padj\left(P\right)\right)=\det\left(P\right)=\det\left(adj\left(P\right)P\right)$. thereby expanding mathematical understanding within QPNFMs.
- **To propose an efficient method for computing determinants in larger QPNFMs:** Introduce a novel technique for calculating determinants of QPNFMs with high dimensionality, aimed at simplifying computation for matrices with more rows and columns.
- **To construct an algorithm for solving decision-making problems:** Develop a systematic approach leveraging QPNFM properties to address complex decision-making scenarios, enhancing practical applications of QPNFMs.
- **To validate the proposed methods with an illustrative example:** Demonstrate the effectiveness and applicability of the determinant theory, methods, and algorithm through a practical example, solidifying the proposed study's relevance.

4. Motivation (Comparative of QPNFM model with the existing soft models).

In earlier research works, the concept of Fuzzy soft sets, Intuitioniasic Fuzzy soft sets, Interval valued Fuzzy soft sets, Interval valued Intuitioniasic Fuzzy soft sets, Neutrosophic Fuzzy soft sets Interval valued Neutrosophic Fuzzy soft sets , Quadri partitions Neutrosophic Fuzzy soft sets, etc. are used successfully to solve decision-making problems that contain parametric uncertain, incomplete, inconsistent, hesitant or indeterminate data. There is no such work that has been done so far where the indeterminacy can be handled parametrically under the neutrosophic environment by keeping T, F, C, and U as dependent quadripartitioned neutrosophic components. So, the present work is devoted to developing a new methodology to handle indeterminacy parametrically by introducing the quadripartitioned neutrosophic Fuzzy Matrices . This study surely provides a more flexible framework for the decision-makers to explore new decision-making approaches to address the issues under the quadripartitioned neutrosophic soft environment with the inherent restrictions. To make the proposed model more visible in the real-life scenario, we give a comparative analysis in the following Table 1 and 2:

Table 1. Comparative of IVQPNFM model with the existing soft models

Types of soft set	Uncertainty	Falsity	Hesitation	Indeterminacy	Indeterminacy is bifurcated	Indeterminacy is bifurcated and restricted
FSS [46]	✓	×	×	×	×	×
IVFSS [47]	✓	×	×	×	×	×
IFSS [48]	✓	✓	✓	×	×	×

IVIFSS [49]	✓	✓	✓	×	×	×
NSS [50]	✓	✓	×	✓	×	×
INSS [51]	✓	✓	×	✓	×	×
QNSS [52]	✓	✓	×	✓	✓	×
QPNFM (Proposed)	✓	✓	✓	✓	✓	✓

Table 2. Validation and Comparison of QPNFM with Existing Models

Aspect	Fuzzy Matrices	Intuitionistic Fuzzy Matrices	Neutrosophic Fuzzy Matrices	QPNFM (Proposed Model)
Advantages	Easy to implement and interpret; suitable for basic uncertainty problems.	Lacks indeterminacy handling.	Basic indeterminate handling; limited for complex data structures.	Handles high indeterminacy with quadri-partitioned structure; suitable for complex decision-making.
Results	Limited to systems where indeterminacy and dynamics are minimal.	Suitable for moderate uncertainty but limited in layered decision analysis.	Adequate in simpler datasets but inconsistent in highly uncertain scenarios.	Improved accuracy in uncertain environments.
Applications	Suitable for straightforward uncertainty handling and decision problems.	Useful in moderate uncertainty; limited for high indeterminacy tasks.	Best for low-complexity decisions.	Ideal for complex, uncertain scenarios.

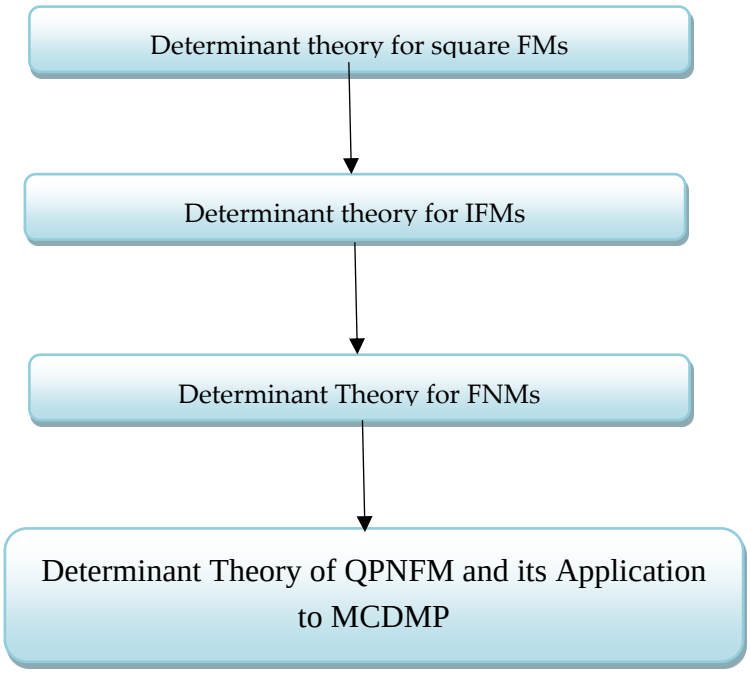
5. Research Gap

The research gap identified in these references centers on the need for a comprehensive framework that extends determinant theory to more complex structures such as Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs) for applications in uncertain and multi-dimensional decision-making environments. While foundational studies by Zadeh [1, 3] and Atanassov [4–6] laid the groundwork for fuzzy and intuitionistic fuzzy set theories, these frameworks do not fully address the complexity introduced in quadri-partitioned systems where neutrosophic uncertainty plays a significant role. Furthermore, Kim and Roush [14] and Lun [24] explored determinant theory in basic fuzzy matrices, and Pal [27] and Im, Lee, and Park [28] extended this to intuitionistic fuzzy matrices, but none of these studies account for the additional partitioning in neutrosophic matrices.

The work of Padder and Murugadas [54] and Uma, Murugadas, and Sriram [55] introduced determinant theories in intuitionistic fuzzy and fuzzy neutrosophic matrices, yet there remains a gap in addressing QPNFMs, where multiple layers of truth, indeterminacy, and falsity in quadri-partitioned setups impact both the matrix structure and the determinant calculation methods. Additionally, existing algorithms for multi-criteria decision-making (e.g., Adak, Bhowmik, and Pal [43]) utilize simpler fuzzy matrix approaches, lacking algorithms tailored for QPNFMs that could leverage their unique partitioning to provide more nuanced decision-support solutions. Addressing this gap could yield a robust determinant theory for QPNFMs, alongside efficient algorithms for high-dimensional decision-making problems involving complex, uncertain data.

Table:1 Review of the Extension of QPNFM.

Ref	**Journal Name**	**Authors Name**	**Extension of NFM.**	**Year**
[20]	Fuzzy Sets Systems	Kim	Determinant theory for square FMs	1989
[55]	Progress in Nonlinear Dynamics and Chaos	Uma et al.	Determinant Theory for NFMs	2016
[54]	Afrika Matematika	Riyaz Ahmad Padder et al	Determinant theory for IFMs	2019
Proposed	Neutrosophic Sets and Systems	Anandhkumar et al.	Determinant Theory of QPNFM and its Application to MCDMP	2024


Determinant theory for square FMs
Determinant theory for IFMs
Determinant Theory for FNMs
Determinant Theory of QPNFM and its Application to MCDMP

6. Novelty

The referenced works collectively contribute to the advancement of fuzzy and intuitionistic fuzzy matrix theory, supporting a range of applications in uncertainty modeling and multi-criteria decision-making. Zadeh [1, 3] laid the foundation of fuzzy set theory by introducing partial membership, which became the basis for further developments in fuzzy systems, and later extended this with linguistic variables for approximate reasoning. Atanassov [4–6] extended fuzzy theory with IFSs, which incorporate both membership and non-membership degrees, creating a richer framework for handling uncertainty. Researchers like Kim and Roush [14], Thomason [15], Kim [16, 19, 20], Ragab and Eman [17], and Lun [24] contributed to the mathematical structure of fuzzy matrices by investigating properties such as idempotence, inverses, determinants, and convergence, which are crucial for stability and transformations in fuzzy systems.

Further developments came from Pal, Bhowmik, and their collaborators [27, 29, 30, 35, 36, 37], who explored operations and properties specific to intuitionistic fuzzy matrices, such as determinants, eigenvalues, and similarity relations. Their work added depth to the theoretical understanding and practical application of these matrices in complex decision-making. Adak, Bhowmik, and Pal [42, 37] applied IFMs to multi-criteria decision-making, highlighting their relevance in real-world applications. Studies by Lee and Jeong [36], Murugadas and Padder [48–49], and Pradhan and Pal [50-52] introduced forms, reductions, and convergence criteria for intuitionistic fuzzy matrices, aiding in simplification and predictability. Collectively, these works by Zadeh, Atanassov, Pal, and others have enriched the mathematical framework and utility of fuzzy and

intuitionistic fuzzy matrices, making them valuable tools in areas requiring nuanced handling of uncertainty and complex decision parameters.

7. Preliminaries

Definition 7.1 [53] Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all QPNSS of X. The collection (F, A) is termed the (QPNSS) over X, where F is a mapping given by F : A⟶ P(X). Here,

$A = \{< x, T_A(x), C_A(x), U_A(x), FA(x) >: x \in U\}$ with T_A , F_A, C_A ,$U_A : X \longrightarrow [0,1]$ and $0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$. In this context

- $T_A(x)$ is the truth membership (TM),
- $C_A(x)$ is contradiction membership (CM),
- $U_A(x)$ is ignorance membership (IM),
- $F_A(x)$ is the false membership (FM).

8. Quadri-Partitioned Neutrosophic Fuzzy Matrices

Definition 8.1 Let $P =< p^T_{ij}, p^C_{ij}, p^U_{ij}, p^F_{ij} >, Q =< q^T_{ij}, q^C_{ij}, q^U_{ij}, q^F_{ij} >\in (QPNFM)_n$

Component-wise addition and multiplication are defined as follows

(i) $$P \oplus Q = \left(\sup\left\{p^T_{ij}, q^T_{ij}\right\}, \sup\left\{p^C_{ij}, q^C_{ij}\right\}, \inf\left\{p^U_{ij}, q^U_{ij}\right\}, \inf\left\{p^F_{ij}, q^F_{ij}\right\}\right)$$

(ii) $$P \square Q = \left(\inf\left\{p^T_{ij}, q^T_{ij}\right\}, \inf\left\{p^C_{ij}, q^C_{ij}\right\}, \sup\left\{p^U_{ij}, q^U_{ij}\right\}, \sup\left\{p^F_{ij}, q^F_{ij}\right\}\right)$$

Definition 8.2. Let $P =< p^T_{ij}, p^C_{ij}, p^U_{ij}, p^F_{ij} >, Q =< q^T_{ij}, q^C_{ij}, q^U_{ij}, q^F_{ij} >\in (QPNFM)_n$

the composition of P and Q is well-defined as

$$P \circ Q = \left(\sum_{k=1}^{n}\left(p^T_{ij} \wedge q^T_{ij}\right), \sum_{k=1}^{n}\left(p^C_{ij} \wedge q^C_{ij}\right), \prod_{k=1}^{n}\left(p^U_{ij} \vee q^U_{ij}\right), \prod_{k=1}^{n}\left(p^F_{ij} \vee q^F_{ij}\right)\right)$$

consistently we can write the same as

$$P \circ Q = \left(\bigcup_{k=1}^{n}\left(p^T_{ij} \wedge q^T_{ij}\right), \bigcup_{k=1}^{n}\left(p^C_{ij} \wedge q^C_{ij}\right), \bigcap_{k=1}^{n}\left(p^U_{ij} \vee q^U_{ij}\right), \bigcap_{k=1}^{n}\left(p^U_{ij} \vee q^U_{ij}\right)\right)$$

The product $P \circ Q$ is defined only when the number of columns in P equals the number of rows in Q. When this condition is met, matrices P and Q are considered conformable for multiplication. For simplicity, we denote the product as $P \circ Q$.

Definition 8.3 The determinant $|P|$ of nxn QPNFM $P=<p^T_{ij}, p^C_{ij}, p^U_{ij}, p^F_{ij}>\in(QPNFM)_n$ is defined as follows

$$|P|=<\bigcup_{\sigma\in S_n} p^T_{1\sigma(1)}\bigcap\cdots\bigcap p^T_{n\sigma(n)}, \bigcup_{\sigma\in S_n} p^C_{1\sigma(1)}\bigcap\cdots\bigcap p^C_{n\sigma(n)}, \bigcap_{\sigma\in S_n} p^U_{1\sigma(1)}\bigcup\cdots\bigcup p^U_{n\sigma(n)},$$
$$\bigcap_{\sigma\in S_n} p^F_{1\sigma(1)}\bigcup\cdots\bigcup p^F_{n\sigma(n)}>$$

Here, S_n represents the symmetric group consisting of all possible permutations of the indices (1,2,... n).

Definition 8.4 The adjoint of an n xn QPFNSM P denoted by adj P, is defined as follows

$q_{ij}=|P_{ji}|$ is the determinant of the (n-1)x(n-1) QPFNSM formed by removing row j and column i from P and Q = adjP

Definition:8.5 Let $P=<p^T_{ij}, p^C_{ij}, p^U_{ij}, p^F_{ij}>\in(QPNFM)_n$ and let Q be a matrix from P by striking out e_1, row e_2,… row e_k and column r_1, column r_2,…, column r_k. we define

$$P\begin{pmatrix} e_1 & e_2 & \ldots & e_k \\ r_1 & r_2 & \ldots & r_k \end{pmatrix}=\det(H).$$

Remark:8.1 We can write the element q_{ij} of adjP = Q = (q_{ij}) as follows:

$q_{ij}=\sum_{\pi\in S_{n_j n_i}}\prod_{t\in n_j}<p^T_{t\pi(t)}, p^C_{t\pi(t)}, p^U_{t\pi(t)}, p^F_{t\pi(t)}>$ where n_j = {1,2,3,…n}\{j} and $S_{n_j n_i}$ is the set of all permutation of set n_j over the set n_i.

9.Properties of the Quadri-Partitioned Neutrosophic Fuzzy Matrices

(i) The determinant's value remains unchanged if any two rows or any two columns are swapped.

(ii) For a Quadri-Partitioned Neutrosophic Fuzzy Matrix (QPNFM), the determinant value is preserved when rows and columns are interchanged.

(iii) For two QPNFMs, P and Q, the property det(PQ)≠det(P)·det(Q) holds.

(iv) If the elements of one row (or column) are added to the corresponding elements of another row (or column), the determinant's value remains the same as the original.

10. Theorems and Results

Theorem:10.1 $P \in (QPNFM)_n$,then

(i) $$\det(P) = |P| = \sum_{t=1}^{n} < p^T_{it}, p^C_{it}, p^U_{it}, p^F_{it} > P_{it}, i \in \{1,2,...,n\}.$$

(ii) $$\det(P) = \sum_{e<f} \begin{vmatrix} < p^T_{1e}, p^C_{1e}, p^U_{1e}, p^F_{1e} > & < p^T_{1f}, p^C_{1f}, p^U_{1f}, p^F_{1f} > \\ < p^T_{2e}, p^C_{2e}, p^U_{2e}, p^F_{2e} > & < p^T_{2f}, p^C_{2f}, p^U_{2f}, p^F_{2f} > \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix}$$

where the summation is taken over all e and f in {1,2,...,n} such that e < f.

Theorem: 10.2

$$\det(P) = \begin{pmatrix} < p^T_{1r_1}, p^C_{1r_1}, p^U_{1r_1}, p^F_{1r_1} > & \cdots & \cdots & < p^T_{1r_k}, p^C_{1r_k}, p^U_{1r_k}, p^F_{1r_k} > \\ < p^T_{2r_1}, p^C_{2r_1}, p^U_{2r_1}, p^F_{2r_1} > & \cdots & \cdots & < p^T_{2r_k}, p^C_{2r_k}, p^U_{2r_k}, p^F_{2r_k} > \\ \cdots & & \cdots & \cdots \\ < p^T_{kr_1}, p^C_{kr_1}, p^U_{kr_1}, p^F_{kr_1} > & \cdots & \cdots & < p^T_{kr_k}, p^C_{kr_k}, p^U_{kr_k}, p^F_{kr_k} > \end{pmatrix} P\begin{pmatrix} 1 & \cdots & k \\ r_1 & \cdots & r_k \end{pmatrix}$$

where the summation is taken over all $r_1, r_2, ..., r_k \in \{1,2,...,n\}$, such that $r_1 < r_2 < ... < r_k$.

Proof: Let $S(r_1, r_2, ..., r_k) = \{\sigma : \{1,2,...,k\} \to \{r_1, r_2, ..., r_k\} / \sigma$ is a bijection}. Then

$$\det(P) = \sum_{\sigma \in S_n} < p^T_{1\sigma(1)}, p^C_{1\sigma(1)}, p^U_{1\sigma(1)}, p^F_{1\sigma(1)} > ... < p^T_{n\sigma(n)}, p^C_{n\sigma(n)}, p^U_{n\sigma(n)}, p^F_{n\sigma(n)} >$$

$$= \sum_{r_1<r_2<...<r_k} \left(\sum_{\sigma\{1,2,...,k\} \in S(r_1,r_2,...,r_k)} < p^T_{1\sigma(1)}, p^C_{1\sigma(1)}, p^U_{1\sigma(1)}, p^F_{1\sigma(1)} > ... < p^T_{n\sigma(n)}, p^C_{n\sigma(n)}, p^U_{n\sigma(n)}, p^F_{n\sigma(n)} > \right)$$

$$= \sum_{r_1<r_2<...<r_k} \left(\sum_{\sigma'\{1,2,...,k\} \in S(r_1,r_2,...,r_k)} < p^T_{1\sigma'(1)}, p^C_{1\sigma'(1)}, p^U_{1\sigma'(1)}, p^F_{1\sigma'(1)} > ... < p^T_{n\sigma'(n)}, p^C_{n\sigma'(n)}, p^U_{n\sigma'(n)}, p^F_{n\sigma'(n)} > \right)$$

$$P\begin{pmatrix} 1 & \cdots & k \\ r_1 & \cdots & r_k \end{pmatrix}$$

$$= \det \sum_{r_1<r_2<...<r_k} \begin{pmatrix} < p^T_{1r_1}, p^C_{1r_1}, p^U_{1r_1}, p^F_{1r_1} > & \cdots & \cdots & < p^T_{1r_k}, p^C_{1r_k}, p^U_{1r_k}, p^F_{1r_k} > \\ < p^T_{2r_1}, p^C_{2r_1}, p^U_{2r_1}, p^F_{2r_1} > & \cdots & \cdots & < p^T_{2r_k}, p^C_{2r_k}, p^U_{2r_k}, p^F_{2r_k} > \\ \cdots & & \cdots & \cdots \\ < p^T_{kr_1}, p^C_{kr_1}, p^U_{kr_1}, p^F_{kr_1} > & \cdots & \cdots & < p^T_{kr_k}, p^C_{kr_k}, p^U_{kr_k}, p^F_{kr_k} > \end{pmatrix} P\begin{pmatrix} 1 & \cdots & k \\ r_1 & \cdots & r_k \end{pmatrix}$$

Hence the theorem.

Lemma 10.1 Let $P=\begin{pmatrix} < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \\ < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \end{pmatrix}$ be a QPNFM.

Then $\det\begin{pmatrix} < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \\ < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \end{pmatrix}\det\begin{pmatrix} < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \\ < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \end{pmatrix}$

$$= \begin{vmatrix} < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \\ < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \end{vmatrix} \begin{vmatrix} < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \\ < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \end{vmatrix} \leq \det(P)$$

Proof: We see that

$$\det\begin{pmatrix} < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \\ < p^T, p^C, p^U, p^F > & < q^T, q^C, q^U, q^F > \end{pmatrix}\det\begin{pmatrix} < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \\ < r^T, r^C, r^U, r^F > & < s^T, s^C, s^U, s^F > \end{pmatrix}$$

$$=< p^T, p^C, p^U, p^F >< q^T, q^C, q^U, q^F >< r^T, r^C, r^U, r^F >< s^T, s^C, s^U, s^F >$$

$$\leq \left(< p^T, p^C, p^U, p^F >< s^T, s^C, s^U, s^F > + < q^T, q^C, q^U, q^F >< r^T, r^C, r^U, r^F > \right)$$

$$\leq \det(P)$$

Hence the theorem.

Theorem:10.3 Let $P \in \left(QPNFM\right)_n$, then

(i) $\det\left(P\left(2 \Rightarrow 1\right)\right)\det\left(P\left(1 \Rightarrow 2\right)\right) \leq \det\left(P\right).$

(ii) $\det\left(P\left(2 \Rightarrow 1\right)\right)\det\left(P\left(3 \Rightarrow 2\right)\right) \leq \det\left(P\right).$

(iii) $\det\left(P\left(p \Rightarrow q\right)\right)\det\left(P\left(q \Rightarrow k\right)\right) \leq \det\left(P\right).$

Proof: To prove (i) $\det\left(P\left(2 \Rightarrow 1\right)\right)\det\left(P\left(1 \Rightarrow 2\right)\right) \leq \det\left(P\right).$

$$= \begin{pmatrix} \begin{vmatrix} < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} > & < p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} > \\ < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} > & < p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} > \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \\ +\ldots+ \begin{vmatrix} < p^T{}_{1e}, p^C{}_{1e}, p^U{}_{1e}, p^F{}_{1e} > & < p^T{}_{1f}, p^C{}_{1f}, p^U{}_{1f}, p^F{}_{1f} > \\ < p^T{}_{1e}, p^C{}_{1e}, p^U{}_{1e}, p^F{}_{1e} > & < p^T{}_{1f}, p^C{}_{1f}, p^U{}_{1f}, p^F{}_{1f} > \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} \\ +\ldots+ \begin{vmatrix} < p^T{}_{1n-1}, p^C{}_{1n-1}, p^U{}_{1n-1}, p^F{}_{1n-1} > & < p^T{}_{1n}, p^C{}_{1n}, p^U{}_{1n}, p^F{}_{1n} > \\ < p^T{}_{1n-1}, p^C{}_{1n-1}, p^U{}_{1n-1}, p^F{}_{1n-1} > & < p^T{}_{1n}, p^C{}_{1n}, p^U{}_{1n}, p^F{}_{1n} > \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix} \end{pmatrix}$$

$$\left(\begin{array}{l}\left|\begin{array}{ll}<p^{T}_{21},p^{C}_{21},p^{U}_{21},p^{F}_{21}> & <p^{T}_{22},p^{C}_{22},p^{U}_{22},p^{F}_{22}>\\ <p^{T}_{21},p^{C}_{21},p^{U}_{21},p^{F}_{21}> & <p^{T}_{22},p^{C}_{22},p^{U}_{22},p^{F}_{22}>\end{array}\right| P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\\ +\ldots+\left|\begin{array}{ll}<p^{T}_{2e},p^{C}_{2e},p^{U}_{2e},p^{F}_{2e}> & <p^{T}_{2f},p^{C}_{2f},p^{U}_{2f},p^{F}_{2f}>\\ <p^{T}_{2e},p^{C}_{2e},p^{U}_{2e},p^{F}_{2e}> & <p^{T}_{2f},p^{C}_{2f},p^{U}_{2f},p^{F}_{2f}>\end{array}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}\\ +\ldots+\left|\begin{array}{ll}<p^{T}_{2n-1},p^{C}_{2n-1},p^{U}_{2n-1},p^{F}_{2n-1}> & <p^{T}_{2n},p^{C}_{2n},p^{U}_{2n},p^{F}_{2n}>\\ <p^{T}_{2n-1},p^{C}_{2n-1},p^{U}_{2n-1},p^{F}_{2n-1}> & <p^{T}_{2n},p^{C}_{2n},p^{U}_{2n},p^{F}_{2n}>\end{array}\right| P\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}\end{array}\right)$$

$$=\left(\sum_{e<f}\left|\begin{array}{ll}<p^{T}_{1e},p^{C}_{1e},p^{U}_{1e},p^{F}_{1e}> & <p^{T}_{1f},p^{C}_{1f},p^{U}_{1f},p^{F}_{1f}>\\ <p^{T}_{1e},p^{C}_{1e},p^{U}_{1e},p^{F}_{1e}> & <p^{T}_{1f},p^{C}_{1f},p^{U}_{1f},p^{F}_{1f}>\end{array}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}\right)$$

$$\left(\sum_{g<h}\left|\begin{array}{ll}<p^{T}_{2g},p^{C}_{2g},p^{U}_{2g},p^{F}_{2g}> & <p^{T}_{2h},p^{C}_{2h},p^{U}_{2h},p^{F}_{2h}>\\ <p^{T}_{2g},p^{C}_{2g},p^{U}_{2g},p^{F}_{2g}> & <p^{T}_{2h},p^{C}_{2h},p^{U}_{2h},p^{F}_{2h}>\end{array}\right| P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}\right)$$

$$\leq\left(\sum_{\substack{e<f\\ g<h}}\left|\begin{array}{ll}<p^{T}_{1e},p^{C}_{1e},p^{U}_{1e},p^{F}_{1e}> & <p^{T}_{1f},p^{C}_{1f},p^{U}_{1f},p^{F}_{1f}>\\ <p^{T}_{2g},p^{C}_{2g},p^{U}_{2g},p^{F}_{2g}> & <p^{T}_{2h},p^{C}_{2h},p^{U}_{2h},p^{F}_{2h}>\end{array}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}\right)$$

We now introduce symbols $\Omega_1,\Omega_2,\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and Ω. Define

$$\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\left|\begin{array}{ll}<p^{T}_{1e},p^{C}_{1e},p^{U}_{1e},p^{F}_{1e}> & <p^{T}_{1f},p^{C}_{1f},p^{U}_{1f},p^{F}_{1f}>\\ <p^{T}_{2g},p^{C}_{2g},p^{U}_{2g},p^{F}_{2g}> & <p^{T}_{2h},p^{C}_{2h},p^{U}_{2h},p^{F}_{2h}>\end{array}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}$$

$$\Omega_1=\sum_{(e,f)=(g,h)}\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\sum_{e<f}\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix},$$

$$\Omega_2=\sum_{(e,f)\neq(g,h)}\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}\quad\text{and}\quad\Omega=\Omega_1+\Omega_2$$

Then we see that

$$\Omega\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}=\left|\begin{array}{ll}<p^{T}_{11},p^{C}_{11},p^{U}_{11},p^{F}_{11}> & <p^{T}_{12},p^{C}_{12},p^{U}_{12},p^{F}_{12}>\\ <p^{T}_{21},p^{C}_{21},p^{U}_{21},p^{F}_{21}> & <p^{T}_{22},p^{C}_{22},p^{U}_{22},p^{F}_{22}>\end{array}\right| P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$$

$$\Omega_1=\det(P)$$

$$\det\left(P(2\Rightarrow 1)\right)\det\left(P(1\Rightarrow 2)\right)\leq\Omega=\det(P)+\Omega_2.$$

We show that $\Omega_2\leq\det(P)$

We consider two separate cases.

Case 1. We consider $p = \Omega\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$, a term of Ω_2.

Let $p_1 = < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} >< p^T_{23}, p^C_{23}, p^U_{23}, p^F_{23} > P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}P\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$

$p_2 = < p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} >< p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} > P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}P\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$

Then $p = p_1 + p_2$,

$$p_1 \le \begin{vmatrix} < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} > & < p^T_{13}, p^C_{13}, p^U_{13}, p^F_{13} > \\ < p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} > & < p^T_{23}, p^C_{23}, p^U_{23}, p^F_{23} > \end{vmatrix} P\begin{pmatrix}1 & 3\\ 1 & 3\end{pmatrix} \le \det(P)$$

$$p_2 \le \begin{vmatrix} < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} > & < p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} > \\ < p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} > & < p^T_{22}, p^C_{22}, p^U_{22}, p^F_{22} > \end{vmatrix} P\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \le \det(P)$$

and $\Omega\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} \le \det(P)$.

Case 2. We take $\Omega\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}$

Let $q_1 = < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} >< p^T_{2n}, p^C_{2n}, p^U_{2n}, p^F_{2n} > P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}P\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}$ and

$q_2 = < p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} >< p^T_{2n-1}, p^C_{2n-1}, p^U_{2n-1}, p^F_{2n-1} > P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}P\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}$.

Then $\Omega\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix} = q_1 + q_2$.

To show that $q_1 \le \det(P)$ and $q_2 = \det(P)$ we observe all coordinates of the elements p_{ij}

involved in $P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$ and $P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$ and $P\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}$

The coordinates of the elements p_{ij} involved in these determinants are all coordinates of the elements of the k th – row P_k of P, for $k \geq 3$. Therefore, if we let $q = p_{3n-1}p_{4n-2}\cdots p_{k+2n-k}\cdots p_{nn-2}$, then we see that

$$q_1 \leq \left(< p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} >< p^T_{2n}, p^C_{2n}, p^U_{2n}, p^F_{2n} >\right)c \leq \det(P).$$

For q_2, let $c = p_{3n}p_{4n-2}p_{5n-3}\cdots p_{n-13}p_{2n-1}$, then we see that

$$q_2 \leq \left(< p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} >< p^T_{2n-1}, p^C_{2n-1}, p^U_{2n-1}, p^F_{2n-1} >\right)c \leq \det(P).$$

For any $\Omega\begin{pmatrix} e & f \\ g & h \end{pmatrix}_{(e,f)\neq(g,h)}$, we apply either the case 1 or the case 2 and we can deduce that $\Omega\begin{pmatrix} e & f \\ g & h \end{pmatrix} \leq \det(P)$. Thus (i) holds. (ii).

First we consider

$$\begin{vmatrix} < q^T_{11}, q^C_{11}, q^U_{11}, q^F_{11} > & < q^T_{12}, q^C_{12}, q^U_{12}, q^F_{12} > \\ < p^T_{31}, p^C_{31}, p^U_{31}, p^F_{31} > & < p^T_{32}, p^C_{32}, p^U_{23}, p^F_{32} > \end{vmatrix} \begin{vmatrix} < q^T_{21}, q^C_{21}, q^U_{21}, q^F_{21} > & < q^T_{22}, q^C_{22}, q^U_{22}, q^F_{22} > \\ < q^T_{21}, q^C_{21}, q^U_{21}, q^F_{21} > & < q^T_{22}, q^C_{22}, q^U_{22}, q^F_{22} > \end{vmatrix}$$

$$\leq \begin{vmatrix} < q^T_{21}, q^C_{21}, q^U_{21}, q^F_{21} > & < q^T_{32}, q^C_{32}, q^U_{32}, q^F_{32} > \\ < p^T_{31}, p^C_{31}, p^U_{31}, p^F_{31} > & < p^T_{32}, p^C_{32}, p^U_{23}, p^F_{32} > \end{vmatrix}$$

$$K\begin{pmatrix} g & h \\ e & f \end{pmatrix} = \begin{vmatrix} < p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} > & < p^T_{32}, p^C_{32}, p^U_{32}, p^F_{32} > \\ < p^T_{31}, p^C_{31}, p^U_{31}, p^F_{31} > & < p^T_{32}, p^C_{32}, p^U_{23}, p^F_{32} > \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$\det\left(P(2 \Rightarrow 1)\right)\det\left(P(3 \Rightarrow 2)\right)$$

$$= \sum_{\substack{e<f \\ g<h}} \begin{vmatrix} < p^T_{1e}, p^C_{1e}, p^U_{1e}, p^F_{1e} > & < p^T_{1f}, p^C_{1f}, p^U_{1f}, p^F_{1f} > \\ < p^T_{3e}, p^C_{3e}, p^U_{3e}, p^F_{3e} > & < p^T_{3f}, p^C_{3f}, p^U_{3f}, p^F_{3f} > \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$\begin{vmatrix} < p^T_{2g}, p^C_{2g}, p^U_{2g}, p^F_{2g} > & < p^T_{2h}, p^C_{2h}, p^U_{2h}, p^F_{2h} > \\ < p^T_{2g}, p^C_{2g}, p^U_{2g}, p^F_{2g} > & < p^T_{2h}, p^C_{2h}, p^U_{2h}, p^F_{2h} > \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix}$$

$$\leq \sum_{\substack{g<h \\ e<f}} \begin{vmatrix} < p^T_{2g}, p^C_{2g}, p^U_{2g}, p^F_{2g} > & < p^T_{2h}, p^C_{2h}, p^U_{2h}, p^F_{2h} > \\ < p^T_{3e}, p^C_{3e}, p^U_{3e}, p^F_{3e} > & < p^T_{3f}, p^C_{3f}, p^U_{3f}, p^F_{3f} > \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$= \sum_{\substack{g<h \\ e<f}} P\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

$$= \sum_{(g,h)=(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix} + \sum_{(g,h)\neq(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

Next we prove that

$$K\begin{pmatrix} g & h \\ e & f \end{pmatrix}_{(g,h)\neq(e,f)} \leq \det(P)$$

.We consider two separate cases.

Case 1. We take $K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. We see that

$$K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \left(< p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} >< p^T{}_{33}, p^C{}_{33}, p^U{}_{33}, p^F{}_{33} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$+\left(< p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22 } >< p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$\leq \begin{vmatrix} < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{23}, p^C{}_{23}, p^U{}_{23}, p^F{}_{23} > \\ < p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} > & < p^T{}_{33}, p^C{}_{33}, p^U{}_{33}, p^F{}_{33} > \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$+\begin{vmatrix} < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22} > \\ < p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} > & < p^T{}_{32}, p^C{}_{32}, p^U{}_{32}, p^F{}_{32} > \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$\leq \det(P) + \det(P) = \det(P)$$

Case 2. We take $K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix}$ We see that

$$K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix} = \left(< p^T{}_{2n-1}, p^C{}_{2n-1}, p^U{}_{2n-1}, p^F{}_{2n-1} >< p^T{}_{32}, p^C{}_{32}, p^U{}_{32}, p^F{}_{32} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$$

$$+\left(< q^T{}_{2n}, q^C{}_{2n}, q^U{}_{2n}, q^F{}_{2n} >< p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}.$$

Considering the coordinates of the elements p_{ij} involved in $P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$, we claim that

$$\left(< p^{T}{}_{2n-1}, p^{C}{}_{2n-1}, p^{U}{}_{2n-1}, p^{F}{}_{2n-1} >< p^{T}{}_{32}, p^{C}{}_{32}, p^{U}{}_{32}, p^{F}{}_{32} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \leq \det(P)$$

and $\left(< q^{T}{}_{2n}, q^{C}{}_{2n}, q^{U}{}_{2n}, q^{F}{}_{2n} >< p^{T}{}_{31}, p^{C}{}_{31}, p^{U}{}_{31}, p^{F}{}_{31} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \leq \det(P)$

Similarly we can prove (iii).

Hence the theorem.

Theorem 10.4. Let $P = \left(p^{T}{}_{ij}, p^{C}{}_{ij}, p^{U}{}_{ij}, p^{F}{}_{ij}\right), Q = \left(q^{T}{}_{ij}, q^{C}{}_{ij}, q^{U}{}_{ij}, q^{F}{}_{ij}\right), R = \left(r^{T}{}_{ij}, r^{C}{}_{ij}, r^{U}{}_{ij}, r^{F}{}_{ij}\right)$

$\in (QPNFM)_n$. Then

(i) If $\left(p^{T}{}_{ii}, p^{C}{}_{ii}, p^{U}{}_{ii}, p^{F}{}_{ii}\right) \geq \left(p^{T}{}_{ik}, p^{C}{}_{ik}, p^{U}{}_{ik}, p^{F}{}_{ik}\right)$ (k= 1,2,3,…,n) for all $1 \leq i \leq n$, then

$$\det(P) =< p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} >< p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \ldots < p^{T}{}_{nn}, p^{C}{}_{nn}, p^{U}{}_{nn}, p^{F}{}_{nn} > .$$

(ii) $\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} \det(P)\det(Q)$ where $(<0,0,1,1>)_n \in (QPNFM)_n$.

(iii) $\det\left(PP^{T}\right) \geq \det(P)$

Proof: (i). We have

$$\left(< p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} >< p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \ldots < p^{T}{}_{nn}, p^{C}{}_{nn}, p^{U}{}_{nn}, p^{F}{}_{nn} >\right) \geq$$

$$\left(< p^{T}{}_{1\sigma(1)}, p^{C}{}_{1\sigma(1)}, p^{U}{}_{1\sigma(1)}, p^{F}{}_{1\sigma(1)} >< p^{T}{}_{2\sigma(2)}, p^{C}{}_{2\sigma(2)}, p^{U}{}_{2\sigma(2)}, p^{F}{}_{2\sigma(2)} > \ldots < p^{T}{}_{n\sigma(n)}, p^{C}{}_{n\sigma(n)}, p^{U}{}_{n\sigma(n)}, p^{F}{}_{n\sigma(n)} >\right)$$

for every $\sigma \in S_n$.

Since $\left(p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11}\right) \geq \left(p^{T}{}_{ik}, p^{C}{}_{ik}, p^{U}{}_{ik}, p^{F}{}_{ik}\right)$ $(k = 1,2,3,\ldots,n)$ for all $1 \leq i \leq n$.

Hence

$$\det(P) = \sum_{\sigma \in S_n} < p^{T}{}_{1\sigma(1)}, p^{C}{}_{1\sigma(1)}, p^{U}{}_{1\sigma(1)}, p^{F}{}_{1\sigma(1)} > \ldots < p^{T}{}_{n\sigma(n)}, p^{C}{}_{n\sigma(n)}, p^{U}{}_{n\sigma(n)}, p^{F}{}_{n\sigma(n)} >$$

$$=< p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} >< p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \ldots < p^{T}{}_{nn}, p^{C}{}_{nn}, p^{U}{}_{nn}, p^{F}{}_{nn} > .$$

This proves (i)

(ii) $\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \left(< s^T{}_{ij}, s^C{}_{ij}, s^U{}_{ij}, s^F{}_{ij} > \right)_{2n}$.

$$\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \sum_{\sigma \in S_{2n}} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} >$$

$$= \sum_{\sigma \in S_{2n}, \sigma(i) \le n \, (if \, i \le n)} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} >$$

$$= \sum_{\sigma \in S_{2n}, \sigma(i) \le n \, (if \, i \le n)} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} > + < 0,0,1,1 >$$

$$+ \sum_{\sigma \in S_{2n}, \exists k > n, \, if \, \sigma(k) \le n} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} >$$

$$= \sum_{\sigma' \in S_n} < s^T{}_{1\sigma'(1)}, s^C{}_{1\sigma'(1)}, s^U{}_{1\sigma'(1)}, s^F{}_{1\sigma'(1)} > \ldots < s^T{}_{n\sigma'(n)}, s^C{}_{n\sigma'(n)}, s^U{}_{n\sigma'(n)}, s^F{}_{n\sigma'(n)} > \det(Q)$$

$$= \left(\sum_{\sigma \in S_n} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{n\sigma(n)}, s^C{}_{n\sigma(n)}, s^U{}_{n\sigma(n)}, s^F{}_{n\sigma(n)} > \right) \det(Q)$$

$$= \det(P)\det(Q).$$

This proves (ii)

(iii) $PP^T = \left(< h^T{}_{ij}, h^C{}_{ij}, h^U{}_{ij}, h^F{}_{ij} > \right)_n$,

We have , for every $\sigma \in S_n$.

$$< h^T{}_{ij}, h^C{}_{ij}, h^U{}_{ij}, h^F{}_{ij} > = \sum_{k=1}^{n} < p^T{}_{ik}, p^C{}_{ik}, p^U{}_{ik}, p^F{}_{ik} > < p^T{}_{kj}, p^C{}_{kj}, p^U{}_{kj}, p^F{}_{kj} > .$$

$$< h^T{}_{11}, h^C{}_{11}, h^U{}_{11}, h^F{}_{11} > < h^T{}_{22}, h^C{}_{22}, h^U{}_{22}, h^F{}_{22} > \ldots < h^T{}_{nn}, h^C{}_{nn}, h^U{}_{nn}, h^F{}_{nn} >$$

$$= \left(\sum_{k=1}^{n} < p^T{}_{ik}, p^C{}_{ik}, p^U{}_{ik}, p^F{}_{ik} > \right) \ldots \left(\sum_{k=1}^{n} < p^T{}_{nk}, p^C{}_{nk}, p^U{}_{nk}, p^F{}_{nk} > \right)$$

$$\ge \left(< p^T{}_{1\sigma(1)}, p^C{}_{1\sigma(1)}, p^U{}_{1\sigma(1)}, p^F{}_{1\sigma(1)} > \ldots < p^T{}_{n\sigma(n)}, p^C{}_{n\sigma(n)}, p^U{}_{n\sigma(n)}, p^F{}_{n\sigma(n)} > \right)$$

Hence

$$\det\left(PP^T\right) \ge \left(< h^T{}_{11}, h^C{}_{11}, h^U{}_{11}, h^F{}_{11} > < h^T{}_{22}, h^C{}_{22}, h^U{}_{22}, h^F{}_{22} > \ldots < h^T{}_{nn}, h^C{}_{nn}, h^U{}_{nn}, h^F{}_{nn} > \right)$$

$$\geq \sum_{\sigma \in S_n} < p^T_{1\sigma(1)}, p^C_{1\sigma(1)}, p^U_{1\sigma(1)}, p^F_{1\sigma(1)} > \ldots < p^T_{n\sigma(n)}, p^C_{n\sigma(n)}, p^U_{n\sigma(n)}, p^F_{n\sigma(n)} >$$

$$= \det(P).$$

This proves (iii)

Hence the Theorem

Theorem 10.5 Let P = (p_{ij}) be a QPNFM. Then we have the following

$$\det\left(Padj(P)\right) = \det(P) = \det\left(adj(P)P\right).$$

Proof: We prove that $\det\left(Padj(P)\right) = \det(P)$.

We first consider n = 2.

Let $P = \begin{pmatrix} < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} > & < p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} > \\ < p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} > & < p^T_{22}, p^C_{22}, p^U_{22}, p^F_{22} > \end{pmatrix}$

$$adj(P) = \begin{pmatrix} < p^T_{22}, p^C_{22}, p^U_{22}, p^F_{22} > & < p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} > \\ < p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} > & < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} > \end{pmatrix}$$

$$\det\left(Padj(P)\right) =$$

$$\begin{vmatrix} \det(P) & < p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} >< p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} > \\ < p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} >< p^T_{22}, p^C_{22}, p^U_{22}, p^F_{22} > & \det(P) \end{vmatrix}$$

$$= \det(P) + \left(< p^T_{11}, p^C_{11}, p^U_{11}, p^F_{11} >< p^T_{12}, p^C_{12}, p^U_{12}, p^F_{12} >\right)$$

$$\left(< p^T_{21}, p^C_{21}, p^U_{21}, p^F_{21} >< p^T_{22}, p^C_{22}, p^U_{22}, p^F_{22} >\right)$$

$$\leq \det(P)$$

Next consider $n > 2$. We can see that

$$Padj(P) =$$

$$\begin{pmatrix} \sum < p^T_{1t}, p^C_{1t}, p^U_{1t}, p^F_{1t} > P_{1t} & \sum < p^T_{1t}, p^C_{1t}, p^U_{1t}, p^F_{1t} > P_{2t} & \ldots & \sum < p^T_{1t}, p^C_{1t}, p^U_{1t}, p^F_{1t} > P_{nt} \\ \sum < p^T_{2t}, p^C_{2t}, p^U_{2t}, p^F_{2t} > P_{1t} & \sum < p^T_{2t}, p^C_{2t}, p^U_{2t}, p^F_{2t} > P_{2t} & \ldots & \sum < p^T_{2t}, p^C_{2t}, p^U_{2t}, p^F_{2t} > P_{nt} \\ \ldots & \ldots & \ldots & \ldots \\ \sum < p^T_{nt}, p^C_{nt}, p^U_{nt}, p^F_{nt} > P_{1t} & \sum < p^T_{nt}, p^C_{nt}, p^U_{nt}, p^F_{nt} > P_{2t} & \ldots & \sum < p^T_{nt}, p^C_{nt}, p^U_{nt}, p^F_{nt} > P_{nt} \end{pmatrix}$$

$$= \left(\sum < p^T_{it}, p^C_{it}, p^U_{it}, p^F_{it} > P_{jt}\right).$$

$$\det\left(Padj(P)\right) = \sum_{\pi \in S_n} \left(\sum < p^T_{1t}, p^C_{1t}, p^U_{1t}, p^F_{1t} > P_{\pi(1)t}\right)\left(\sum < p^T_{2t}, p^C_{2t}, p^U_{2t}, p^F_{2t} > P_{\pi(2)t}\right)$$

$$...\left(\sum < p^{T}_{nt}, p^{C}_{nt}, p^{U}_{nt}, p^{F}_{nt} > P_{\pi(n)t}\right).$$

It is evident that each diagonal entry of the matrix Padj(P) equals det(P). We demonstrate this result as follows.

(i) Let us define

$$T_{\pi} = \left(\sum < p^{T}_{1t}, p^{C}_{1t}, p^{U}_{1t}, p^{F}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^{T}_{2t}, p^{C}_{2t}, p^{U}_{2t}, p^{F}_{2t} > P_{\pi(2)t}\right)$$
$$...\left(\sum < p^{T}_{nt}, p^{C}_{nt}, p^{U}_{nt}, p^{F}_{nt} > P_{\pi(n)t}\right).$$

for $\pi \in S_n$. Let e be the identity of the group S_n. If $\pi = e,$ then $T_{\pi} = \det(P).$

Suppose that there exists $k \in \{1, 2, ..., n\}$ such that $\pi(k) = k.$ Then we see that

$$= \sum < p^{T}_{kt}, p^{C}_{kt}, p^{U}_{kt}, p^{F}_{kt} > P_{\pi(k)t} = \sum < p^{T}_{kt}, p^{C}_{kt}, p^{U}_{kt}, p^{F}_{kt} > P_{kt}$$

$$= \det(P)$$

$$\Omega_n = \left(\sum < p^{T}_{1t}, p^{C}_{1t}, p^{U}_{1t}, p^{F}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^{T}_{2t}, p^{C}_{2t}, p^{U}_{2t}, p^{F}_{2t} > P_{\pi(2)t}\right)...\det(P)...$$
$$\left(\sum < p^{T}_{nt}, p^{C}_{nt}, p^{U}_{nt}, p^{F}_{nt} > P_{\pi(n)t}\right) \le \det(P).$$

(ii) Let π be a permutation in S_n. Assume that $\pi(k) \neq k.$ for all $k \in \{1, 2, ..., n\}$. We know that every permutation π can be written as a product of disjoint cycles π_i and let $\pi = \pi_1 \pi_2 ... \pi_k$

We further assume that $\pi_1 = (1 \quad 2)$ transposition. Then Ω_{π} has two factors $\sum < p^{T}_{1t}, p^{C}_{1t}, p^{U}_{1t}, p^{F}_{1t} > P_{\pi(1)t}$ and $\sum < p^{T}_{2t}, p^{C}_{2t}, p^{U}_{2t}, p^{F}_{2t} > P_{\pi(2)t}$, and from these we see that

$$\left(\sum < p^{T}_{1t}, p^{C}_{1t}, p^{U}_{1t}, p^{F}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^{T}_{2t}, p^{C}_{2t}, p^{U}_{2t}, p^{F}_{2t} > P_{\pi(2)t}\right)$$
$$= \left(\sum < p^{T}_{1t}, p^{C}_{1t}, p^{U}_{1t}, p^{F}_{1t} > P_{2t}\right)\left(\sum < p^{T}_{2t}, p^{C}_{2t}, p^{U}_{2t}, p^{F}_{2t} > P_{1t}\right)$$

$$\det\left(P\left(2 \Rightarrow 1\right)\right)\det\left(P\left(1 \Rightarrow 2\right)\right) \le \det(P)$$

(iii) If $\pi = \pi_1 \pi_2 ... \pi_k$ and $\pi_1(s,t),$ then we can prove that $\Omega_n \le \det(P)$ by an argument used in (ii). Consider Ω for $\pi = \pi_1 \pi_2 ... \pi_k$. If $\pi = (k, e, f, ...),$ then we see that

$$\Omega_n = \left(\sum < p^{T}_{kt}, p^{C}_{kt}, p^{U}_{kt}, p^{F}_{kt} > P_{\pi(k)t}\right)\left(\sum < p^{T}_{et}, p^{C}_{et}, p^{U}_{et}, p^{F}_{et} > P_{\pi(e)t}\right)...$$

$$= \left(\sum < p^{T}{}_{kt}, p^{C}{}_{kt}, p^{U}{}_{kt}, p^{F}{}_{kt} > P_{et}\right)\left(\sum < p^{T}{}_{et}, p^{C}{}_{et}, p^{U}{}_{et}, p^{F}{}_{et} > P_{ft}\right)...$$

$$= \det\left(P\left(e \Rightarrow k\right)\right)\det\left(P\left(f \Rightarrow e\right)\right)...$$

we obtain that $\det\left(P\left(e \Rightarrow k\right)\right)\det\left(P\left(f \Rightarrow e\right)\right) \le \det\left(P\right)$ and so that $\Omega_{n} \le \det\left(P\right)$. This proves that $\det\left(Padj\left(P\right)\right) = \det\left(P\right)$. Equally, we can prove that $\det\left(adj\left(P\right)P\right) = \det\left(P\right)$.

Hence the theorem.

Theorem 10.6 Let $P, Q \in (QPNFM)_{n}$. Then

(i) $\det(PQ) \ge \det(P)\det(Q)$

(ii) $\det(PQ) \le \det(P+Q)$,

Where $\det(PQ) = \det\left(\sum_{k=1}^{n} p^{T}{}_{ik} \wedge q^{T}{}_{kj}, \sum_{k=1}^{n} p^{C}{}_{ik} \wedge q^{C}{}_{kj}, \prod_{k=1}^{n} p^{U}{}_{ik} \vee q^{U}{}_{kj}, \prod_{k=1}^{n} p^{F}{}_{ik} \vee q^{F}{}_{kj}\right)$

Proof: $P+Q = \left(\sup\left\{p_{ij}{}^{T}, q_{ij}{}^{T}\right\}, \sup\left\{p_{ij}{}^{C}, q_{ij}{}^{C}\right\}, \inf\left\{p_{ij}{}^{U}, q_{ij}{}^{U}\right\}, \inf\left\{p_{ij}{}^{F}, q_{ij}{}^{F}\right\}\right)$

$$= \sum_{\sigma \in S_{n}}\left(\sum_{k=1}^{n} p^{T}{}_{1k} \wedge q^{T}{}_{k\sigma(1)}, \sum_{k=1}^{n} p^{C}{}_{1k} \wedge q^{C}{}_{k\sigma(1)}, \prod_{k=1}^{n} p^{U}{}_{1k} \vee q^{U}{}_{k\sigma(1)}, \prod_{k=1}^{n} p^{F}{}_{1k} \vee q^{F}{}_{k\sigma(1)}\right), ...,$$

$$\left(\sum_{k=1}^{n} p^{T}{}_{nk} \wedge q^{T}{}_{k\sigma(n)}, \sum_{k=1}^{n} p^{C}{}_{nk} \wedge q^{C}{}_{k\sigma(n)}, \prod_{k=1}^{n} p^{U}{}_{nk} \vee q^{U}{}_{k\sigma(n)}, \prod_{k=1}^{n} p^{F}{}_{nk} \vee q^{F}{}_{k\sigma(n)}\right)$$

$$= \sum_{\sigma \in S_{n}}\left(\sum_{k_{1},k_{2},...,k_{n}} p^{T}{}_{1k_{1}} \wedge p^{T}{}_{2k_{2}} ... \wedge p^{T}{}_{nk_{n}} \wedge q^{T}{}_{k_{1}\sigma(1)} \wedge q^{T}{}_{k_{2}\sigma(2)} ... \wedge q^{T}{}_{k_{n}\sigma(n)}\right)$$

$$\left(\sum_{k_{1},k_{2},...,k_{n}} p^{C}{}_{1k_{1}} \wedge p^{C}{}_{2k_{2}} ... \wedge p^{C}{}_{nk_{n}} \wedge q^{C}{}_{k_{1}\sigma(1)} \wedge q^{C}{}_{k_{2}\sigma(2)} ... \wedge q^{C}{}_{k_{n}\sigma(n)}\right)$$

$$\left(\prod_{k_{1},k_{2},...,k_{n}} p^{U}{}_{1k_{1}} \vee p^{U}{}_{2k_{2}} ... \vee p^{U}{}_{nk_{n}} \vee q^{U}{}_{k_{1}\sigma(1)} \vee q^{U}{}_{k_{2}\sigma(2)} ... \vee q^{U}{}_{k_{n}\sigma(n)}\right)$$

$$\left(\prod_{k_{1},k_{2},...,k_{n}} p^{F}{}_{1k_{1}} \vee p^{F}{}_{2k_{2}} ... \vee p^{F}{}_{nk_{n}} \vee q^{F}{}_{k_{1}\sigma(1)} \vee q^{F}{}_{k_{2}\sigma(2)} ... \vee q^{F}{}_{k_{n}\sigma(n)}\right)$$

$$\ge \sum_{(k_{1},k_{2},...,k_{n}) \in S_{n}} < p^{T}{}_{1k_{1}}, p^{C}{}_{1k_{1}}, p^{U}{}_{1k_{1}}, p^{F}{}_{1k_{1}} > ... < p^{T}{}_{nk_{n}}, p^{C}{}_{nk_{n}}, p^{U}{}_{nk_{n}}, p^{F}{}_{nk_{n}} >$$

$$\sum_{\sigma\in S_n} < q^T{}_{1k_1}, q^C{}_{1k_1}, q^U{}_{1k_1}, q^F{}_{1k_1} > \ldots < q^T{}_{nk_n}, q^C{}_{nk_n}, q^U{}_{nk_n}, q^F{}_{nk_n} >$$

$$= \left(\sum_{(k_1,k_2,\ldots,k_n)\in S_n} < p^T{}_{1k_1}, p^C{}_{1k_1}, p^U{}_{1k_1}, p^F{}_{1k_1} > \ldots < p^T{}_{nk_n}, p^C{}_{nk_n}, p^U{}_{nk_n}, p^F{}_{nk_n} > \right) \det(Q)$$

$$= \det(P)\det(Q)$$

(iii) We know that

$$\det(PQ) = \det\left(\sum_{k=1}^{n} p^T{}_{ik} \wedge q^T{}_{kj}, \sum_{k=1}^{n} p^C{}_{ik} \wedge q^C{}_{kj}, \prod_{k=1}^{n} p^U{}_{ik} \vee q^U{}_{kj}, \prod_{k=1}^{n} p^F{}_{ik} \vee q^F{}_{kj} \right)$$

$$\sum_{\sigma\in S_n} (\sum_{k=1}^{n} p^T{}_{ik} \wedge q^T{}_{k\sigma(1)}, \sum_{k=1}^{n} p^C{}_{ik} \wedge q^C{}_{k\sigma(1)}, \prod_{k=1}^{n} p^U{}_{ik} \vee q^U{}_{k\sigma(1)}, \prod_{k=1}^{n} p^F{}_{ik} \vee q^F{}_{k\sigma(1)} \ldots$$

$$\sum_{k=1}^{n} p^T{}_{nk} \wedge q^T{}_{k\sigma(n)}, \sum_{k=1}^{n} p^C{}_{nk} \wedge q^C{}_{k\sigma(n)}, \prod_{k=1}^{n} p^U{}_{nk} \vee q^U{}_{k\sigma(n)}, \prod_{k=1}^{n} p^F{}_{nk} \vee q^F{}_{k\sigma(n)})$$

$$= \sum_{\sigma\in S_n} (\bigcap_{t\le s, t\le n} \left(p^T{}_{1s} \vee q^T{}_{t\sigma(1)} \right), \bigcap_{t\le s, t\le n} \left(p^C{}_{1s} \vee q^C{}_{t\sigma(1)} \right), \bigcup_{t\le s, t\le n} \left(p^U{}_{1s} \wedge q^U{}_{t\sigma(1)} \right), \bigcup_{t\le s, t\le n} \left(p^F{}_{1s} \wedge q^F{}_{t\sigma(1)} \right) \ldots$$

$$\bigcap_{t\le s, t\le n} \left(p^T{}_{ns} \vee q^T{}_{t\sigma(n)} \right), \bigcap_{t\le s, t\le n} \left(p^C{}_{ns} \vee q^C{}_{t\sigma(n)} \right), \bigcup_{t\le s, t\le n} \left(p^U{}_{ns} \wedge q^U{}_{t\sigma(n)} \right), \bigcup_{t\le s, t\le n} \left(p^F{}_{ns} \wedge q^F{}_{t\sigma(n)} \right))$$

$$\le \sum_{\sigma\in S_n}^{n} \left(p^T{}_{1\sigma(1)} \vee q^T{}_{1\sigma(1)} \right) \wedge \left(p^C{}_{1\sigma(1)} \vee q^C{}_{1\sigma(1)} \right) \wedge \left(p^U{}_{1\sigma(1)} \vee q^U{}_{1\sigma(1)} \right) \wedge \left(p^F{}_{1\sigma(1)} \vee q^F{}_{1\sigma(1)} \right) \ldots.$$

$$\bigcup_{t\le s, t\le n} \left(p^T{}_{n\sigma(n)} \wedge q^T{}_{n\sigma(n)} \right) \wedge \left(p^C{}_{n\sigma(n)} \wedge q^C{}_{n\sigma(n)} \right) \wedge \left(p^U{}_{n\sigma(n)} \wedge q^U{}_{n\sigma(n)} \right) \wedge \left(p^F{}_{n\sigma(n)} \wedge q^F{}_{n\sigma(n)} \right)$$

$$= \det\left(\left(\left(p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} \right) + \left(q^T{}_{ij}, q^C{}_{ij}, q^U{}_{ij}, q^F{}_{ij} \right) \right)_n \right)$$

$$= \det\left(P + Q \right)$$

Hence the theorem.

Corollary 10.1. Let A be a QPNFM, $P_r = \left(p_{ij} \right) \in (QPNFM)_n$ (r = 1,2,3,…,m).Then

(i) $\det(P_1)\det(P_1)\ldots\det(P_m) \le \det\left(\sum_{r=1}^{m} P_r \right)$ whrere $\sum_{r=1}^{m} P_r = \left(\sum_{r=1}^{m} a^r{}_{ij} \right)_n \in (QPNFM)_n.$

(ii) $\det(P^{(r)}) = \det(P)$, where $P = \left(p_{ij} \right) \in (QPNFM)_n$ and $r \in N$.

Example 10.1. Consider the 4x4 QPNFM

$$P=\begin{pmatrix} <p^T_{11},p^C_{11},p^U_{11},p^F_{11}> & <p^T_{12},p^C_{12},p^U_{12},p^F_{12}> & \cdots & <p^T_{14},p^C_{14},p^U_{14},p^F_{14}> \\ <p^T_{21},p^C_{21},p^U_{21},p^F_{21}> & <p^T_{22},p^C_{22},p^U_{22},p^F_{22}> & \cdots & <p^T_{24},p^C_{24},p^U_{24},p^F_{24}> \\ <p^T_{31},p^C_{31},p^U_{31},p^F_{31}> & <p^T_{31},p^C_{31},p^U_{31},p^F_{31}> & \cdots & <p^T_{34},p^C_{34},p^U_{34},p^F_{34}> \\ <p^T_{41},p^C_{41},p^U_{41},p^F_{41}> & <p^T_{42},p^C_{42},p^U_{42},p^F_{42}> & \cdots & <p^T_{44},p^C_{44},p^U_{44},p^F_{44}> \end{pmatrix}$$

We compute the determinant of the matrix above using the following QPNFM.

$$=\begin{vmatrix} <p^T_{11},p^C_{11},p^U_{11},p^F_{11}> & <p^T_{12},p^C_{12},p^U_{12},p^F_{12}> \\ <p^T_{21},p^C_{21},p^U_{21},p^F_{21}> & <p^T_{22},p^C_{22},p^U_{22},p^F_{22}> \end{vmatrix}_{1<2}$$

$$\begin{vmatrix} <p^T_{33},p^C_{33},p^U_{33},p^F_{33}> & <p^T_{34},p^C_{34},p^U_{34},p^F_{34}> \\ <p^T_{43},p^C_{43},p^U_{43},p^F_{43}> & <p^T_{44},p^C_{44},p^U_{44},p^F_{44}> \end{vmatrix}$$

$$+\begin{vmatrix} <p^T_{11},p^C_{11},p^U_{11},p^F_{11}> & <p^T_{13},p^C_{13},p^U_{13},p^F_{13}> \\ <p^T_{21},p^C_{21},p^U_{21},p^F_{21}> & <p^T_{23},p^C_{23},p^U_{23},p^F_{23}> \end{vmatrix}_{1<3}$$

$$\begin{matrix} <p^T_{32},p^C_{32},p^U_{32},p^F_{32}> & <p^T_{34},p^C_{34},p^U_{34},p^F_{34}> \\ <p^T_{42},p^C_{42},p^U_{42},p^F_{42}> & <p^T_{44},p^C_{44},p^U_{44},p^F_{44}> \end{matrix}$$

$$+\begin{vmatrix} <p^T_{11},p^C_{11},p^U_{11},p^F_{11}> & <p^T_{14},p^C_{14},p^U_{14},p^F_{14}> \\ <p^T_{21},p^C_{21},p^U_{21},p^F_{21}> & <p^T_{24},p^C_{24},p^U_{24},p^F_{24}> \end{vmatrix}_{1<4}$$

$$\begin{vmatrix} <p^T_{32},p^C_{32},p^U_{32},p^F_{32}> & <p^T_{33},p^C_{33},p^U_{33},p^F_{33}> \\ <p^T_{42},p^C_{42},p^U_{42},p^F_{42}> & <p^T_{43},p^C_{43},p^U_{43},p^F_{43}> \end{vmatrix}$$

$$+\begin{vmatrix} <p^T_{12},p^C_{12},p^U_{12},p^F_{12}> & <p^T_{13},p^C_{13},p^U_{13},p^F_{13}> \\ <p^T_{22},p^C_{22},p^U_{22},p^F_{22}> & <p^T_{23},p^C_{23},p^U_{23},p^F_{23}> \end{vmatrix}_{2<3}$$

$$\begin{vmatrix} <p^T_{31},p^C_{31},p^U_{31},p^F_{31}> & <p^T_{34},p^C_{34},p^U_{34},p^F_{34}> \\ <p^T_{41},p^C_{41},p^U_{41},p^F_{41}> & <p^T_{44},p^C_{44},p^U_{44},p^F_{44}> \end{vmatrix}$$

$$+\begin{vmatrix} <p^T_{12},p^C_{12},p^U_{12},p^F_{12}> & <p^T_{14},p^C_{14},p^U_{14},p^F_{14}> \\ <p^T_{22},p^C_{22},p^U_{22},p^F_{22}> & <p^T_{24},p^C_{24},p^U_{24},p^F_{24}> \end{vmatrix}_{2<4}$$

$$\begin{vmatrix} <p^T_{31},p^C_{31},p^U_{31},p^F_{31}> & <p^T_{33},p^C_{33},p^U_{33},p^F_{33}> \\ <p^T_{41},p^C_{41},p^U_{41},p^F_{41}> & <p^T_{43},p^C_{43},p^U_{43},p^F_{43}> \end{vmatrix}$$

$$+\begin{vmatrix} <p^T_{13},p^C_{13},p^U_{13},p^F_{13}> & <p^T_{14},p^C_{14},p^U_{14},p^F_{14}> \\ <p^T_{23},p^C_{23},p^U_{23},p^F_{23}> & <p^T_{24},p^C_{24},p^U_{24},p^F_{24}> \end{vmatrix}_{2<4}$$

$$\begin{vmatrix} < p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} > & < p^T{}_{32}, p^C{}_{32}, p^U{}_{32}, p^F{}_{32} > \\ < p^T{}_{41}, p^C{}_{41}, p^U{}_{41}, p^F{}_{41} > & < p^T{}_{42}, p^C{}_{42}, p^U{}_{42}, p^F{}_{42} > \end{vmatrix}$$

By applying this method, we can determine the determinant of the given QPNFM.

$$P = \begin{pmatrix} <0.8,0.7,0.3,0.5> & <0.7,0.5,0.4,0.2> & <0.7,0.8,0.3,0.2> & <0.3,0.6,0.3,0.1> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> & <0.4,0.7,0.8,0.2> & <0.1,0.3,0.5,0.9> \\ <0.4,0.6,0.8,0.7> & <0.2,0.3,0.4,0.5> & <0.6,0.3,0.5,0.7> & <0.9,0.8,0.5,0.4> \\ <0.5,0.3,0.4,0.8> & <0.4,0.8,0.5,0.2> & <0.1,0.3,0.6,0.8> & <0.7,0.3,0.2,0.1> \end{pmatrix}$$

$$= \begin{vmatrix} <0.8,0.7,0.3,0.5> & <0.7,0.5,0.4,0.2> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> \end{vmatrix} \begin{vmatrix} <0.6,0.3,0.5,0.7> & <0.9,0.8,0.5,0.4> \\ <0.1,0.3,0.6,0.8> & <0.7,0.3,0.2,0.1> \end{vmatrix}$$

$$+ \begin{vmatrix} <0.8,0.7,0.3,0.5> & <0.7,0.8,0.3,0.2> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> \end{vmatrix} \begin{vmatrix} <0.2,0.3,0.4,0.5> & <0.9,0.8,0.5,0.4> \\ <0.4,0.8,0.5,0.2> & <0.7,0.3,0.2,0.1> \end{vmatrix}$$

$$+ \begin{vmatrix} <0.8,0.7,0.3,0.5> & <0.3,0.6,0.3,0.1> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> \end{vmatrix} \begin{vmatrix} <0.2,0.3,0.4,0.5> & <0.6,0.3,0.5,0.7> \\ <0.4,0.8,0.5,0.2> & <0.1,0.3,0.6,0.8> \end{vmatrix}$$

$$+ \begin{vmatrix} <0.7,0.5,0.4,0.2> & <0.7,0.8,0.3,0.2> \\ <0.1,0.3,0.5,0.9> & <0.4,0.7,0.8,0.2> \end{vmatrix} \begin{vmatrix} <0.4,0.6,0.8,0.7> & <0.9,0.8,0.5,0.4> \\ <0.5,0.3,0.4,0.8> & <0.7,0.3,0.2,0.1> \end{vmatrix}$$

$$+ \begin{vmatrix} <0.7,0.5,0.4,0.2> & <0.3,0.6,0.3,0.1> \\ <0.1,0.3,0.5,0.9> & <0.1,0.3,0.5,0.9> \end{vmatrix} \begin{vmatrix} <0.4,0.6,0.8,0.7> & <0.6,0.3,0.5,0.7> \\ <0.5,0.3,0.4,0.8> & <0.1,0.3,0.6,0.8> \end{vmatrix}$$

$$+ \begin{vmatrix} <0.7,0.8,0.3,0.2> & <0.3,0.6,0.3,0.1> \\ <0.4,0.7,0.8,0.2> & <0.1,0.3,0.5,0.9> \end{vmatrix} \begin{vmatrix} <0.4,0.6,0.8,0.7> & <0.2,0.3,0.4,0.5> \\ <0.5,0.3,0.4,0.8> & <0.2,0.3,0.4,0.5> \end{vmatrix}$$

$$= [<0.1,0.3,0.5,0.9> + <0.5,0.5,0.4,0.3>][<0.6,0.3,0.5,0.7> + <0.1,0.3,0.6,0.8>] +$$

$$[<0.1,0.3,0.5,0.9> + <0.5,0.8,0.4,0.3>][<0.2,0.3,0.4,0.8> + <0.1,0.3,0.6,0.8>] +$$

$$[<0.1,0.3,0.5,0.9> + <0.3,0.6,0.4,0.3>][<0.1,0.3,0.6,0.8> + <0.4,0.3,0.5,0.7>] +$$

$$[<0.4,0.5,0.8,0.2> + <0.1,0.3,0.5,0.9>][<0.4,0.3,0.8,0.7> + <0.5,0.3,0.8,0.7>] +$$

$$[<0.1,0.3,0.5,0.9> + <0.1,0.3,0.5,0.9>][<0.1,0.3,0.8,0.8> + <0.5,0.3,0.5,0.8>] +$$

$$[<0.1,0.3,0.5,0.9> + <0.3,0.6,0.8,0.2>][<0.2,0.3,0.8,0.7> + <0.2,0.3,0.4,0.8>]$$

$$= <0.5,0.5,0.4,0.3><0.6,0.3,0.5,0.7> + <0.5,0.8,0.4,0.3><0.2,0.3,0.4,0.8> +$$

$< 0.3, 0.6, 0.4, 0.3 >< 0.4, 0.3, 0.5, 0.7 > + < 0.4, 0.5, 0.5, 0.2 >< 0.5, 0.3, 0.8, 0.7 > +$
$< 0.1, 0.3, 0.5, 0.9 >< 0.5, 0.3, 0.5, 0.8 > + < 0.3, 0.6, 0.5, 0.2 >< 0.2, 0.3, 0.4, 0.7 >$
$=< 0.5, 0.3, 0.5, 0.7 > + < 0.2, 0.3, 0.4, 0.8 > + < 0.3, 0.3, 0.5, 0.7 > + < 0.4, 0.3, 0.8, 0.7 > +$
$< 0.1, 0.3, 0.5, 0.9 > + < 0.2, 0.3, 0.5, 0.7 >$
$=< 0.5, 0.3, 0.4, 0.7 >$

11. An Algorithm Based on QPNFM in a Decision-Making Problem

Definition 11.1. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. Then, for an QPNSS (η,E) over Y the degree of TM and the degree of − CM of an element y_i to $\eta(e_j)$ denoted by $T_{\eta(e_j)}(y_i)$ and $C_{\eta(e_j)}(y_i)$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{T_{\eta(e_j)}}(y_i) = \sum_{k=1}^{N}\left[T_{\eta(e_j)}(y_i) - T_{\eta(e_j)}(y_k)\right]$$

$$S_{C_{\eta(e_j)}}(y_i) = \sum_{k=1}^{N}\left[C_{\eta(e_j)}(y_i) - C_{\eta(e_j)}(y_k)\right]$$

Definition 11.2. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. Then, for an QPNSS (η,E) over Y the degree of UM and the degree of − FM of an element xi to $\eta(e_j)$ denoted by $U_{\eta(e_j)}(y_i)$ and $F_{\eta(e_j)}(y_i)$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{U_{\eta(e_j)}}(y_i) = -\sum_{k=1}^{N}\left[U_{\eta(e_j)}(y_i) - U_{\eta(e_j)}(y_k)\right]$$

$$S_{F_{\eta(e_j)}}(y_i) = \sum_{K=1}^{N}\left[F_{\eta(e_j)}(y_i) - F_{\eta(e_j)}(y_k)\right]$$

Definition 11.3. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. For an QPNSS (η,E) over Y, the scores of the TM, CM, UM, and FM of y_i for each e_j be denoted by $S_{T_{\eta(e_j)}}(y_i), S_{C_{\eta(e_j)}}(y_i), S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ respectively. Then, the total score of x_i for each e_j is denoted by $T_{\eta(e_j)}(y_i) = S_{T_{\eta(e_j)}}(y_i) + S_{C_{\eta(e_j)}}(y_i) + S_{U_{\eta(e_j)}}(y_i) + S_{F_{\eta(e_j)}}(y_i)$

Based on the above definitions, we give the steps of the proposed algorithm as follows:

Algorithm:

Step 1. For the universal set Y = {$y_1, y_2, \ldots, y_n$} and the parameter set $E = \{e_1, e_2, \ldots, e_m\}$ input the matrix representation of an QPNSS (η,E) in tabular form, according to a decision-maker.

Step 2. Reference to the input matrix obtained in step 1 and using the Definitions 11.1 and 11.2, we compute $S_{T_{\eta(e_j)}}(y_i), S_{C_{\eta(e_j)}}(y_i), S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ for xi for each ej where i = 1 to n; j = 1 to m.

Step 3. Taking the results obtained in step 2 and using the Definition 11.3, compute the score $\square_{T_{\eta(e_j)}}(x_i)$ of xi for each $ej$where i = 1 to n; j = 1 to m.

Step 4. Compute the overall score ui for xiin such a way that

$$v_i = \square_{T_{\eta(e1)}}(y_i) + \square_{C_{\eta(e2)}}(y_i) + \square_{U_{\eta(e3)}}(y_i) + \ldots + \square_{F_{\eta(em)}}(y_i)$$

Step 5. Find k, for which $v_k = \max_{xi \in X}\{v_i\}$. Then, $x_k \in X$ is the optimal choice.

Step 6. In case of a tie, either we take both as an optimal choice or we reassess all the values with the expert's advice and repeat all the previous steps.

12. To demonstrate the practical application of the algorithm, we present the following example.

To implement the proposed algorithm successfully in a real-life context, we consider the following problem:

An agricultural planner wants to select the most suitable crop for cultivation in a specific region. However, limited expertise in agronomy complicates the decision-making process due to various conflicting factors involved in crop selection. To address this challenge, the planner consults a group of experts and decision makers (DMs) who specialize in agricultural sciences. The DMs evaluate five alternative crops according to a set of defined parameters.

Define Parameters

First, the five critical parameters for evaluating the crop options are identified as follows:

1. **Soil Fertility**: Measures the nutrient content and suitability of the soil for crop growth.
2. **Water Requirements**: Assesses the water needs of each crop relative to available irrigation and rainfall.
3. **Climate Suitability**: Evaluates how well the local climate matches the optimal growing conditions for each crop.
4. **Pest Resistance**: Indicates the crop's resilience against local pests and diseases, impacting yield stability.
5. **Economic Viability**: Considers market price, demand, and yield potential to determine the profitability of each crop.

Evaluation Procedure

To select the best crop alternative, the assessment procedure is executed as follows:

Step 1: Consider a set of five crops represented as C={ c_1,c_2,c_3,c_4,c_5}, where:

c_1 = Rice, c_2 = Wheat, c_3 = Maize, c_4 = Millet and c_5 = Barley

And define the set of parameters as P={p_1, p_2, p_3, p_4, p_5}, where:

p_1= Soil Fertility, p_2= Water Requirements, p_3 = Climate Suitability, p_4 = Pest Resistance

p_5 = Economic Viability

Based on the evaluations from the DMs, the decision matrix reflecting the five crops and five evaluation criteria under the Multi-Criteria Decision-Making framework is presented in Table 2.

Table 2. Tabular illustration of QPNFM to describe the set of five crops

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	<0.8,0.8,0.9,0.6>	<0.9,0.5,0.4,0.1>	<0.6,0.4,0.5,0.2>	<0.9,0.7,0.2,0.3>	<0.6,0.2,0.1,0.3>
y_2	<0.8,0.4,0.3,0.2>	<0.6,0.1,0.2,0.3>	<0.8,0.3,0.7,0.6>	<0.4,0.1,0.3,0.2>	<0.5,0.4,0.3,0.1>
y_3	<0.9,0.8,0.9,0.6>	<0.4,0.8,0.9,0.1>	<0.3,0.8,0.1,0.6>	<0.5,0.2,0.3,0.6>	<0.5,0.8,0.1,0.3>
y_4	<0.7,0.1,0.2,0.3>	<0.5,0.7,0.1,0.6>	<0.9,0.8,0.2,0.3>	<0.4,0.1,0.3,0.5>	<0.7,0.3,0.2,0.1>
y_5	<0.6,0.3,0.5,0.1>	<0.5,0.4,0.3,0.2>	<0.9,0.2,0.3,0.4>	<0.8,0.5,0.2,0.6>	<0.2,0.4,0.1,0.2>

Step 2. The score of the truth-membership degrees $S_{T_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 3.

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	0.2	1.6	-0.5	1.5	0.5
y_2	0.2	0.1	0.5	-1	0
y_3	0.7	-0.9	-2	-0.5	0
y_4	-0.3	-0.4	1	-1	1
y_5	-0.8	-0.4	1	1	-1.5

The score of the contradiction-membership degrees $S_{C_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 4.

Table 4. Tabular illustration of the score of contradiction-membership degree

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	1.6	0	-0.5	1.9	-1.1
y_2	-0.4	-2	-1	-1.1	-0.1
y_3	1.6	1.5	1.5	-0.6	1.9
y_4	-1.9	1	1.5	-1.1	-0.6
y_5	-0.9	-0.5	-1.5	0.9	-0.1

The score of the unknown-membership degrees $S_{U_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 5.

Table 5. Tabular illustration of the score of unknown-membership degree

Y/E	e1	e2	e3	e4	e5
y_1	1.7	0.1	0.7	-0.3	-0.3
y_2	-1.3	-0.9	1.7	0.2	0.7
y_3	1.7	2.6	-1.3	0.2	-0.3
y_4	-1.8	-1.4	-0.8	0.2	0.2
y_5	-0.3	-0.4	-0.3	-0.3	-0.3

The score of the false-membership degrees $S_{F_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 6.

Table 6. Tabular illustration of the score of false-membership degree

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	1.2	-0.8	-1.1	-0.8	0.5
y_2	-0.8	0.2	0.9	-1.3	-0.5
y_3	1.2	-0.8	0.9	0.7	0.5
y_4	-0.3	1.7	-0.6	0.2	-0.5
y_5	-1.3	-0.3	-0.1	0.7	0

Step 3. By using Table 3 to Table 6, the score $\eta_{(ej)}(x_i)$ for (η,E) is exhibited in Table 7.

Table 7. Tabular illustration of the score $\eta_{(ej)}(x_i)$.

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	4.7	0.9	-0.5	2.3	-0.4
y_2	-2.3	-2.6	2.1	-3.2	0.1
y_3	5.2	3.4	-0.9	-0.2	2.1
y_4	-4.3	0.9	1.1	-1.7	0.1
y_5	-3.3	-1.6	-0.9	2.3	-1.9

Step 4. Now, we calculate the overall score given as:

$v_1 = 7$, $v_2 = -5.9$, $v_3 = 9.6$, $v_4 = -3.9$, $v_5 = -5.4$.

Step 5. Thus, $v_k = \max_{y_i \in Y}\{v_1, v_2, v_3, v_4, v_5\} = v_3$. Therefore, v_3 is the optimal choice object for the decision maker.

13. Conclusion and Future Work

In this study, we investigated determinant theory for Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs), establishing foundational properties and proofs for key determinant relationships, such as det(Padj(P)) = det(P) = det(adj(P)P). Additionally, we proposed an efficient method to calculate determinants for large QPNFMs, addressing computational challenges posed by matrices with higher dimensions. The introduction of a structured algorithm for decision-making problems further enhances the applicability of QPNFMs in complex decision scenarios. Our

illustrative example demonstrates the practical relevance of our approach, verifying the effectiveness and robustness of our methodology in real-world applications.

Future research could expand on this work by exploring determinant properties in other classes of neutrosophic fuzzy matrices, including different partitioning schemes beyond quadri-partitioned structures. Investigating QPNFMs under various types of operations, such as matrix inversions and generalized matrix functions, could also deepen understanding of their theoretical and practical utility. Additionally, developing more sophisticated algorithms for multi-criteria decision-making using QPNFMs could broaden the scope of applications across different domains, such as engineering, economics, and artificial intelligence. Finally, computational optimizations leveraging parallel processing could make determinant calculation methods for QPNFMs more efficient, especially for high-dimensional matrices, further enhancing their viability in large-scale applications.

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Statistical Analysis of Foreign Investment Using Neutrosophic Interval Based Approach

Jamil Afzal[1], Naim Mathlouthi[2], Mohamed Saer Rahal[3], Chen Yongmei[4*], Aftab Haider[5], Muhammad Adeel Afzal[6]

[1]School of International Law, Southwest University of Political Science and Law, China; sirjamilafzal@gmail.com

[2]Law department, Prince Mohammad Bin Fahad University (PMU), Khobar, Saudi Arabia; nmathlouthi@pmu.edu.sa

[3]Law department, Prince Mohammad Bin Fahd University, Saudi Arabia; mrahal@pmu.edu.sa

[4]School of International Law, Southwest University of Political Science and Law, China; chenyongmei@swupl.edu.cn

[5]School of International Law, Southwest University of Political Science and Law, China; aftabhaider@awkum.edu.pk

[6]College of Economics & Management Science, China Three Gorges University, Yichang, China; adeel@ctgu.edu.cn

Abstract

Pakistan, a nation at the crossroads of South Asia, has been actively striving to attract foreign investment to boost its economic growth and development. As a country rich in natural resources, with a young and vibrant workforce, and a strategic geographic location, Pakistan offers significant opportunities for foreign investors seeking to explore new markets and tap into its potential. The foreign investment policies in Pakistan have undergone substantial reforms over the years to create a more investor-friendly environment. These policies are designed to facilitate and incentivize foreign direct investment while ensuring the protection of investors' rights and contributing to the overall economic progress of the country. The government of Pakistan, recognizing the critical role of foreign investment in driving economic transformation, has implemented various initiatives and measures to attract investors from around the world. The purpose of this study was to investigate the use of Neutrosophic Analysis during Statistical Analysis of Foreign Investment using Imprecise Information, and this approach provides remarkable results. The neutrosophic approach has provided a comprehensive understanding of the various perspectives and uncertainties involved in foreign investment strategies. This study's results demonstrated that this Interval-Based Approach may be used for better understanding instead of the classical approach.

Keywords: Foreign Investment; Imprecise Information; Neutrosophic Approach; Statistical Analysis

1. Introduction

The policy of the host country plays an important role and influences foreign investment; host nations may implement measures to encourage international investment or may impose various restrictions on foreign engagement in their economies[1]. The amount of investment by these corporations is influenced by the political statements of host countries. In addition, policies of the host country can direct investors to invest in those areas that are thought to be particularly crucial for the development of the nation[2]. Globalization has made the world a small, interconnected community, and emerging nations are working to adapt foreign investment policies[3]. They constantly work to entice foreign investment inflow into their economy in order to overcome the shortage of technology and cash. Both the investor and the invested party take advantage of the foreign investment in their country. The ability to withdraw skill resources from the host nation is made possible by the local market[4]. As compared to local investment in the country, foreign direct investment helped its contribution to financial growth by creating employment possibilities for local people. Foreign direct investment flows are crucial for both emerging and developed countries. Because capital is scarce in emerging countries, the peripheral productivity of capital is higher. Investors in the industrialized world, on the other hand, are looking for substantial returns on their money; there is an advantage to both parties when capital moves internationally[5].

In Pakistan, over the past ten years, foreign direct investment has grown at a rate that is at least twice as fast as trade; the government's top priority is to increase foreign direct investment[6]. Pakistan is a desirable location for international investment due to its geographical location. With a population of more than 241 million, Pakistan offers a sizable market for consumer goods as well as a developing middle class with sufficient purchasing power, access to cheap labor to lower production costs, and a strategic location in South Asia[7]. Pakistan is a rich country due to its geographical location but lack of resource management. Now Pakistan makes favorable policies to encourage foreign direct investment. Investment is typically focused on industries, which leads to the development of the economy and increases productivity. Foreign investment is a combination of finance, technological management, and market access; as investors earn profit they frequently reinvest their earnings rather than sending them elsewhere. Another advantage of

foreign investment is boosting confidence. Foreign investment inflows increase domestic investment confidence, which could lead to positive feedback on both domestic and foreign investment as well as international commerce and output[8]. The impact of foreign investment on the host nation's balance payments has ramifications. Foreign direct investment in developing countries helps to make their economy strong. Every field of the state grows up with the help of FDI. The state makes such policies that attract foreign investment so that they invest in every field and the state is responsible for facilitating the FDI investor[9].

The focus of this study was to investigate the use of Neutrosophic Analysis during Statistical Analysis of Foreign Investment using Imprecise Information. This study will provide a deep insight into the;

- Use of Neutrosophic Approach for Interval-Based Data Sets of Foreign Investment in Pakistan
- Comparison of Classical and Neutrosophic Approaches Used for Statistical Analysis of Foreign Investment
- To provide Methodological Innovation for Decision-Making under Uncertainty for Statistical Analysis

2. Prior Investigation

Foreign investment policies play a crucial role in shaping a country's economic development and attracting foreign capital[10]. In the case of Pakistan, there has been significant academic research and prior investigation exploring various aspects of foreign investment policies and their implications.

A common theme in the literature is the importance of liberalization and policy reforms to attract foreign investment[11]. Pakistan has undertaken several reforms to create a more favorable investment climate, such as the establishment of Special Economic Zones (SEZs), deregulation, and simplification of investment procedures. Pakistan also has taken steps to liberalize its economy and create a more open investment climate[12]. The country has implemented policies to facilitate foreign investment, reduce bureaucratic hurdles, and simplify investment procedures. These measures aim to encourage foreign investors to enter various sectors of the economy. Research emphasizes the role of investment incentives in attracting foreign capital. These incentives include

tax breaks, exemptions, repatriation of profits, and protection against expropriation. Pakistan offers a range of investment incentives to attract foreign capital. These incentives include tax exemptions, repatriation of profits and dividends, and remittance of capital. In certain sectors or regions, additional incentives may be provided, such as reduced customs duties or land allocation. Pakistan has established Special Economic Zones to provide a conducive environment for foreign investment. These zones offer various benefits to investors, including tax incentives, streamlined regulations, and infrastructure support[13]. The government has identified several priority sectors for investment within these zones, such as manufacturing, information technology, and export-oriented industries. To attract foreign investment in strategic sectors, Pakistan has established sector-specific policies. These policies aim to encourage investment in areas such as energy, infrastructure development, telecommunications, agriculture, and tourism[14]. The government has also implemented changes to make doing business in industries easier. The focus is on evaluating the impact of these policies on investment inflows, technology transfer, employment generation, and overall scrotal development. The institutional framework and governance structures also receive attention in the literature. Scholars assess the efficiency and effectiveness of institutions responsible for implementing foreign investment policies in Pakistan. They analyze issues related to bureaucratic procedures, legal framework, corruption[15], and transparency, which can affect investor confidence[16]. Pakistan has entered bilateral investment treaties with multiple countries to encourage and safeguard foreign investment. These agreements offer legal protection to foreign investors, encompassing provisions related to dispute resolution and compensation in the event of expropriation[17]. Ensuring effective intellectual property rights protection is crucial for attracting foreign investment. Pakistan has made efforts to strengthen its intellectual property regime and align it with international standards. The government has taken steps to enhance enforcement mechanisms and raise awareness about the importance of intellectual property rights[18]. Pakistan has been working to improve investor protection mechanisms. The country has established the Securities and Exchange Commission of Pakistan (SECP) to regulate the capital market and protect investor interests[19]. Efforts have also been made to strengthen corporate governance practices and enhance transparency in business operations.

3. Contradictory and Incomplete Information

The nature of foreign investment data often involves contradictory or incomplete information. Neutrosophic Theory allows for the consideration of these contradictions within its logical framework, providing a more comprehensive analysis compared to traditional methods.

3.1 Theoretical justification

Neutrosophic Theory is the most suitable theoretical framework for this it aligns well with the document's objectives of analyzing foreign investment data under conditions of uncertainty, imprecision, and indeterminacy. Its ability to handle interval-based data and contradictory information makes it a superior choice for this specific analysis[20].

3.2 Handling Uncertainty and Imprecision

The study deals with the analysis of foreign investment data that is inherently uncertain and imprecise. Neutrosophic Theory is designed to handle such conditions by allowing for the inclusion of indeterminate, true, and false values simultaneously. This flexibility is crucial in analyzing complex, real-world data where traditional binary logic might fall short.

3.3 Interval-Based Analysis

The study employs interval-based data analysis, where each data point is represented as an interval rather than a single value. Neutrosophic Theory excels in this area by providing a framework that can accommodate the vagueness and uncertainty inherent in interval data, making it a more suitable approach than classical statistical methods.

3.4 Decision-Making under Uncertainty

The study emphasizes making informed decisions in the face of uncertainties, such as foreign investment strategies in Pakistan. Neutrosophic Theory supports decision-making in such uncertain environments by offering a more nuanced and adaptable analytical framework. Following fig. 1 represents conceptual framework of this study;

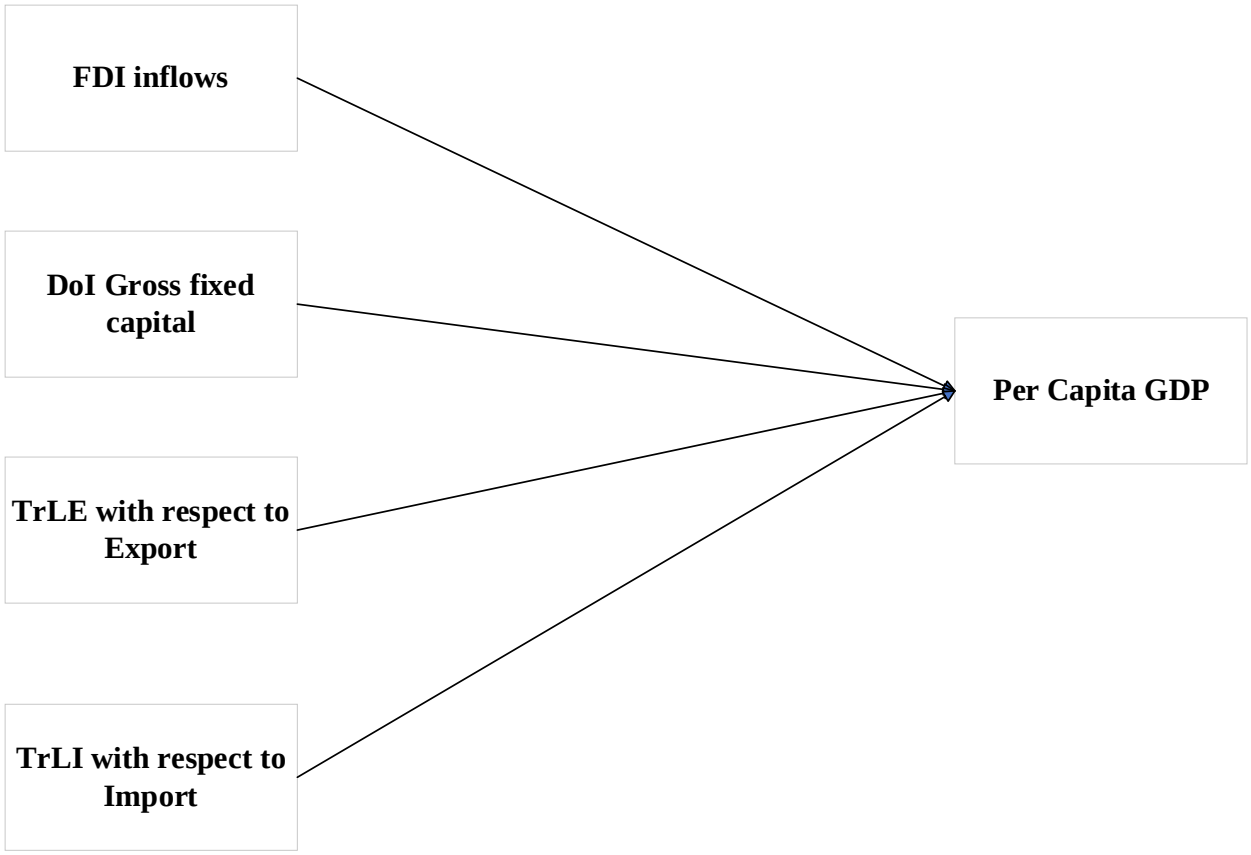

Fig 1: Conceptual Framework

4. Methodology and Analysis

This research explores the correlation between financial growth in Pakistan and foreign direct investment (FDI), offering more up-to-date analysis than previous research. The study utilizes annual data covering the period from 2013 to 2023 to explore the co-integration, or link, between domestic product and foreign direct investment. While monthly or quarterly data would have provided more observations, such data is not widely available for most variables in this study. In this study, the focus lies on the dependent variable, which is the economic output; FDI (Foreign Direct Investment), DoI (Domestic Investment), and TrL (Trade Liberalization) are among the independent variables, represented by imports and exports combined. The data used for analysis is sourced from the *World Bank* Development Indicators.

4.1 Classical Data Analysis

This study investigates the correlation between financial growth (Y) and the variables of FDI, DoI, and TrL; to examine the relationship between economic development and Foreign Direct

Investmen. The following table 1 shows a detail list of variables used to complete this study along with their classical data analysis;

Sr #	Details of Variable	Mean	Median	Mode	Standard dev.
01	Dependent Variable = Per Capita GDP of Pakistan	1444.9	1437.2	1669.6	120.3
02	Independent Variable = FDI inflows with respect to annual percentage of GDP	0.58	0.60	0.50	0.24
03	Independent Variable = DoI Gross fixed capital formation percentage of GDP	14.05	15.4	15.4	4.69
04	Independent Variable = TrLE with respect to Export	2.04	-1.30	-11.10	10.79
05	Independent Variable = TrLI with respect to Import	6.39	2.50	-28.90	16.91

Table 1: Classical Data Analysis of Variables

4.2 Neutrosophic Analysis

The neutrosophic method provides a framework to analyze and reason with incomplete, uncertain, or contradictory information[21]. In many real-world scenarios, especially in domains like decision-making, risk assessment, and expert systems, uncertainties and contradictions are prevalent. The neutrosophic method helps in better understanding and representing such complex situations[22].

Sr#	Year	Interval	Neutrosophic Analysis
1	2013	[1236.9, 1259.6]	1236.9 + 1259.6 I_G; $I_G \epsilon$[0, 0.018]
2	2014	[1259.6,1303.2]	1259.6 + 1303.2 I_G; $I_G \epsilon$[0, 0.033]
3	2015	[1302.2,1421.9]	1303.2 + 1421.9 I_G; $I_G \epsilon$[0, 0.083]
4	2016	[1421.9,1468.9]	1421.9 + 1468.9 I_G; $I_G \epsilon$[0, 0.031]
5	2017	[1468.9,1567.7]	1468.9 + 1567.7 I_G; $I_G \epsilon$[0, 0.063]
6	2018	[1567.7,1620.8]	1567.7 + 1620.8 I_G; $I_G \epsilon$[0, 0.032]
7	2019	[1620.8,1437.2]	1620.8 + 1437.2 I_G; $I_G \epsilon$[0, -0.127]
8	2020	[1437.2,1324.1]	1437.2 + 1324.1 I_G; $I_G \epsilon$[0, -0.085]
9	2021	[1324.1,1507.9]	1324.1 + 1507.9I_G; $I_G \epsilon$[0, 0.121]
10	2022	[1507.9,1592.1]	1507.9+ 1592.1I_G; $I_G \epsilon$[0, 0.052]
11	2023	[1592.1,1390.8]	1592.1+ 1390.8I_G; $I_G \epsilon$[0, -0.144]

Table 2: Per Capita GDP of Pakistan

Table 2 is highlighting the neutrosophic analysis of Per Capita GDP of Pakistan. Neutrosophic formula provides a flexible equation for each interval with indeterminacy interval. In 2013 the maximum value of interval is 1259.6 and the minimum value is 1236.9.

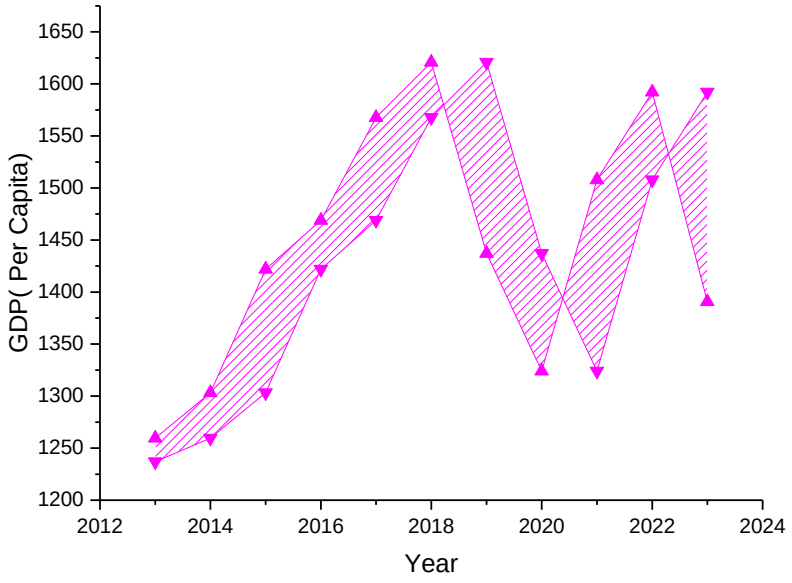

Fig 2: Graphical Representation of Per Capita GDP of Pakistan

The neutrosophic Graphical Representation of Per Capita GDP of Pakistan is shown in fig 2. The value lies between 1259.6 and 1303.2 in the year 2014, which was calculated through the neutrosophic formula by putting value from its undetermined interval.

Sr#	Year	Interval	Neutrosophic Analysis
1	2013	[0.3,0.5]	$0.3+ 0.5I_F$; $I_F\epsilon[0, 0.4]$
2	2014	[0.5,0.6]	$0.5+ 0.6I_F$; $I_F\epsilon[0, 0.166]$
3	2015	[0.6,0.3]	$0.6+ 0.3I_F$; $I_F\epsilon[0, -1]$
4	2016	[0.3,0.8]	$0.3+ 0.8I_F$; $I_F\epsilon[0, 0.625]$
5	2017	[0.8,0.7]	$0.8+ 0.7I_F$; $I_F\epsilon[0, -0.142]$
6	2018	[0.7,0.8]	$0.7+ 0.8I_F$; $I_F\epsilon[0, 0.125]$
7	2019	[0.8,0.9]	$0.8+ 0.9I_F$; $I_F\epsilon[0, 0.111]$
8	2020	[0.9,0.7]	$0.9+ 0.7I_F$; $I_F\epsilon[0, -0.285]$
9	2021	[0.7,0.5]	$0.7+ 0.5I_F$; $I_F\epsilon[0, -0.4]$
10	2022	[0.5,0.5]	$0.5+ 0.5I_F$; $I_F\epsilon[0, 0]$
11	2023	[0.5,0.1]	$0.5+ 0.1I_F$; $I_F\epsilon[0, -4]$

Table 3: FDI inflows with respect to annual percentage of GDP

Table 3 is highlighting the neutrosophic analysis of FDI inflows with respect to annual percentage of GDP. Neutrosophic formula provides a flexible equation for each interval with indeterminacy interval. In 2016 the maximum value of interval is 0.8 and the minimum value is 0.3.

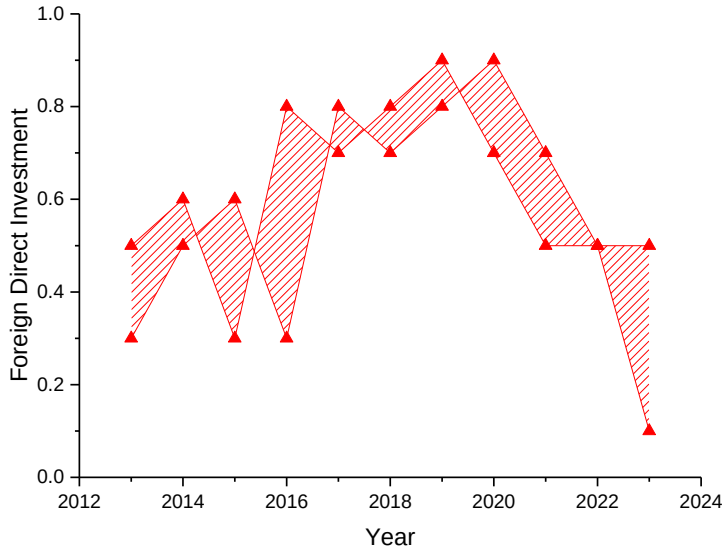

Fig 3: Graphical representation of FDI inflows with respect to annual percentage of GDP

The neutrosophic Graphical Representation of FDI inflows with respect to annual percentage of GDP are shown in fig 3. The value lies between 0.8 and 0.7 in the year 2017; that was calculated through the neutrosophic formula by putting value from its undetermined interval.

Sr#	Year	Interval	Neutrosophic Analysis
1	2013	[16.9,16.1]	16.9+ 16.1I_D; $I_D\epsilon$[0, -0.049]
2	2014	[16.1,15.6]	16.1+ 15.6I_D; $I_D\epsilon$[0, -0.032]
3	2015	[15.6,15.4]	15.6+ 15.4I_D; $I_D\epsilon$[0, -0.012]
4	2016	[15.4,14.7]	15.4+ 14.7I_D; $I_D\epsilon$[0, -0.047]
5	2017	[14.7,15.3]	14.7+ 15.3I_D; $I_D\epsilon$[0, 0.039]
6	2018	[15.3,16.6]	15.3+ 16.6I_D; $I_D\epsilon$[0, 0.078]
7	2019	[16.6,15.7]	16.6+ 15.7I_D; $I_D\epsilon$[0, -0.057]
8	2020	[15.7,15]	15.7+ 15I_D; $I_D\epsilon$[0, -0.046]
9	2021	[15,15.4]	15+ 15.4I_D; $I_D\epsilon$[0, 0.025]
10	2022	[15.4,14.8]	15.4+ 14.8I_D; $I_D\epsilon$[0, -0.040]
11	2023	[14.8,0]	14.8+ 0 I_D; $I_D\epsilon$[0, 0]

Table 4: Domestic Investment Gross fixed capital formation percentage of GDP (private sectors)

Table 4 is highlighting the neutrosophic analysis of Domestic Investment Gross fixed capital formation percentage of GDP (private sectors). In 2014 the maximum value of interval is 15.6 and the minimum value is 16.1.

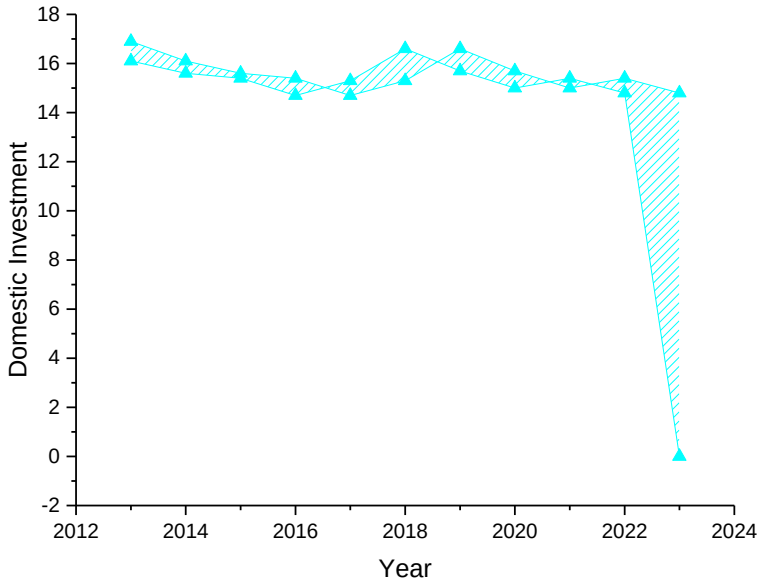

Fig 4: Graphical representation of DoI Gross fixed capital formation percentage of GDP

The neutrosophic Graphical Representation of DoI Gross fixed capital formation percentage of GDP are shown in fig 4. The value lies between 15.3 and 16.6 in the year 2018, which was calculated through the neutrosophic formula by putting value from its undetermined interval.

Sr#	Year	Interval	Neutrosophic Analysis
1	2013	[-4.4,6]	-4.4+ 6I_{TE}; $I_{TE}\epsilon$[0, 1.733]
2	2014	[6,-3.5]	6 -3.5I_{TE}; $I_{TE}\epsilon$[0, 2.714]
3	2015	[-3,5,-1.5]	-3.5- 1.5I_{TE}; $I_{TE}\epsilon$[0, -1.333]
4	2016	[-1.5,-8.5]	-1.5- 8.5I_{TE}; $I_{TE}\epsilon$[0, 0.823]
5	2017	[-8.5,1.8]	-8.5+ 1.8I_{TE}; $I_{TE}\epsilon$[0, 5.722]
6	2018	[1.8,9.7]	1.8+ 9.7I_{TE}; $I_{TE}\epsilon$[0, 0.814]
7	2019	[9.7,-1.3]	9.7-1.3I_{TE}; $I_{TE}\epsilon$[0, 8.461]
8	2020	[-1.3,-7.4]	-1.3-7.4I_{TE}; $I_{TE}\epsilon$[0, 0.824]
9	2021	[-7.4,12.9]	-7.4+12.96I_{TE}; $I_{TE}\epsilon$[0, 1.573]
10	2022	[12.9,25.4]	12.9+ 25.4I_{TE}; $I_{TE}\epsilon$[0, 0.492]
11	2023	[25.4,-11.1]	25.4- 11.1I_{TE}; $I_{TE}\epsilon$[0, 3.288]

Table 5: Trade Liberalization with respect to Export

Table 5 is highlighting the neutrosophic analysis of Trade Liberalization with respect to Export. In 2021 the maximum value of interval is 12.9 and the minimum value is -7.4.

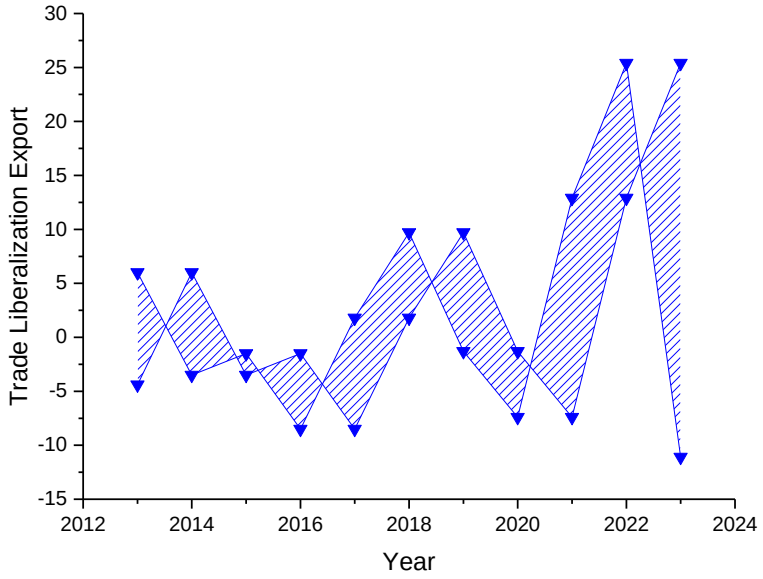

Fig 5: Graphical Analysis of Trade Liberalization with respect to Export

The neutrosophic Graphical Representation of Trade Liberalization with respect to Export are shown in fig 5. The value lies between 9.7 and -1.3 in the year 2019, which was calculated through the neutrosophic formula by putting value from its undetermined interval.

Sr#	Year	Interval	Neutrosophic Analysis
1	2013	[11.7,-0.5]	11.7 -0.5I_{TI}; $I_{TI}\epsilon$[0, 24.4]
2	2014	[-0.5,2.5]	-0.5 +2.5I_{TI}; $I_{TI}\epsilon$[0, 1.2]
3	2015	[2.5,1.1]	2.5+1.1 I_{TI}; $I_{TI}\epsilon$[0, -1.272]
4	2016	[1.1-0.2]	1.1 -0.2I_{TI}; $I_{TI}\epsilon$[0, 6.5]
5	2017	[-0.2,16.9]	-0.2 +16.9I_{TI}; $I_{TI}\epsilon$[0, 1.011]
6	2018	[16.9,16]	16.9 +16I_{TI}; $I_{TI}\epsilon$[0, -0.056]
7	2019	[16,-7.6]	16 -7.6I_{TI}; $I_{TI}\epsilon$[0, 3.105]
8	2020	[-7.6,16.6]	-7.6 +16.6I_{TI}; $I_{TI}\epsilon$[0, 1.457]
9	2021	[16.6,19.7]	16.6 +19.7I_{TI}; $I_{TI}\epsilon$[0, 0.157]
10	2022	[19.7,34.7]	19.7 +34.7I_{TI}; $I_{TI}\epsilon$[0, 0.4322]
11	2023	[34.7,-28.9	34.7 -28.9I_{TI}; $I_{TI}\epsilon$[0, 2.200]

Table 6: Trade Liberalization with respect to Import

Table 6 is highlighting the neutrosophic analysis of Trade Liberalization with respect to Import. In 2017 the maximum value of interval is 16.9 and the minimum value is -0.2.

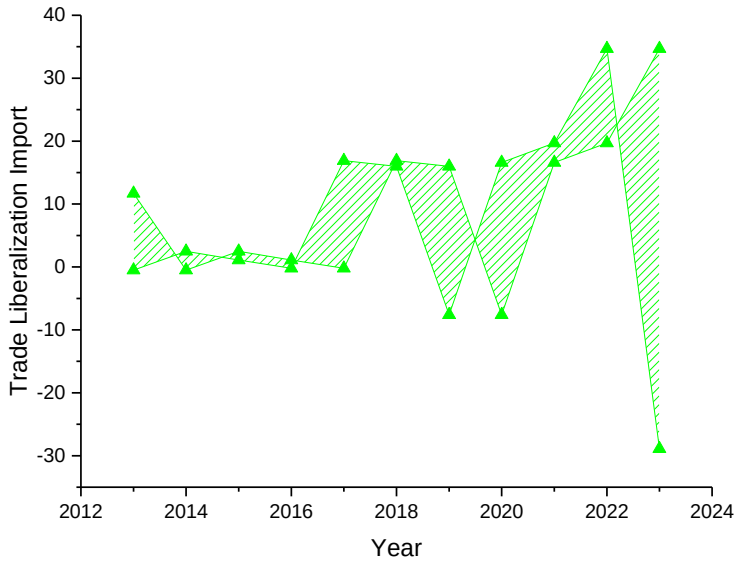

Fig 6: Graphical Representation of Trade Liberalization with respect to Import

The neutrosophic Graphical Representation of Trade Liberalization with respect to Import are shown in fig 6. The value of neutrosophic lies between -7.6 and 16.6 in the year 2020, which was calculated through the neutrosophic formula by putting value from its undetermined interval. The below mentioned table 7 shows the neutrosophic analysis of all variables sleeted to complete this study.

Sr#	Year	Var_1	Var_2	Var_3	Var_4	Var_5
1	2013	1236.9 + 1259.6 I_G; $I_G\epsilon$[0, 0.018]	0.3+ 0.5I_F; $I_F\epsilon$[0, 0.4]	16.9+ 16.1I_D; $I_D\epsilon$[0, -0.049]	-4.4+ 6I_{TE}; $I_{TE}\epsilon$[0, 1.733]	11.7 -0.5I_{TI}; $I_{TI}\epsilon$[0, 24.4]
2	2014	1259.6 + 1303.2 I_G; $I_G\epsilon$[0, 0.033]	0.5+ 0.6I_F; $I_F\epsilon$[0, 0.166]	16.1+ 15.6I_D; $I_D\epsilon$[0, -0.032]	6 -3.5I_{TE}; $I_{TE}\epsilon$[0, 2.714]	-0.5 +2.5I_{TI}; $I_{TI}\epsilon$[0, 1.2]
3	2015	1303.2 + 1421.9 I_G; $I_G\epsilon$[0, 0.083]	0.6+ 0.3I_F; $I_F\epsilon$[0, -1]	15.6+ 15.4I_D; $I_D\epsilon$[0, -0.012]	-3.5- 1.5I_{TE}; $I_{TE}\epsilon$[0, -1.333]	2.5+1.1 I_{TI}; $I_{TI}\epsilon$[0, -1.272]
4	2016	1421.9 + 1468.9 I_G; $I_G\epsilon$[0, 0.031]	0.3+ 0.8I_F; $I_F\epsilon$[0, 0.625]	15.4+ 14.7I_D; $I_D\epsilon$[0, -0.047]	-1.5- 8.5I_{TE}; $I_{TE}\epsilon$[0, 0.823]	1.1 -0.2I_{TI}; $I_{TI}\epsilon$[0, 6.5]
5	2017	1468.9 + 1567.7 I_G; $I_G\epsilon$[0, 0.063]	0.8+ 0.7I_F; $I_F\epsilon$[0, -0.142]	14.7+ 15.3I_D; $I_D\epsilon$[0, 0.039]	-8.5+ 1.8I_{TE}; $I_{TE}\epsilon$[0, 5.722]	-0.2 +16.9I_{TI}; $I_{TI}\epsilon$[0, 1.011]
6	2018	1567.7 + 1620.8 I_G; $I_G\epsilon$[0, 0.032]	0.7+ 0.8I_F; $I_F\epsilon$[0, 0.125]	15.3+ 16.6I_D; $I_D\epsilon$[0, 0.078]	1.8+ 9.7I_{TE}; $I_{TE}\epsilon$[0, 0.814]	16.9 +16I_{TI}; $I_{TI}\epsilon$[0, -0.056]
7	2019	1620.8 + 1437.2 I_G; $I_G\epsilon$[0, -0.127]	0.8+ 0.9I_F; $I_F\epsilon$[0, 0.111]	16.6+ 15.7I_D; $I_D\epsilon$[0, -0.057]	9.7-1.3I_{TE}; $I_{TE}\epsilon$[0, 8.461]	16 -7.6I_{TI}; $I_{TI}\epsilon$[0, 3.105]
8	2020	1437.2 + 1324.1 I_G; $I_G\epsilon$[0, -0.085]	0.9+ 0.7I_F; $I_F\epsilon$[0, -0.285]	15.7+ 15I_D; $I_D\epsilon$[0, -0.046]	-1.3-7.4I_{TE}; $I_{TE}\epsilon$[0, 0.824]	-7.6 +16.6I_{TI}; $I_{TI}\epsilon$[0, 1.457]
9	2021	1324.1 + 1507.9I_G; $I_G\epsilon$[0, 0.121]	0.7+ 0.5I_F; $I_F\epsilon$[0, -0.4]	15+ 15.4I_D; $I_D\epsilon$[0, 0.025]	-7.4+12.96I_{TE}; $I_{TE}\epsilon$[0, 1.573]	16.6 +19.7I_{TI}; $I_{TI}\epsilon$[0, 0.157]

10	2022	1507.9+ 1592.1I_G; $I_G\epsilon$[0, 0.052]	0.5+ 0.5I_F; $I_F\epsilon$[0, 0]	15.4+ 14.8I_D; $I_D\epsilon$[0, -0.040]	12.9+ 25.4I_{TE}; $I_{TE}\epsilon$[0, 0.492]	19.7 +34.7I_{TI}; $I_{TI}\epsilon$[0, 0.4322]
11	2023	1592.1+ 1390.8I_G; $I_G\epsilon$[0, -0.144]	0.5+ 0.1I_F; $I_F\epsilon$[0, -4]	14.8+ 0 I_D; $I_D\epsilon$[0, 0]	25.4- 11.1I_{TE}; $I_{TE}\epsilon$[0, 3.288]	34.7 -28.9I_{TI}; $I_{TI}\epsilon$[0, 2.200]

Table 7: Neutrosophic Analysis of all variables

5. Comparison of Classical and Neutrosophic Analysis

It is clear from the above mentioned analysis of data that unlike the classical method, the neutrosophic method allows for more nuanced reasoning. The neutrosophic method is designed to handle incomplete or partial data, where classical approaches may struggle[23]. It provides a means to deal with missing or imprecise information and can be useful in situations where limited or uncertain data is available for analysis. This flexibility allows the neutrosophic method to adapt to various scenarios where classical approaches may be too rigid or insufficient. It offers tools and techniques that can be tailored to address specific requirements and complexities[24]; this approach is best suited for interval-based data analysis. The neutrosophic approach offers several advantages over the classical approach when it comes to dealing with uncertainty and decision-making.

5.1 Handling Ambiguity and Vagueness

One of the key advantages of the neutrosophic approach is its ability to handle ambiguity and vagueness effectively. Traditional decision-making frameworks like the classical approach struggle with imprecise, incomplete, or contradictory information. Neutrosophic logic allows decision-makers to explicitly represent and reason with uncertainty, unknowns, and contradictions. This enables a more comprehensive analysis of complex problems where information is not clear-cut.

5.2 Flexibility in Decision-Making

In contrast to the rigid and deterministic nature of the classical approach, the neutrosophic approach offers greater flexibility in decision-making. Decision-makers can account for various levels of certainty, uncertainty, and indeterminacy when evaluating alternatives. This flexibility allows for a more nuanced consideration of trade-offs, conflicting objectives, and diverse perspectives, leading to more informed and adaptive decisions. Neutrosophic logic embraces contradictions and paradoxes as inherent aspects of complex systems. This is in stark contrast to the classical approach, which typically assumes that contradictions are unacceptable. By

acknowledging the presence of contradictions and evaluating truth values as true, false, and indeterminate, the neutrosophic approach enables decision-makers to grapple with conflicting information and make more robust decisions in scenarios where contradictory data is prevalent.

5.3 Tackling Uncertainty

Uncertainty is a fundamental aspect of decision-making, and the neutrosophic approach excels in addressing varying degrees of uncertainty. Unlike the classical approach, which often struggles to incorporate indeterminate or unknown factors, neutrosophic logic allows for a more comprehensive treatment of uncertainty. This enables decision-makers to make well-informed choices even in situations where information is incomplete or evolving. The neutrosophic approach enhances problem-solving capabilities by providing a richer and more nuanced representation of reality. By embracing ambiguity, uncertainty, and contradictions, decision-makers can explore a wider range of possibilities and outcomes. This can lead to innovative solutions, improved risk management strategies, and a deeper understanding of complex systems that may elude traditional decision-making approaches like the classical method.

6. Discussion and Contribution of the Study

The research presented in *Statistical Analysis of Foreign Investment Using Interval-Based Approach* provides a distinctive methodology by applying Neutrosophic Analysis, particularly in addressing uncertainties and imprecise information related to foreign direct investment (FDI) in Pakistan. A comparison with previous studies highlights several key differences in approach and results, as well as a more nuanced understanding of FDI impacts.

6.1 Handling Uncertainty and Contradictions

Previous research on FDI in Pakistan and other developing nations typically relied on classical statistical methods, which often simplify the complexity of real-world data. For instance, studies like those by Reiter and Steensma[1] and Konara and Ganotakis[25] employ classical regression models that focus on identifying direct relationships between FDI and economic variables such as GDP, employment, and productivity. These approaches assume a certain degree of stability and completeness in data, which can overlook the uncertainties inherent in emerging markets.

In contrast, the current study's use of Neutrosophic Analysis offers a more adaptable framework; Neutrosophic logic accounts for indeterminacy by allowing values to be simultaneously true, false, or indeterminate, which is particularly useful in analyzing contradictory or incomplete data, a common issue in developing economies. This distinction provides a more flexible analytical tool

compared to the rigid assumptions of classical methods. The inclusion of indeterminacy allows policymakers to better interpret FDI impacts in environments where data reliability and consistency are often compromised.

6.2 Investment Policy Evaluation and Sectoral Focus

Previous studies, such as those by Biglaiser and DeRouen[11] and Mmieh and Owusu-Frimpong[9], emphasized the need for liberalization and the removal of investment barriers to attract FDI. These studies outlined how tax incentives, profit repatriation allowances, and sector-specific policies help boost investor confidence. However, they often did not delve into the indeterminate nature of how these policies are perceived or how they play out under real-world conditions marked by political and economic uncertainties.

The current study advances the conversation by not only analyzing the role of liberalization but also using interval-based Neutrosophic methods to capture the variability in the effectiveness of these policies. For example, while previous research points out the positive impact of SEZs and tax breaks, the Neutrosophic Analysis provides a clearer view of the uncertainty investors might feel due to fluctuating political conditions and governance issues. This gives a more comprehensive picture of how FDI incentives work under varying levels of economic and political stability.

6.3. Sector-Specific Impacts and Data Precision

Previous studies, such as by Baharumshah and Thanoon[8], have explored the importance of FDI in sectors like manufacturing, infrastructure, and energy, often using precise but static data sets. These studies typically show how FDI leads to technology transfer and job creation, enhancing productivity in target sectors. However, they do not account for the long-term variability or possible contradictions in the success of such investments over time.

The current research takes this sectoral focus further by introducing Neutrosophic intervals that highlight fluctuations and uncertainties in FDI inflows across different years. For instance, the study points out how FDI in Pakistan fluctuated between 0.3% to 0.9% of GDP from 2013 to 2023, with significant variations depending on global market conditions and domestic political stability. By using an interval-based approach, the study gives a more dynamic view of how sectoral investments perform over time.

6.4. Methodological Innovation

Another significant contrast is the focus on decision-making frameworks. Traditional studies often provide static conclusions based on average or median values, such as GDP growth or employment figures. These studies, while informative, tend to offer limited flexibility when applied to uncertain environments like Pakistan's. Decision-making frameworks in previous research rely heavily on classical approaches that might fail to consider future risks and uncertainties. The Neutrosophic approach, however, explicitly incorporates uncertainty into the decision-making process. By allowing for the consideration of both contradictory and indeterminate information, it offers a more refined tool for policymakers. This is particularly relevant in Pakistan's case, where political instability and economic volatility can quickly alter the investment landscape. This method provides a clearer framework for predicting outcomes and adjusting policies in real time, something classical approaches do not handle well. In comparison to previous research, the current study offers significant advancements in understanding FDI inflows in Pakistan. The Neutrosophic Analysis goes beyond traditional methods by addressing data uncertainty and contradictions more effectively, which is crucial for countries like Pakistan with fluctuating political and economic environments[26, 27]. By incorporating flexible decision-making and recognizing the indeterminacy of certain variables, this study provides a more comprehensive tool for policymakers. It bridges the gaps left by earlier studies, offering deeper insights into how foreign investments can be attracted and sustained amid uncertainties.

7. Conclusion

In conclusion, the study has utilized the neutrosophic approach to assess and analyze the various aspects of foreign investment in the country. The neutrosophic approach has helped deal with the uncertainties, contradictions, and indeterminacies in this complex area. From the analysis of foreign investment trends, it is evident that Pakistan has witnessed a steady increase in foreign investment inflows over the years. The neutrosophic approach has provided a comprehensive understanding of the various perspectives and uncertainties involved in foreign investment strategies for sustainable development in Pakistan. The neutrosophic approach offers distinct advantages over the classical approach by providing a more flexible, adaptive, and comprehensive framework for decision-making in the face of uncertainty, ambiguity, and contradictions. By leveraging neutrosophic logic, decision-makers can navigate complex, unpredictable situations with greater confidence and arrive at more robust and nuanced solutions that account for the inherent complexities of real-world problems.

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Boosted MACBETH Technique for Effectiveness Evaluation of Mental Health Education in College Students Based on Neutrosophic Cubic Sets

Xueyuan Sun*
Guangdong Jiangmen Preschool Education College, Jiangmen, Guangdong, 529000, China
*Corresponding author, E-mail: sunxueyuan0011@163.com

Abstract: The evaluation of the effectiveness of college students' mental health education (MHE) aims to assess the impact of related educational activities on students' mental well-being. Using both quantitative and qualitative methods, it evaluates improvements in students' psychological resilience, knowledge acquisition, emotional management, and the prevention of mental health issues. Effective evaluation helps optimize educational programs, enhancing students' mental health and promoting their overall development. The effectiveness evaluation of college students' MHE is multiple-attribute decision-making (MADM). Recently, the MACBETH technique and average model were dealt with MADM. The neutrosophic cubic sets (NCSs) are utilized as a better technique for expressing uncertain data during the effectiveness evaluation of college students' MHE. In this study, the NC-MACBETH model is put forward the MADM with NCSs. The average technique is utilized to deal with weight. Finally, a numerical example of the effectiveness evaluation of college students' MHE is put forward through some comparisons to verify the NC-MACBETH technique.

Keywords: MADM; NC-MACBETH approach; effectiveness evaluation; Mental Health Education.

1. Introduction

The evaluation of the effectiveness of MHE for college students aims to comprehensively assess the actual impact of such programs on students' mental well-being. This process helps determine whether the education effectively enhances students' psychological resilience, emotional regulation, and stress management skills. The evaluation typically employs a multi-dimensional approach, combining quantitative and qualitative data, such as mental health scales, behavioral observations, surveys, and interviews. Key areas of assessment include students' understanding of mental health knowledge, the effectiveness of preventing psychological issues, improvements in emotional management, and changes in their ability to cope with stress or challenges. Additionally, attention should be given to students' mental health development in areas such as academics, social interactions, and career planning, particularly regarding the long-term effects after the program's implementation. A systematic evaluation provides valuable feedback to educators, helping them refine and optimize the content and format of MHE, ensuring it is more targeted and effective. This not only supports students' psychological growth but also promotes their overall development in both life and academics. The literature review for the effectiveness evaluation of college students' MHE is listed in Table 1.

Table 1. Literature review for effectiveness evaluation of college students' MHE

Author Information	Methods	Research Content
Wei [1]	SCL-90 questionnaire, comparison of mental health interventions across different types of students	Explores differences in mental health status between police academy students and other university students, and intervention effects.
Dong [2]	Survey via questionnaires on students' attitudes and course evaluations	Investigates students' attitudes toward MHE courses to improve course construction and effectiveness.
Zeng [3]	Discussion on principles, content, and methods of evaluation	Proposes evaluation methods for assessing the teaching effectiveness of university MHE courses.
Li and Shen [4]	Whole-group random sampling, intervention using group counseling vs. lecture-based teaching	Investigates the effects of group counseling and traditional teaching on reducing mobile phone dependence among vocational students.
Wang, Li, Zhang and Li [5]	Comparative study, using emotion management and subjective well-being as assessment indicators	Evaluate the effectiveness of action-based experiential MHE courses in improving emotional management.
Sun, Luo and Chen [6]	SCL-90 questionnaire, comparison between experiential and lecture-based teaching	Compares the effects of experiential teaching and traditional lecture-based teaching in MHE courses.
Zhu [7]	Empirical study using the SPOC platform, analysis of data from teaching activities	Studies the impact of SPOC-based flipped classroom teaching on students' MHE and learning outcomes.
Tang, Liang, and Yin [8]	Comparative study between group counseling and MOOC-based teaching, analysis of course records and performance	Compares the effectiveness of group counseling vs. MOOC in university MHE.
Xu and Zhu [9]	Retrospective study, a comparison of 100 students divided into two groups (emotion intervention and regular intervention)	Evaluates the effectiveness of emotion intervention theory in improving mental health compared to regular interventions.

Tang, Lu and Zhang [10]	Cluster sampling, comparison of project-based vs. traditional teaching, evaluation through multiple metrics	Investigate the application of project-based teaching in medical students' MHE courses.
Xu and Liao [11]	Grounded theory, in-depth interviews with teachers and students, Nvivo12.0 coding	Constructs a model of factors affecting the teaching effectiveness of MHE courses in universities.
Yuan [12]	Content analysis of 119 self-reported assignments from vocational college students	Explores the educational outcomes of university MHE courses and offers suggestions for improvement.

The effectiveness evaluation of college students' MHE could be put forward by MADM. Recently, MACBETH [13] was utilized to solve MADM. The NCSs [14] put forward a better technique for expressing uncertain data during the effectiveness evaluation of college students' MHE. In this study, the NC-MACBETH model deals with MADM with NCSs. The average method is to put forward the weight with NCSs.

Finally, a numerical example of the effective evaluation of college students' MHE is put forward with different comparisons to prove the NC-MACBETH approach. Thus, the key contributions of the NC-MACBETH model are summarized: (1) the MACBETH model is enhanced under NCSs; (2) the NC-MACBETH model is dealt with MADM with NCSs; (3) Finally, a numerical example for effectiveness evaluation of college students' MHE is put forward with different comparisons to verify NC-MACBETH technique.

The framework of such a study is put forward: Sect. 2 lists the NCSs. Sect. 3 put forward the MADM technique in light of the NC-MACBETH model. Sect. 4 put forward the NC-MACBETH through numerical study for effectiveness evaluation of college students' MHE with some comparisons. Conclusions are dealt with in Sect. 5.

2. Neutrosophic Definitions

This section shows the operations of NC.

Definition 1

Let a linguistic term set be $S = \{s_j | j \in [0, P]\}$ Where p+1 is an odd number. The NC h can be defined in S as $h = (u, c), u = ([S_{Ta}, S_{Tb}], [S_{Ia}, S_{Ib}], [S_{Fa}, S_{Fb}])$ Refers to the truth, indeterminacy, and falsity of membership functions.

Definition 2

Let $h = (u, c), u = ([S_{Ta}, S_{Tb}], [S_{Ia}, S_{Ib}], [S_{Fa}, S_{Fb}])$ then

h an internal NC if $Ta \le T \le Tb, Ia \le I \le Ib, Fa \le F < Fb$

h an internal NC if $T \notin (Ta, Tb), I \notin (Ia, Ib), F \notin (Fa, Fb)$

Definition 3

Let two NC numbers as:

$$h_1 = \left(\left[S_{Ta_1}, S_{Tb_1}\right], \left[S_{Ia_1}, S_{Ib_1}\right], \left[S_{Fa_1}, S_{Fb_1}\right], \left(S_{T_1}, S_{I_1}, S_{F_1}\right)\right)$$

$$h_2 = \left(\left[S_{Ta_2}, S_{Tb_2}\right], \left[S_{Ia_2}, S_{Ib_2}\right], \left[S_{Fa_2}, S_{Fb_2}\right], \left(S_{T_2}, S_{I_2}, S_{F_2}\right)\right)$$

$$h_1 \oplus h_1 = \left(\begin{pmatrix} \left[S_{Ta_1} + T_{a_2} - \frac{S_{Ta_1}S_{Ta_2}}{p}, S_{Tb_1} + S_{Tb_2} - \frac{S_{Tb_1}S_{Tb_2}}{p}\right], \\ \left[\frac{S_{Ia_1}S_{Ia_2}}{p}, \frac{S_{Ib_1}S_{Ib_2}}{P}\right], \\ \left[\frac{S_{Fa_1}S_{Fa_2}}{P}.\frac{S_{Fb_1}S_{Fb_2}}{P}\right], \left(S_{T_1} + S_{T_2} - \frac{S_{T_1}S_{T_2}}{P}, \frac{S_{I_1}S_{I_2}}{P}, \frac{S_{F_1}S_{F_1}}{P}\right) \end{pmatrix}\right), \tag{1}$$

$$h_1 \otimes h_2 = \begin{pmatrix} \left[\frac{S_{Ta_1}S_{Ta_2}}{P}, \frac{S_{Tb_1}\dot{S}_{Tb_2}}{P}\right], \\ \left[S_{Ia_1} + S_{Ia_2} - \frac{S_{Ia_1}S_{Ia_2}}{P}, S_{Ib_1} + S_{Ib_2} - \frac{S_{Ib_1}S_{Ib_2}}{P}\right], \\ \left[S_{Fa_1} + S_{Fa_2} - \frac{S_{Fa_1}S_{Fa_2}}{P}, S_{Fb_1} + S_{Fb_2} - \frac{S_{Fb_1}S_{Fb_2}}{P}\right], \\ \left(\frac{S_{T_1}S_{T_2}}{P}, S_{I_1} + S_{I_2} - \frac{S_{I_1}S_{I_2}}{P}, S_{F_1} + S_{F_2} - \frac{S_{F_1}S_{F_2}}{P}\right) \end{pmatrix} \tag{2}$$

$$\tau h_1 = \begin{pmatrix} \left[S_p - S_p\left(1 - \frac{S_{Ta_1}}{P}\right)^{\tau}, S_p - S_p\left(1 - \frac{S_{Tb_1}}{P}\right)^{\tau}\right], \\ \left[S_P\left(\frac{S_{Ia_1}}{P}\right)^{\tau}, S_P\left(\frac{S_{Ib_1}}{P}\right)^{\tau}\right], \\ \left[S_P\left(\frac{S_{Fa_1}}{P}\right)^{\tau}, S_P\left(\frac{S_{Fb_1}}{P}\right)^{\tau}\right], \\ \left(S_p - Sp\left(\left(1 - \frac{S_{T_1}}{P}\right)^{\tau}\right), \left(\frac{S_{I_1}}{P}\right)^{\tau}, \left(\frac{S_{F_1}}{P}\right)^{\tau}\right) \end{pmatrix} \tag{3}$$

Definition 4

The score function can be defined as:

$$S(h_1) = \frac{6 + S_{Ta_1} + S_{TB_1} - S_{Ia_1} - S_{Ib_1} - S_{Fa_1} - S_{Fb_1} - S_{T_1} - S_{I_1} - S_{F_1}}{9} \tag{4}$$

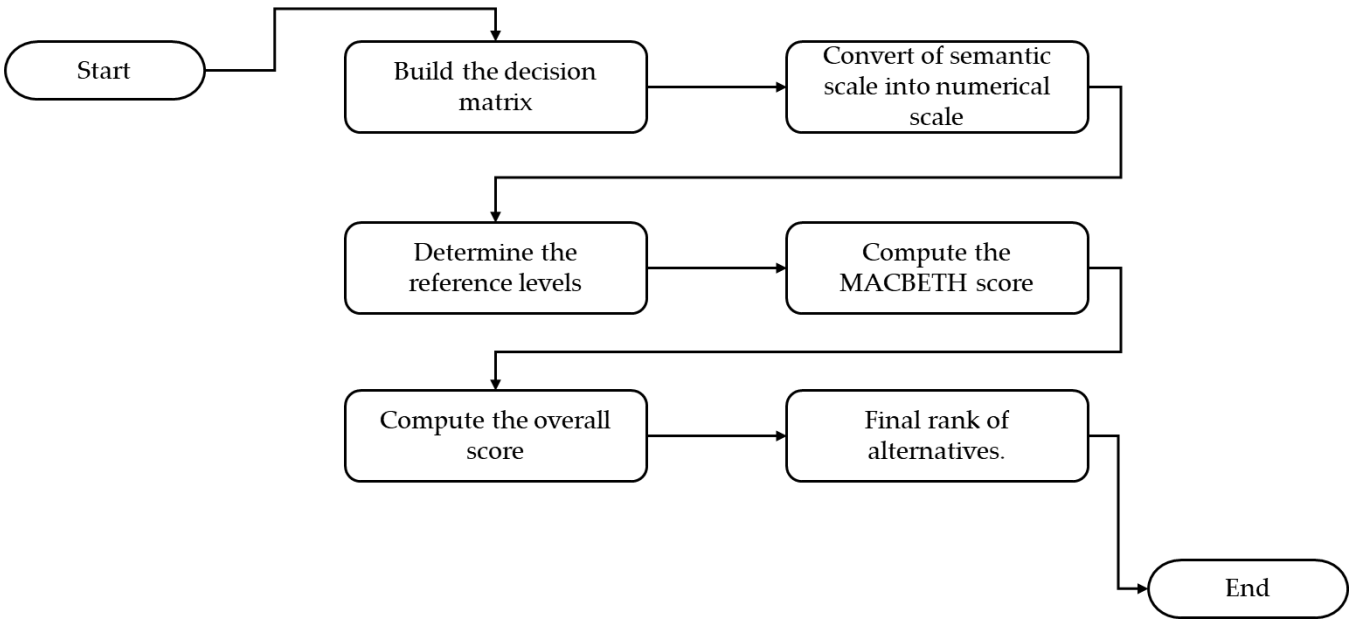

Figure 1. The steps of the neutrosophic MACBETH.

3. Neutrosophic Framework

This section shows the steps of the proposed methodology. Figure 1 shows the steps of the proposed methodology.

Step 1. Build the decision matrix

Step 2. Convert of semantic scale into numerical scale.

Each semantic scale can change to the numerical scale for positive and negative criteria.

Step 3. Determine the reference levels.

The reference point is computed by the minimum and maximum values as:

$$y_j^- = \min x_{ij} \quad (5)$$

$$y_j^- = \max x_{ij} \quad (6)$$

Step 4. Compute the MACBETH score

$$U(y_{ij}) = U(y_j^-) + \frac{x_{ij}-y_j^-}{y_j^+-y_j^-}\left[U(y_j^+) - U(y_j^-)\right] \quad (7)$$

Where $U(y_j^-)\ equal\ to\ zero$ and $U(y_j^+)\ equal\ to$ 100.

Step 5. Compute the overall score

$$O_i = \sum_{j=1}^{n} U(y_{ij}) w_j \quad (8)$$

Step 6. Final rank of alternatives.

4. Results

A numerical example is put forward effectiveness evaluation of college students' MHE through NC-MACBETH. Six preschool education colleges are assessed by employing 12 attributes. The following are descriptions of 12 criteria:

12 criteria for evaluating the effectiveness of mental health education programs for college students, ranked by importance (highest to lowest), with detailed descriptions for each.

1. Psychological Well-Being Improvement

The degree to which students demonstrate enhanced emotional stability, self-confidence, and reduced symptoms of anxiety, depression, or stress after participating in mental health programs. This is the ultimate goal of mental health education and the clearest indicator of effectiveness.

2. Awareness of Mental Health Issues

The extent to which students understand common mental health challenges, recognize symptoms in themselves or peers and are aware of available support systems. Awareness ensures early intervention and prevention.

3. Accessibility of Counseling Services

The availability and ease of accessing mental health resources, such as counseling centers, workshops, or helplines. A highly accessible system encourages more students to seek help when needed.

4. Student Participation and Engagement

The level of active involvement of students in mental health-related activities, workshops, or initiatives. Greater participation reflects the program's ability to capture interest and meet students' needs.

5. Development of Coping Mechanisms

The extent to which students develop effective stress-management and problem-solving skills through mental health education. These skills help students navigate academic and personal challenges more effectively.

6. Reduction in Stigma

The degree to which mental health education programs reduce the stigma associated with seeking mental health support. Lower stigma encourages more open conversations and greater utilization of resources.

7. Training of Faculty and Staff

The effectiveness of mental health training for faculty and staff to identify at-risk students and provide initial support or referrals. Properly trained personnel can expand the program's reach and impact.

8. Peer Support System Development

The presence and effectiveness of peer mentoring or peer support groups provide students with a safe space to share concerns and seek advice. Peer systems complement formal services by creating an inclusive and empathetic environment.

9. Academic Performance Correlation

The relationship between students' mental health improvements and their academic performance, such as grades, attendance, or focus. Better mental health often leads to improved academic outcomes.

10. Customization to Student Needs

The degree to which programs are tailored to address the specific mental health challenges faced by college students, such as exam stress, social isolation, or financial worries. A targeted approach increases program effectiveness.

11. Parental and Family Involvement

The effectiveness of strategies to involve parents or family members in supporting students' mental health. Family support reinforces the education program's impact.

12. Long-Term Impact and Follow-Up

The sustainability of the program's effects, including whether students continue to apply learned skills and maintain mental health improvements after the program ends. Regular follow-ups help evaluate and maintain the program's effectiveness.

Table 2. Th opinions of three experts.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))
C_2	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
C_3	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))
C_4	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))
C_5	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))
C_6	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_7	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))

C_8	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))
C_9	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))
C_{10}	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
C_{11}	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_{12}	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))
	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))
C_2	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
C_3	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))
C_4	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_5	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))
C_6	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_7	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))
C_8	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))
C_9	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))
C_{10}	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_{11}	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_{12}	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))
	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))
C_2	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
C_3	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))
C_4	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))
C_5	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))
C_6	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_7	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))
C_8	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.38, 0.71], [0.29, 0.49], [0.26, 0.61]), (0.70, 0.39, 0.47))
C_9	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.42, 0.69], [0.31, 0.55], [0.26, 0.48]), (0.66, 0.41, 0.50))
C_{10}	(([0.23, 0.57], [0.17, 0.36], [0.11, 0.45]), (0.48, 0.21, 0.37))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))
C_{11}	(([0.35, 0.62], [0.28, 0.47], [0.19, 0.39]), (0.59, 0.32, 0.44))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))
C_{12}	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))	(([0.19, 0.44], [0.15, 0.38], [0.13, 0.30]), (0.43, 0.26, 0.34))	(([0.22, 0.49], [0.19, 0.37], [0.15, 0.40]), (0.46, 0.30, 0.39))	(([0.13, 0.40], [0.14, 0.33], [0.22, 0.50]), (0.38, 0.28, 0.35))	(([0.12, 0.48], [0.21, 0.39], [0.15, 0.43]), (0.37, 0.28, 0.42))

Step 1. The decision matrix was built using the neutrosophic numbers as shown in Table 2. Then we obtain crisp values. Then we combine these values into one matrix. Then we compute the criteria weights using the average method. Table 3 shows the criteria weights.

Table 3. The weights of criteria.

Criteria	**Weights**	**Rank**
C_1	0.082262	1
C_2	0.083135	5

C_3	0.084097	11
C_4	0.082674	2
C_5	0.083381	6
C_6	0.082841	3
C_7	0.083067	4
C_8	0.083636	10
C_9	0.083459	8
C_{10}	0.084451	12
C_{11}	0.083548	9
C_{12}	0.083449	7

Step 2. Then we convert semantic scales into numerical scales.

Step 3. Then we determine the reference levels.

Minimum values

C1= 0.495555556, C2=0.495555556, C3=0.507037037, C4=0.506666667, C5=0.510740741, C6= 0.506666667, C7=0.51037037, C8=0.495555556, C9=0.506666667, C10=0.50925926, C11= 0.514814815, C12=0.507037037.

Maximum values

C1=0.531111111, C2=0.538888889, C3=0.540740741, C4=0.532222222, C5=0.548888889, C6= 0.536666667, C7=0.536666667, C8=0.538518518, C9=0.548518518, C10,0.554444444, C11=0.538888889, C12= 0.536666667.

Step 4. Then we computed the MACBETH score

Step 5. Then we compute the overall score

Step 6. Final rank of alternatives.

A6>A3>A1>A4>A5>A2

We show the final rank of alternatives in Table 4. We show alternative 6 is the best and alternative 2 is the worst.

Table 4. Final rank of alternatives.

Alternatives	**Rank**
A_1	4
A_2	1
A_3	5
A_4	3
A_5	2
A_6	6

The evaluation of the effectiveness of MHE for college students is an important process to assess how well such education improves students' psychological resilience, enhances stress management, and improves emotional regulation. With increasing social competition and growing pressures from life and academics, the mental health of college students has become a

significant concern. As a result, MHE has become an indispensable part of university education. Therefore, a scientific and systematic evaluation of its effectiveness is crucial in optimizing educational content and improving students' mental well-being. The evaluation process typically combines both quantitative and qualitative methods. Quantitative evaluation employs psychological measurement tools such as mental health scales and emotional regulation questionnaires to collect data on students' psychological states before and after participating in educational activities. By comparing this data, it objectively reflects the extent to which MHE improves students' psychological well-being. Qualitative evaluation, on the other hand, includes interviews, questionnaires, and case studies to gain deeper insights into students' understanding and application of the educational content in their daily lives. The core indicators of the evaluation include students' mastery of mental health knowledge, improvements in psychological adjustment skills, better emotional management and stress-coping abilities, and the prevention of mental health issues. Additionally, the evaluation should consider students' psychological growth in areas such as learning, interpersonal relationships, and career planning. Effective evaluation of MHE not only helps educators identify strengths and weaknesses in the current teaching but also provides valuable insights for designing future curricula. By analyzing the results, universities can continuously optimize their MHE programs, better meet students' psychological needs, and enhance their overall psychological well-being, thereby promoting their holistic development and growth. The effectiveness evaluation of college students' MHE is MADM. We show that alternative 6 is the best and alternative 2 is the worst.

4.1 Comparative analysis

The NC-MACBETH approach is compared with the N-VIKOR technique [16], the N-CE approach [17], the N-GRA model [18], and the N-MABAC model [15]. The order is expressed in Table 5.

Table 5. Order for different approaches

Methods	order
N-VIKOR technique	A6>A3>A4>A1>A5>A2
N-CE approach	A6>A1>A3>A4>A5>A2
N-GRA model	A6>A3>A1>A5>A4>A2
N-MABAC technique	A6>A3>A1>A4>A5>A2
Our method	A6>A3>A1>A4>A5>A2

In light with detailed WS coefficient values [19, 20], the final derived WS coefficient values between N-VIKOR, N-CE technique, N-GRA model, N-MABAC approach, and NC-MACBETH technique are 0.942, 0.942, 0.942, 1.0000. Therefore, the NC-MACBETH technique is an effective and reliable MADM method.

The NC-MACBETH model excels in handling uncertain information, multi-attribute decision-making, and flexible weight allocation, making it a powerful tool for evaluating the

effectiveness of college students' MHE. These advantages make it highly practical and accurate in environments characterized by complexity and fuzzy information.

Here are the three main advantages of the NC-MACBETH model:

(1). Ability to Handle Uncertainty and Fuzzy Information

The NC-MACBETH model utilizes NCSs to handle uncertainty and fuzzy information, which is particularly important in complex decision-making scenarios. Traditional decision-making models often assume that data is precise, but in reality, much information—such as the effectiveness evaluation of MHE—is vague, subjective, or uncertain. The NC-MACBETH model is better equipped to represent and manage this fuzzy and uncertain data, thereby improving the accuracy and reliability of evaluations.

(2). Effectiveness in MADM

This model is based on the MADM framework, which allows it to consider multiple influencing factors simultaneously (such as psychological resilience, emotional management, knowledge acquisition, etc.). This multi-dimensional evaluation is particularly crucial in complex fields like MHE, where numerous interrelated factors must be considered.

(3). Flexible Weight Allocation Mechanism

The NC-MACBETH model uses the average technique to dynamically determine weights, allowing it to adaptively allocate different attribute weights based on the degree of uncertainty in the data. This method is more flexible and objective compared to traditional fixed-weight allocation methods. Such a dynamic weight allocation mechanism ensures that the model is more adaptable and accurate when dealing with different datasets, especially when the importance of various attributes may vary. This flexibility allows the model to better reflect the complexity of real-world situations.

4.2 Managerial Implications

Evaluating the effectiveness of mental health education programs for college students provides actionable insights to improve program delivery, resource allocation, and student outcomes.

Tailoring Mental Health Interventions

The evaluation identifies which interventions are most effective for specific groups of students or mental health challenges. Customize programs to address the unique needs of diverse student demographics (e.g., first-year students, international students, or students with disabilities).

Enhancing Resource Allocation

Managers can identify high-impact areas that require more funding and resources, ensuring optimal utilization of limited resources. Invest in areas like counseling services, peer mentoring programs, or digital mental health tools based on evaluation findings.

Improving Accessibility and Equity

Insights into accessibility barriers, such as lack of awareness or stigma, allow managers to implement solutions to increase program reach. Enhance outreach efforts, expand online resources, and ensure mental health services are inclusive and culturally sensitive.

Strengthening Training for Faculty and Staff

Evaluation highlights the need for better training for faculty and staff to identify and support students in distress. Develop and implement comprehensive mental health training programs for university personnel to act as first responders.

Promoting a Supportive Campus Culture

Effective mental health education reduces stigma and fosters an open, supportive environment for students to seek help. Organize awareness campaigns, peer support groups and community-building activities to normalize conversations around mental health.

Tracking Long-Term Outcomes

Evaluation reveals whether mental health programs lead to sustained improvements in students' mental health and academic performance over time. Establish mechanisms for long-term follow-ups, such as alumni surveys or ongoing wellness checks, to assess lasting program impacts.

Linking Mental Health to Academic Success

Evidence of improved academic outcomes (e.g., better grades, reduced dropout rates) from effective mental health programs strengthens their justification. Highlight the link between mental health and academic performance to secure institutional buy-in and increased funding.

5. Conclusion

The evaluation of the effectiveness of MHE for college students aims to assess the actual impact of educational programs on students' mental health, focusing on improvements in psychological resilience, emotional regulation, and stress management. The evaluation is typically conducted through various methods such as psychological scales, surveys, and interviews, incorporating both quantitative analysis and qualitative feedback. Key indicators include students' mastery of mental health knowledge, improvements in emotional management, and the effectiveness of preventing psychological distress. Effective evaluation provides targeted feedback for educators, helping to optimize the content and format of education, better-supporting students' mental health needs, and promoting their overall development. The effectiveness evaluation of college students' MHE is MADM. Currently, the MACBETH approach and average models are utilized to deal with MADM. The NCSs put forward better techniques for putting forward uncertain data during the effectiveness evaluation of college students' MHE. In this study, the NC-MACBETH model in light is put forward the MADM with NCSs. The average technique is utilized to deal with weight numbers in light. Finally, a numerical example of the effectiveness evaluation of college students' MHE is put forward through some comparisons to prove the NC-MACBETH technique.

Here are the four key contributions of the study:

(1) Application of NCSs for Uncertainty Handling: The study introduces NCS as a more effective technique for managing uncertainty in the effectiveness evaluation of college students' MHE, improving the precision of decision-making in uncertain environments.

(2) Development of the NC-MACBETH Model: A new NC-MACBETH model is proposed. This model is specifically designed for MADM under uncertain conditions, leveraging the strength of MACBETH and NCSs techniques.

(3) Use of average for Weight Determination: The study utilizes the average technique to handle weight assignment in the decision-making process. This method objectively determines the importance of different attributes, enhancing the reliability of the evaluation when dealing with complex, multi-attribute criteria.

(4) Validation through Numerical Example and Comparative Analysis: The study validates the NC-MACBETH model with a numerical example of the effective evaluation of college students' MHE. Additionally, it includes a comparative analysis, demonstrating the model's superiority and effectiveness over other approaches in addressing MADM problems.

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The First Resolution of the Travelling Salesman Problem under Neutrosophic Octagonal Fuzzy Environment

Mariem Miledi[a], Taicir Loukil[b], Souhail Dhouib[a,*],

[a] OLID Laboratory, Higher Institute of Industrial Management,

University of Sfax, TUNISIA

[b] MODILS Lab, Faculty of Management and Economic Sciences,

University of Sfax, TUNISIA

*Corresponding author: souhail.dhouib@gmail.com

Abstract: The Travelling Salesman Problem (TSP) is a challenging combinatorial optimization problem classified as NP-hard. Its objective is to identify the shortest cycle that visits every city precisely once before returning to the initial city. To the best of our knowledge there is no one in the literature who solved the TSP under the neutrosophic octagonal fuzzy environment. That's why, in this paper the novel heuristic namely Dhouib-Matrix-TSP1 (DM-TSP1) is exploited to optimize the TSP under the neutrosophic octagonal fuzzy domain. So, this research work represents the first application of DM-TSP1 on this mentioned environment. A defuzzification function is used to convert neutrosophic octagonal fuzzy numbers to crisp ones then the four simple steps of DM-TSP1 are launched. A numerical example illustrating a step-by-step application of DM-TSP1 on novel created benchmark instances is provided to prove its performance and efficiency in solving the neutrosophic octagonal fuzzy TSP.

Keywords: Urban development, Infrastructure governance, Artificial Intelligence, Intelligent Computing and Algorithms, Data science applications, Neutrosophic Set.

1. Introduction

The Travelling Salesman Problem (TSP) is one of the most important problems in the operational research and it is extensively used in the real-world. It aims to generate the shortest cycle between all nodes (cities, customers, suppliers, … etc.) namely the Hamiltonian cycle: each node is visited only once excepting the starting node which will be also the last visited node.

The mathematical formulation of the TSP is as follows:

Minimize:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij} \tag{1}$$

Subject to:

$$\begin{aligned} &\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,...,n \\ &\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,...,n \\ &x_{ij} = 0 \text{ or } 1, \ i = 1,...,n \ , \ \ \mathrm{j} = 1,...,n \end{aligned} \tag{2}$$

x_{ij} is the binary variable used to specify if city i is linked to city j (so $x_{ij} = 1$) or if cities i and j are not linked ($x_{ij} = 0$). d is the parameter that represents the distance (cost, time, … etc.) while d_{ij} denotes the distance between cities i and j.

In industrial companies, data are usually uncertain and imprecise that's why fuzzy, intuitionistic and neutrosophic numbers are considered in almost of optimization problems (instead of crisp numbers). The neutrosophic concept is firstly proposed by Smarandache [1]. The neutrosophic components are: the Truth (T), the Indeterminacy (I) and the Falsity (F) with T, I and F are subsets of [-0,1+].

Several research works are interested in solving the TSP under the neutrosophic environment. The neutrosophic TSP using hexagonal fuzzy numbers is optimized in [2] via the nearest neighbor technique which is a local approximative method that uses the ancient values closed to the current one to estimate its trajectory by the means of the trajectories of its neighbors. The neutrosophic TSP with trapezoidal fuzzy numbers is solved in [3] by the use of the branch and bound method that divides the problem into several sub-problems and uses a function to remove sub-problems that cannot have the optimal solution. A formulation and a genetic algorithm are proposed in [4] to optimize the neutrosophic TSP with the neutrosophic edge weight and a new heuristic crossover as well as a heuristic mutation for the proposed GA are designed. A novel heuristic for a new pentagonal neutrosophic TSP is introduced in [5]. In fact, the novel heuristic Dhouib-Matrix-TSP1 (DM-TSP1) is applied using a ranking function in order to transform the fuzzy numbers to crisp data and a range function to select cities. The Pascal's graded mean method and the ones assignment approach are used to unravel the neutrosophic TSP in [6]. The novel column-row heuristic namely Dhouib-Matrix-TSP1 is enhanced by the means of the center of gravity ranking function and the standard deviation metric to optimize the neutrosophic TSP in [7]. The Yager's ranking function is applied to convert the neutrosophic triangular fuzzy numbers to crisp numbers then the neutrosophic TSP is solved by the means of the Dhouib-Matrix-TSP1 heuristic in [8].

Other research papers are focused on the optimization of the Transportation Problem (TP) using neutrosophic numbers. The neutrosophic TP is modelled as a parametric Linear Programming Problem (LPP) then it is solved by a dynamic optimal solution approach in [9]. The constraints containing pentagonal neutrosophic quantities are converted into crisp constraints with the help of the score function then the pentagonal neutrosophic TP with an interval cost is solved in [10]. A modeling study of the TP using the neutrosophic logic and neutrosophic mathematical formulas for the TP are presented in [11]. A neutrosophic approach for the TP by the use of single-valued trapezoidal neutrosophic numbers is introduced in [12] and the obtained optimal solution is compared with solutions given by other approaches. The bilevel TP in the neutrosophic environment is optimized in [13] using a goal programming strategy to arrive at a satisfactory solution. The neutrosophic multi-objective green four-dimensional fixed-charge TP is optimized in [14] by the means of two new approaches based on the neutrosophic programming and the pythagorean hesitant fuzzy programming. The single-valued trapezoidal neutrosophic TP is solved in [15] by enriching the Dhouib-Matrix-TP1 heuristic with two functions namely the score and the metric functions.

Concerning the resolution of the Assignment Problem (AP) under the neutrosophic domain, many methods are proposed. The neutrosophic AP is optimized by the means of interval-valued trapezoidal neutrosophic numbers and the use of the order relations in [16]. The AP with

neutrosophic costs is studied in [17] and its solution methodology is presented where two real problems are solved to prove the efficiency of the proposed method. An approach to optimize the multi-objective AP with neutrosophic numbers is introduced in [18]. This approach is used by the help of a weighting Tchebycheff problem which is applied by defining relative weights and ideal targets. A new method to rank pentagonal neutrosophic numbers using its magnitude and its applications to solve the AP is proposed in [19]. The neutrosophic set theory is applied on the generalized AP and after evaluating the score function matrix the problem is solved by the use of the Extremum Difference Method (EDM) to get the optimal assignment in [20]. A solution for the multi-criteria AP is obtained through the neutrosophic set theory and by applying two different methods in [21].

To optimize the Shortest Path Problem (SPP) using neutrosophic numbers, few techniques are introduced. The neutrosophic SPP is solved in [22] by applying the Dijkstra algorithm which is redesigned to handle the case where the most of parameters of a network are given as neutrosophic numbers. An overview about the SPP in fuzzy, intuitionistic and neutrosophic environments and advantages as well as limitations of different types of sets are given in [23]. A mathematical model of the SPP in the neutrosophic environment and three different approaches for solving the neutrosophic SPP are presented in [24]. The neutrosophic SPP is optimized in [25] by the means of the Bellman algorithm and a comparative analysis is carried out with other existing methods which proves the performance of the new technique. The basic concepts about neutrosophic sets and interval valued neutrosophic graphs are reviewed then an algorithm is suggested in [26] to find the shortest path and distance in an interval valued neutrosophic graph. A novel linear programming approach for solving the gaussian valued neutrosophic SPP without using any ranking method is proposed in [27]. An algorithm is developed in [28] to find the shortest path on a network in which the weights of the edges are represented as bipolar neutrosophic numbers.

In this research work, the TSP under the neutrosophic octagonal fuzzy environment is optimized by the means of the novel heuristic namely Dhouib-Matrix-TSP1. To the best of our knowledge, this research work marks the inaugural resolution of the TSP under the neutrosophic octagonal fuzzy domain and that's why novel benchmark instances are created. Additionally, this paper presents the first application of DM-TSP1 on this mentioned environment. The remaining of this paper is structured as follows. Section 2 contains all details about the neutrosophic octagonal fuzzy numbers. Section 3 describes how to solve the TSP under the neutrosophic octagonal fuzzy environment using the novel heuristic namely DM-TSP1. Section 4 summarizes a numerical example. Finally, section 5 gives a conclusion and further research perspectives.

2. The neutrosophic octagonal fuzzy number

In this section, some definitions concerning the neutrosophic octagonal fuzzy numbers are presented.

A neutrosophic octagonal fuzzy number $\tilde{S}_N$ is described as follows:

$$\tilde{S}_N = < [(a_1; b_1; c_1; d_1; e_1; f_1; g_1; h_1) ; \rho] ;$$
$$[(a_2; b_2; c_2; d_2; e_2; f_2; g_2; h_2) ; \sigma] ;$$
$$[(a_3; b_3; c_3; d_3; e_3; f_3; g_3; h_3 ; \omega] >$$

Where $\rho, \sigma, \omega \in [0,1]$.

Three membership functions are defined for $\tilde{S}_N$:

1) The Truth $T_{\tilde{S}_N} : R \to [0, \rho]$ (see Figure 1.a),

2) The Indeterminacy membership function $I_{\tilde{S}} : R \to [\sigma, 1]$ (see Figure 1.b),

3) The Falsity membership function $F_{\tilde{S}} : R \to [\omega, 1]$ (see Figure 1.c).

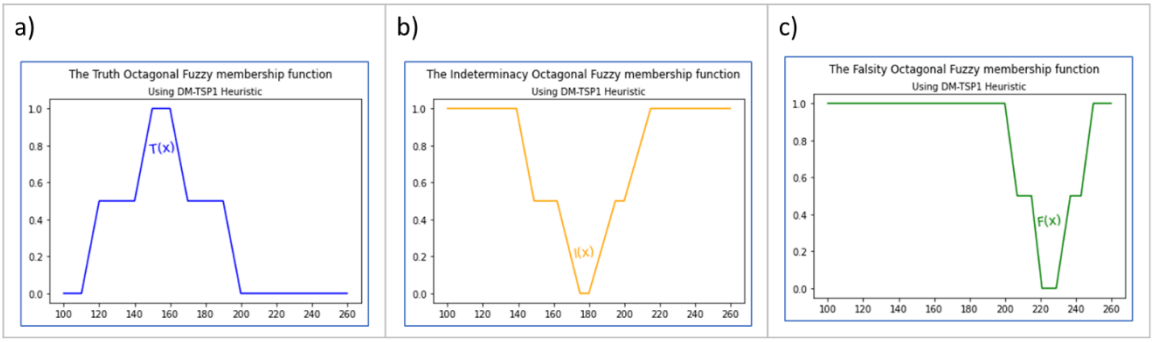

Figure 1. A graphical representation of: a) The thruth membership function b) The Indeterminacy membership function c) The Falsity membership function

These membership functions are defined as follows:

$$T_{\tilde{S}_N}(x)=\begin{cases}\dfrac{x-\mathrm{a}_1}{b_1-\mathrm{a}_1}, & \mathrm{a}_1\le x\le b_1\\ \dfrac{x-b_1}{c_1-b_1}, & \mathrm{b}_1\le x\le c_1\\ \dfrac{x-c_1}{d_1-c_1}, & \mathrm{c}_1\le x\le d_1\\ 1, & x=d_1\\ 1, & x=e_1\\ \dfrac{f_1-x}{f_1-e_1}, & \mathrm{e}_1\le x\le f_1\\ \dfrac{g_1-x}{g_1-f_1}, & f_1\le x\le g_1\\ \dfrac{h_1-x}{h_1-g_1}, & g_1\le x\le h_1\\ 0, & otherwise\end{cases}$$

$$I_{\tilde{S}_N}(x)=\begin{cases}\dfrac{b_2-x}{b_2-\mathrm{a}_2}, & \mathrm{a}_2\le x\le b_2\\ \dfrac{c_2-x}{c_2-b_2}, & \mathrm{b}_2\le x\le c_2\\ \dfrac{d_2-x}{d_2-c_2}, & \mathrm{c}_2\le x\le d_2\\ 0, & x=d_2\\ 0, & x=e_2\\ \dfrac{x-e_2}{f_2-e_2}, & \mathrm{e}_2\le x\le f_2\\ \dfrac{x-f_2}{g_2-f_2}, & f_2\le x\le g_2\\ \dfrac{x-g_2}{h_2-g_2}, & g_2\le x\le h_2\\ 1, & otherwise\end{cases}$$

$$F_{\tilde{S}_N}(x)=\begin{cases}\dfrac{b_3-x}{b_3-a_3}, & a_3\le x\le b_3\\ \dfrac{c_3-x}{c_3-b_3}, & b_3\le x\le c_3\\ \dfrac{d_3-x}{d_3-c_3}, & c_3\le x\le d_3\\ 0, & x=d_3\\ 0, & x=e_3\\ \dfrac{x-e_3}{f_3-e_3}, & e_3\le x\le f_3\\ \dfrac{x-f_3}{g_3-f_3}, & f_3\le x\le g_3\\ \dfrac{x-g_3}{h_3-g_3}, & g_3\le x\le h_3\\ 1, & otherwise\end{cases}$$

Figure 2 represents a graphical representation of a neutrosophic octagonal fuzzy number where $-0\le T_{\tilde{S}_N}(x)+I_{\tilde{S}_N}(x)+F_{\tilde{S}_N}(x)\le 3+$.

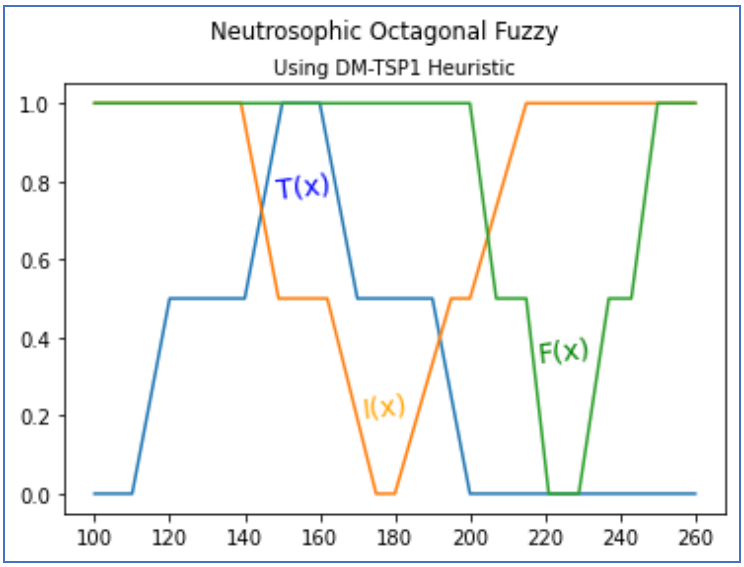

Figure 2. A graphical representation of a neutrosophic octagonal fuzzy number

3. Solving the neutrosophic octagonal fuzzy TSP using Dhouib-Matrix-TSP1

Recently and more precisely in 2021, the novel theory of optimization methods named Dhouib-Matrix (DM) has been invented. DM encompasses multiple optimization methods (see Figure 3). Regarding the approximative techniques, a new multi-start method entitled Dhouib-Matrix-4 (DM4) using statistical metrics is proposed in [29, 30, 31, 32, 33]. Furthermore, a new iterated stochastic metaheuristic named Dhouib-Matrix-3 (DM3) is introduced in [34, 35, 36, 37]. In addition, a new local search algorithm entitled Far-to-Near (FtN) using a guided search procedure is presented in [38]. Hence, concerning the optimization of the Assignment Problem two heuristics namely Dhouib-Matrix-AP1 (DM-AP1) and Dhouib-Matrix-AP2 (DM-AP2) are proposed in [39, 40, 41, 42]. Besides, the technique entitled Dhouib-Matrix-TP1 (DM-TP1) is designed in [43, 44] to optimize the Transportation Problem. The methods named Dhouib-Matrix-TSP1 (DM-TSP1) and Dhouib-Matrix-TSP2 (DM-TSP2) are developed in [45, 46, 47, 48, 49] to solve the Travelling Salesman Problem. Concerning the optimal techniques, two novel methods are developed: Dhouib-Matrix-SPP (DM-SPP) which aims to optimize the standard shortest path problem in [50] as

well as the autonomous mobile robot shortest path planning problem in [51, 52, 53]; and Dhouib-Matrix-ALL-SPP (DM-ALL-SPP) to solve the all pairs (Multiple pair) shortest path problem [54]. Thus, a new technique named Dhouib-Matrix-MSTP (DM-MSTP) is proposed to unravel the Minimum Spanning Tree problem in [55].

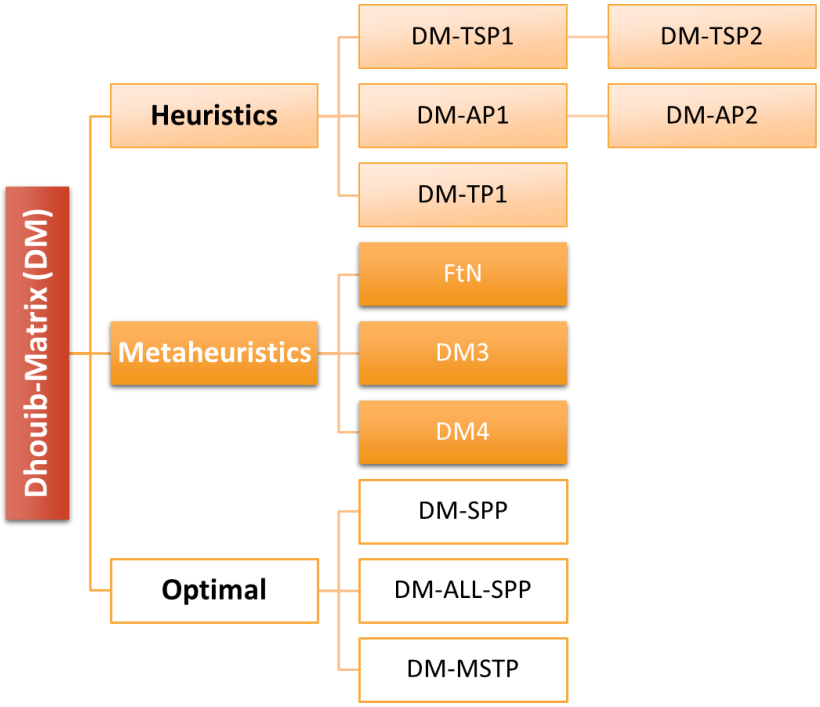

Figure 3. The Dhouib-Matrix techniques

At first, the heuristic DM-TSP1 is proposed to optimize the generalized TSP. Then, DM-TSP1 is adapted to solve the trapezoidal fuzzy TSP [56] as well as the octagonal fuzzy TSP [57] and the intuitionistic fuzzy TSP [58]. Furthermore, new versions of DM-TSP1 are introduced in [59, 60] to minimize the total distance.

As far as we know, this research work presents the first resolution of the TSP under the neutrosophic octagonal fuzzy environment as well as the first application of the novel heuristic namely DM-TSP1 on this mentioned environment. DM-TSP1 starts by applying the defuzzification function (see equation 1) used by [61].

$$D_{\tilde{S}_N} = \frac{D_{T_{\tilde{S}_N}} + D_{I_{\tilde{S}_N}} + D_{F_{\tilde{S}_N}}}{3} \tag{1}$$

Where:

$$D_{T_{\tilde{S}_N}} = \frac{1}{6}(a_1 + h_1) + k(b_1 + c_1 + f_1 + g_1) + (d_1 + e_1)$$

$$D_{I_{\tilde{S}_N}} = \frac{1}{6}(a_2 + h_2) + k(b_2 + c_2 + f_2 + g_2) + (d_2 + e_2)$$

$$D_{F_{\tilde{S}_N}} = \frac{1}{6}(a_3 + h_3) + k(b_3 + c_3 + f_3 + g_3) + (d_3 + e_3)$$

DM-TSP1 is composed of four steps where step 2 and step 3 are iterated n times (with n is the number of cities). Whereas steps 1 and 4 are executed only once. Actually, DM-TSP1 is characterized by its flexibility to use different descriptive statistical metrics. In this research work, the range metric is used. DM-TSP1 starts by converting the neutrosophic octagonal fuzzy numbers to crisp ones using the defuzzification function then the four steps of DM-TSP1 are executed (see Figure 4).

Step1: Compute the range function (max-min) for each row and write it on the right-hand side of the matrix. Next, find the minimal range and select its row. Then, select the smallest element in this row which will specify the two first cities x and y to be inserted in the list *List-cities* $\{x, y\}$. Finally, discard the respected columns of cities x and y.

Step 2: Find the minimal element for city x and for city y and select the smallest distance which will indicate city z.

Step 3: Add city z to the list *List-cities* and discard its column. Next, if there is no column go to Step 4 otherwise go to Step 2.

Step 4: Modify the realizable solution in *List-cities* in order to generate a cycle (the starting city in the cycle has to be also the last one). First, to ensure that the starting city will be at the first position, translate all the cities (one by one) before the starting one at the end of the list. Second, duplicate the starting city at the last position.

Figure 4. Steps of DM-TSP1

DM-TSP1 is a heuristic and not a metaheuristic so it is dedicated to solve only the TSP and not other problems such as the Transportation Problem. That's why, another heuristic is developed for the Transportation Problem namely DM-TP1 inspired from DM-TSP1. In addition, DM-TSP1 is used to generate a starting solution for different metaheuristics such as the Artificial Bee Colony in [49], the novel Dhouib-Matrix-3 in [36, 37] and Dhouib-Matrix-4 in in [29, 30].

4. Numerical example

A salesman is looking for the Hamiltonian cycle joining six cities to visit. It is required to visit each city only once and generating the shortest cycle. The distances between the six cities are represented as neutrosophic octagonal fuzzy numbers as represented in Figure 5.

$$\begin{bmatrix} \infty & d_{01} & d_{02} & d_{03} & d_{04} & d_{05} \\ d_{10} & \infty & d_{12} & d_{13} & d_{14} & d_{15} \\ d_{20} & d_{21} & \infty & d_{23} & d_{24} & d_{25} \\ d_{30} & d_{31} & d_{32} & \infty & d_{34} & d_{35} \\ d_{40} & d_{41} & d_{42} & d_{43} & \infty & d_{45} \\ d_{50} & d_{51} & d_{52} & d_{53} & d_{54} & \infty \end{bmatrix}$$

Figure 5. Neutrosophic octagonal fuzzy distance matrix

Where:

$$d_{01} = \langle 4,8,10,15,16,16.5,18,22;8.5,9.5,10.5,12,14,16.5,17.5,18.5;11,12,15,16,19,21,22,23 \rangle$$

$$d_{02} = \langle 5,6,7,8,9,10,11,12;3.5,4.5,5.5,6.5,7.5,8.5,9.5,10.5;7,8,9,10,11,12,13,14 \rangle$$

$$d_{03} = \langle 3,4,5,6,7,8,9,10;4.5,5.5,6.5,7.5,8.5,9.5,10.5,11.5;9,11,12,13,14,15,16,17 \rangle$$

$$d_{03} = \langle 0,1,2,3,5,6,7,8; 2.5,2.5,4.5,5.5,6.5,7.5,8.5,9.5; 5,7,9,11,12,13,14,15 \rangle$$

$$d_{04} = \langle 4,5,6,7,8,9,10,11; 5.5,6.5,7.5,8.5,9.5,11.5,13.5,14.5; 9,11,12,13,14,15,16,17 \rangle$$

$$d_{05} = \langle 5,6,7,8,10,12,13,14; 6.5,7.5,8.5,9.5,11.5,12,13,14; 11,12,13,14,15,16,17,18 \rangle$$

$$d_{10} = \langle 4,8,10,15,16,16.5,18,22; 8.5,9.5,10.5,12,14,16.5,17.5,18.5; 11,12,15,16,19,21,22,23 \rangle$$

$$d_{12} = \langle 12,13,15,19,20,21,22,24; 13,15,16,17,19,21,22,23; 14,15,16,17,18,19,21,22 \rangle$$

$$d_{13} = \langle 7,8,11,12,13,14,15,16; 8.5,9.5,10.5,11.5,12.5,13.5,15.5,17.5; 13,14,15,16,17,18,19,20 \rangle$$

$$d_{14} = \langle 11,12,13,14,15,16,17,18; 7.5,8.5,9.5,10.5,11.5,12.5,13.5,14.5; 12,13,14,15,16,17,18,19 \rangle$$

$$d_{15} = \langle 10,11,12,13,14,15,16,17; 9.5,10.5,11.5,12.5,13.5,14.5,15.5,17.5; 12,13,14,15,16,17,18,19 \rangle$$

$$d_{20} = \langle 5,6,7,8,9,10,11,12; 3.5,4.5,5.5,6.5,7.5,8.5,9.5,10.5; 7,8,9,10,11,12,13,14 \rangle$$

$$d_{21} = \langle 12,13,15,19,20,21,22,24; 13,15,16,17,19,21,22,23; 14,15,16,17,18,19,21,22 \rangle$$

$$d_{23} = \langle 7,8,11,15,16,17,18,20; 10.5,11.5,12.5,13.5,15.5,17.5,18.5,19.5; 14,15,16,17,18,20,21,22 \rangle$$

$$d_{24} = \langle 1,2,3,4,5,6,7,8; 2.5,3.5,4.5,5.5,6.5,7.5,8.5,9.5; 4,5,7,9,11,12,13,14 \rangle$$

$$d_{25} = \langle 6,8,11,15,16,17,18,20; 7.5,10.5,12.5,14.5,16.5,17.5,18.5,19.5; 13,14,15,17,18,20,21,23 \rangle$$

$$d_{30} = \langle 3,4,5,6,7,8,9,10; 4.5,5.5,6.5,7.5,8.5,9.5,10.5,11.5; 9,11,12,13,14,15,16,17 \rangle$$

$$d_{31} = \langle 7,8,11,12,13,14,15,16; 8.5,9.5,10.5,11.5,12.5,13.5,15.5,17.5; 13,14,15,16,17,18,19,20 \rangle$$

$$d_{32} = \langle 7,8,11,15,16,17,18,20; 10.5,11.5,12.5,13.5,15.5,17.5,18.5,19.5; 14,15,16,17,18,20,21,22 \rangle$$

$$d_{34} = \langle 9,10,14,17,19,20,21,22; 8,9,11,12,13,15,16,17; 11,12,13,14,15,16,17,18 \rangle$$

$$d_{35} = \langle 11,13,16,17,18,20,21,22; 11,12,13,14,15,16,17,18; 12,13,14,15,16,17,18,19 \rangle$$

$$d_{40} = \langle 4,5,6,7,8,9,10,11; 5.5,6.5,7.5,8.5,9.5,11.5,13.5,14.5; 9,11,12,13,14,15,16,17 \rangle$$

$$d_{41} = \langle 11,12,13,14,15,16,17,18; 7.5,8.5,9.5,10.5,11.5,12.5,13.5,14.5; 12,13,14,15,16,17,18,19 \rangle$$

$$d_{42} = \langle 1,2,3,4,5,6,7,8; 2.5,3.5,4.5,5.5,6.5,7.5,8.5,9.5; 4,5,7,9,11,12,13,14 \rangle$$

$$d_{43} = \langle 9,10,14,17,19,20,21,22;8,9,11,12,13,15,16,17;11,12,13,14,15,16,17,18 \rangle$$

$$d_{45} = \langle 5,6,7,8,11,12,13,14;6.5,7.5,8.5,9.5,10.5,11.5,12.5,13.5;10,11,12,13,14,15,16,17 \rangle$$

$$d_{50} = \langle 5,6,7,8,10,12,13,14;6.5,7.5,8.5,9.5,11.5,12,13,14;11,12,13,14,15,16,17,18 \rangle$$

$$d_{51} = \langle 10,11,12,13,14,15,16,17;9.5,10.5,11.5,12.5,13.5,14.5,15.5,17.5;12,13,14,15,16,17,18,19 \rangle$$

$$d_{52} = \langle 6,8,11,15,16,17,18,20;7.5,10.5,12.5,14.5,16.5,17.5,18.5,19.5;13,14,15,17,18,20,21,23 \rangle$$

$$d_{53} = \langle 11,13,16,17,18,20,21,22;11,12,13,14,15,16,17,18;12,13,14,15,16,17,18,19 \rangle$$

$$d_{54} = \langle 5,6,7,8,11,12,13,14;6.5,7.5,8.5,9.5,10.5,11.5,12.5,13.5;10,11,12,13,14,15,16,17 \rangle$$

The first step is to convert all neutrosophic octagonal fuzzy numbers to crisp numbers using the defuzzification function. The generated crisp distance matrix is represented in Figure 6.

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix}$$

Figure 6. The crisp distance matrix

Now, DM-TSP1 is launched. The range for each row is computed and the smallest element (4.45) is selected and its corresponding element is designated at position d_{13} (see Figure 7).

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix} \begin{matrix} 6.23 \\ 4.45 \\ 11.45 \\ 7.17 \\ 7.91 \\ 4.75 \end{matrix}$$

Figure 7. The smallest range is selected

Next, cities 1 and 3 are inserted in the list *List-cities* {1-3} and their corresponding columns are discarded. Subsequently, the smallest elements at row 1 (which is 13.67) and at row 3 (which is 8.58) are selected and their smallest element (8.58 at position d_{03}) is nominated for the next step (see Figure 8).

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix}$$

Figure 8. The smallest elements of rows 1 and 3 are selected

Hence, city 0 is inserted in the list *List-cities* {1-3-0} and its column is discarded and the smallest elements at row 1 (13.67) and at row 0 (8.67) are selected (see Figure 9) and their minimal element (8.67 at position d_{02}) is nominated for the next step.

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix}$$

Figure 9. The smallest elements of rows 0 and 1 are selected

Following, city 2 is inserted in the list *List-cities* {1-3-0-2} and its column is discarded and the smallest elements at row 1 (13.67) and at row 2 (6.63) are selected (see Figure 10) and their minimal element (6.63 at position d_{24}) is nominated for the next step.

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix}$$

Figure 10. The smallest elements of row 1 and 2 are selected

Subsequently, city 4 is inserted in the list *List-cities* {1-3-0-2-4} and its column is discarded and the smallest elements at row 1 (14.04) and at row 4 (11) are selected (see Figure 11) and their minimal element (11 at position d_{45}) is nominated for the next step.

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix}$$

Figure 11. The smallest elements of row 1 and 2 are selected

Hence, city 5 is inserted in the list *List-cities* {1-3-0-2-4-5} and its column is discarded (see Figure 12).

$$\begin{bmatrix} \infty & 14.81 & 8.67 & 8.58 & 10.17 & 11.40 \\ 14.81 & \infty & 18.08 & 13.63 & 13.67 & 14.04 \\ 8.67 & 18.08 & \infty & 15.58 & 6.63 & 15.42 \\ 8.58 & 13.63 & 15.58 & \infty & 14.54 & 15.75 \\ 10.17 & 13.67 & 6.63 & 14.54 & \infty & 11 \\ 11.40 & 14.04 & 15.42 & 15.75 & 11 & \infty \end{bmatrix}$$

Figure 12. All the columns are discarded

Now, all the columns are discarded and step 4 is launched in order to create a cycle from *List-cities* {1-3-0-2-4-5}. Thus, all cities before city 0 are translated at the end of *List-cities*: At first, city 1 is added at the last position of *List-cities* {3-0-2-4-5-1} and at second city 3 is added at the end of *List-cities* {0-2-4-5-3-1}. Finally, the starting city is added at the last position of *List-cities* {0-2-4-5-1-3-0}. Accordingly, the optimal solution generated by DM-TSP1 is $x_{02}=1;\ x_{24}=1;\ x_{45}=1;\ x_{51}=1;\ x_{13}=1; x_{30}=1$ with the path crisp total cost $z=8.67+6.63+11+14.04+13.63+8.58=62.55$.

Consequently, the neutrosophic octagonal fuzzy solution is:

$z^N =< 31,37,45,51,59,65,71,77;35,41,47,53,59,65,72,80;55,62,69,76,83,89,95,101 >$ and its graphical representation is illustrated in Figure 13.

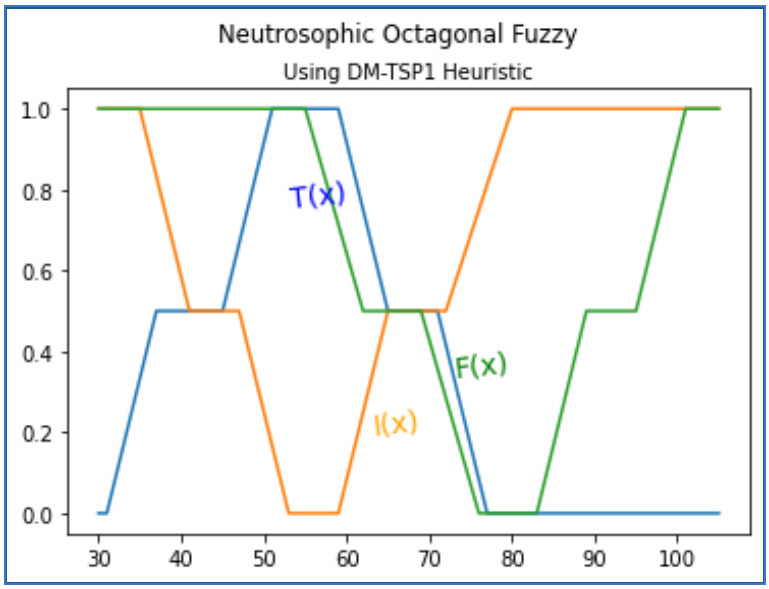

Figure 13. The neutrosophic octagonal fuzzy optimal solution generated by DM-TSP1

The corresponding truth membership function is presented by Equation 5.

$$T_{\tilde{S}_N}(x)=\begin{cases}\dfrac{x-31}{37-31}, & 31\le x\le 37\\ \dfrac{x-37}{45-37}, & 37\le x\le 45\\ \dfrac{x-45}{51-45}, & 45\le x\le 51\\ 1, & x=51\\ 1, & x=59\\ \dfrac{65-x}{65-59}, & 59\le x\le 65\\ \dfrac{71-x}{71-65}, & 65\le x\le 71\\ \dfrac{77-x}{77-71}, & 71\le x\le 77\\ 0, & otherwise\end{cases} \tag{5}$$

In the same way, the indeterminacy membership function is denoted by Equation 6.

$$I_{\tilde{S}_N}(x)=\begin{cases}\dfrac{41-x}{41-35}, & 35\le x\le 41\\ \dfrac{47-x}{47-41}, & 41\le x\le 47\\ \dfrac{53-x}{53-47}, & 47\le x\le 53\\ 0, & x=53\\ 0, & x=59\\ \dfrac{x-59}{65-59}, & 59\le x\le 65\\ \dfrac{x-65}{72-65}, & 65\le x\le 72\\ \dfrac{x-72}{80-72}, & 72\le x\le 80\\ 1, & otherwise\end{cases} \tag{6}$$

And the falsity membership function is offered by Equation 7.

$$F_{\tilde{S}_N}(x)=\begin{cases}\dfrac{62-x}{62-55}, & 55\le x\le 62\\ \dfrac{69-x}{69-62}, & 62\le x\le 69\\ \dfrac{76-x}{76-69}, & 69\le x\le 76\\ 0, & x=76\\ 0, & x=83\\ \dfrac{x-83}{89-83}, & 83\le x\le 89\\ \dfrac{x-89}{95-89}, & 89\le x\le 95\\ \dfrac{x-95}{101-95}, & 95\le x\le 101\\ 1, & otherwise\end{cases} \tag{7}$$

5. Conclusion

The Travelling Salesman Problem (TSP) stands as a challenging problem in the field of the operational research with widespread real-world applications. Its objective is to determine the shortest Hamiltonian cycle that connects all nodes which can represent cities, customers, suppliers and more. This cycle ensures that each node is visited exactly once excepting the starting node which also serves as the final point. In this current research work, a new variant of the TSP under the neutrosophic octagonal fuzzy domain is optimized using the novel heuristic DM-TSP1. As far as we know, this paper represents the first resolution of the neutrosophic octagonal fuzzy TSP in addition to the inaugural application of the new heuristic DM-TSP1 on this new variant of the TSP. That's why new benchmark instances are created. All neutrosophic octagonal fuzzy numbers are converted to crisp ones using a defuzzification function. Then, the four simple steps of DM-TSP1 are launched. A numerical example shows the rapidity and the efficiency of DM-TSP1 in optimizing the neutrosophic octagonal fuzzy TSP. Further research works aim to apply DM-TSP1 on other variants of the TSP: the generalized TSP, the multi-objective TSP and the TSP under a dynamic environment.

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Integrating KEMIRA with Interval-Valued Neutrosophic Numbers to Assess University English Teaching Quality: A Multi-Attribute Decision-Making Model

Li Ding*

Shaanxi Xueqian Normal University, Xi'an, Shaanxi, 710100, China

*Corresponding author, E-mail: Barbara.lily@163.com

Abstract

Evaluating the quality of university English teaching is essential for improving learning outcomes. Key factors include teaching methods, curriculum design, and teacher-student interaction. Effective teachers use engaging approaches that help students develop skills in reading, writing, listening, and speaking. Tailored course content and regular feedback mechanisms, such as surveys and assessments, enhance learning efficiency and address challenges. Teachers' enthusiasm and ability to motivate students greatly influence the quality of teaching. Emphasizing practical communication skills prepares students for real-world applications. This paper introduces the KEmeny Median Indicator Ranks Accordance (KEMIRA) method, combined with interval-valued neutrosophic sets (INSs), to address multiple-attribute decision-making (MADM) problems. A numerical example evaluates the quality of university English teaching, showcasing the advantages of the interval-valued neutrosophic KEMIRA (INN-KEMIRA) approach. Key contributions include extending the KEMIRA model to INSs, determining attribute weights using the average method, and applying INN-KEMIRA to complex MADM problems. A practical case study demonstrates the effectiveness of this approach, with comparative analyses and sensitivity tests validating its accuracy and reliability. This framework provides a robust solution for decision-making under uncertainty, offering valuable insights for educational quality evaluation and similar challenges.

Keywords: Multiple-attribute decision-making (MADM); interval-valued neutrosophic sets (INSs); KEMIRA approach; university English teaching quality evaluation

1. Introduction

The quality of college English teaching reflects the effectiveness of instructional strategies, curriculum design, and student learning outcomes. High-quality teaching ensures that students not only improve their language proficiency but also develop critical thinking, communication, and

cross-cultural skills. It involves a well-structured curriculum, engaging teaching methods, and the integration of innovative approaches, such as blended learning and student-centered instruction. The quality is also influenced by the teacher's professional competence, teaching attitude, and ability to adapt to diverse student needs. With the rise of online education, the use of digital tools and platforms has become a key factor in enhancing teaching quality. Moreover, the alignment of teaching goals with application-oriented and globalized demands has led to reforms that emphasize practical skills and real-world application. Ultimately, quality teaching aims to create an interactive, inclusive, and effective learning environment that equips students with the skills needed for academic and professional success. The quality of college English teaching reflects the effectiveness of instructional strategies, curriculum design, and student learning outcomes. High-quality teaching ensures that students not only improve their language proficiency but also develop critical thinking, communication, and cross-cultural skills. It involves a well-structured curriculum, engaging teaching methods, and the integration of innovative approaches, such as blended learning and student-centered instruction. The quality is also influenced by the teacher's professional competence, teaching attitude, and ability to adapt to diverse student needs. With the rise of online education, the use of digital tools and platforms has become a key factor in enhancing teaching quality. Moreover, the alignment of teaching goals with application-oriented and globalized demands has led to reforms that emphasize practical skills and real-world application. Ultimately, quality teaching aims to create an interactive, inclusive, and effective learning environment that equips students with the skills needed for academic and professional success. In recent years, research on the quality evaluation of college English teaching has deepened, with studies exploring the construction of evaluation systems, methodological improvements, and practical applications. The following is a chronological summary of these studies. Wang and Zhang [1] were among the first to propose the application of fuzzy comprehensive evaluation methods to assess college English classroom teaching quality and emphasized the scientific nature and feasibility of transforming qualitative indicators into quantitative ones. Li [2] analyzed the strengths and weaknesses of the classroom teaching quality monitoring mechanism at Jishou University, suggesting that only a scientific and standardized evaluation system could effectively improve teaching quality. Gao [3] explored the feasibility of integrating ISO9000 standards into college English teaching quality evaluation, offering a new

perspective for building comprehensive evaluation systems. In 2017, Tu [4] introduced an interval fuzzy comprehensive evaluation method for college English teaching, arguing that this approach was more accurate and comprehensive than general fuzzy evaluation methods and effectively reduced information loss. Geng [5] constructed an all-process, full-procedure evaluation system for college English teaching under the framework of Outcome-Based Education (OBE). He highlighted the importance of student-centered and outcome-oriented evaluation methods. Li [6] focused on the application-oriented talent training model, proposing strategies for constructing a quality assurance evaluation system through teaching reforms to meet students' professional development needs. In the same year, Yan et al. [7] developed a college English teaching quality evaluation index system based on the SPOC model, aiming to cultivate students' autonomous learning and innovation abilities and argued that online teaching provided significant opportunities for improving teaching quality. Cui [8] established a quality evaluation system for college English teaching based on the concept of curriculum ideology, including aspects such as teaching plans, teaching processes, and teaching outcomes, providing valuable references for curriculum reform. Zhang [9] introduced a value-driven evaluation model, comparing students' "expectations" with their "actual experiences" to analyze the role of value-based thinking in college English teaching and proposed measures for improvement. Finally, He [10] examined the impact of formative assessment on the quality of blended online and offline teaching and argued that the widespread use of internet technologies has significantly facilitated teaching quality evaluation. In conclusion, these studies, approaching the subject from various dimensions and methodologies, have gradually constructed and refined the evaluation system for college English teaching quality, providing theoretical support and practical guidance for improving teaching quality.

The problem of evaluating university English teaching quality is MADM task. Recently, the KEMIRA approach [11,12,13] has been applied to address various MADM problems effectively. Meanwhile, INSs [14] have emerged as a powerful tool for representing and handling uncertain, imprecise, or incomplete information, which is often encountered in real-world evaluations such as teaching quality assessment. However, to date, no research has extended the modified KEMIRA approach to the framework of INSs [14].

1.1 Motivation of This Study

I. The primary motivation and research aim of this paper are to bridge this gap by extending the modified KEMIRA approach to INSs and developing a corresponding MADM model for university English teaching quality evaluation.

II. Specifically, this paper proposes the INN- KEMIRA approach, which integrates the KEMIRA concept with INSs to handle MADM problems under uncertainty.

III. By combining the KEMIRA approach with INSs, the INN- KEMIRA approach is designed to provide a robust and systematic methodology for evaluating teaching quality.

IV. To demonstrate the effectiveness and practicality of the established approach, a numerical example is provided that focuses on university English teaching quality evaluation. Moreover, comparative analyses are conducted to highlight the advantages and rationality of the INN- KEMIRA approach.

V. The results show that the established method is capable of handling uncertain information more effectively than traditional methods, making it a valuable tool for decision-making in uncertain environments.

1.2 Main Contributions of This Study

This study provides a novel framework for addressing MADM problems and offers a practical solution for evaluating teaching quality in uncertain and complex environments. The main contributions and research objectives of this paper are summarized:

(1) The KEMIRA model is extended to the INSs framework.

(2) The average approach is adopted to construct attribute weights under INSs, ensuring a more objective evaluation process.

(3) The INN- KEMIRA approach is developed to address MADM problems in the context of INSs, providing a novel solution for uncertain decision-making.

(4) A practical case study on university English teaching quality evaluation is presented to illustrate the validity and applicability of the established method.

(5) Comparative algorithms are implemented to verify the rationality, effectiveness, and advantages of the INN- KEMIRA approach.

The structure of this paper is organized as follows. Section 2 introduces the fundamental concepts of INSs, providing the theoretical basis for the study. In Section 3, the KEMIRA approach

is developed to address MADM problems within the framework of INSs. Section 4 presents a case study focused on evaluating university English teaching quality, demonstrating the practical application of the established method. Additionally, a comparative analysis is conducted to validate the effectiveness and advantages of the approach. Finally, the study concludes in Section 5, summarizing the findings and highlighting potential directions for future research.

2. Preliminaries

This section introduces the fundamental operations of interval-valued neutrosophic sets (INSs) and provides an overview of the KEMIRA approach. These concepts form the basis for the proposed methodology, integrating INSs with KEMIRA to address MADM problems.

2.1 Definition [15]. The SVNSs is established:

$$KA=\left\{\left(\vartheta,KT_A\left(\vartheta\right),KI_A\left(\vartheta\right),KF_A\left(\vartheta\right)\right)\middle|\vartheta\in\Phi\right\} \tag{1}$$

where the $KT_A\left(\vartheta\right),KI_A\left(\vartheta\right),KF_A\left(\vartheta\right)$ portrays the truth-membership (TM), indeterminacy-membership (IM) and falsity-membership (FM), $KT_A\left(\vartheta\right),KI_A\left(\vartheta\right),KF_A\left(\vartheta\right)\in\left[0,1\right]$ and satisfies $0\le KT_A\left(\vartheta\right)+KI_A\left(\vartheta\right)+KF_A\left(\vartheta\right)\le3$.

Wang et al. [14] established the INSs.

2.2 Definition [14]. The INSs is established:

$$SA=\left\{\left(\vartheta,ST_A\left(\vartheta\right),SI_A\left(\vartheta\right),SF_A\left(\vartheta\right)\right)\middle|\vartheta\in\Phi\right\} \tag{2}$$

where the $ST_A\left(\vartheta\right),SI_A\left(\vartheta\right),SF_A\left(\vartheta\right)$ portray the TM, IM and FM, $ST_A\left(\vartheta\right),SI_A\left(\vartheta\right),SF_A\left(\vartheta\right)\subseteq\left[0,1\right]$ and satisfies $0\le\sup ST_A\left(\vartheta\right)+\sup SI_A\left(\vartheta\right)+\sup SF_A\left(\vartheta\right)\le3$.

The interval neutrosophic number (INN) is established: $SA=\left(ST_A,SI_A,SF_A\right)=\left(\left[STL_A,STU_A\right],\left[SIL_A,SIU_A\right],\left[SFL_A,SFU_A\right]\right)$, where $ST_A,SI_A,SF_A\subseteq\left[0,1\right]$, and $0\le STU_A+SIU_A+SFU_A\le3$.

2.3 Definition [16]. Let $SA=\left(\left[STL_A,STU_A\right],\left[SIL_A,SIU_A\right],\left[SFL_A,SFU_A\right]\right)$ be INN, the score value is established:

$$SV(SA)=\frac{(2+STL_A-SIL_A-SFL_A)+(2+STU_A-SIU_A-SFU_A)}{6},$$

$$SV(SA)\in[0,1]. \quad (2)$$

2.4 Definition [16]. Let $SA=([STL_A,STU_A],[SIL_A,SIU_A],[SFL_A,SFU_A])$ be INN, an accurate value is established:

$$AV(SA)=\frac{(STL_A+STU_A)-(SFL_A+SFU_A)}{2},\ AV(SA)\in[-1,1]. \quad (3)$$

The larger $AV(SA)$ is, the more $SA=([STL_A,STU_A],[SIL_A,SIU_A],[SFL_A,SFU_A])$ is.

Huang, Wei and Wei [16] established the order for INNs.

2.5 Definition [16]. Let $SA=([STL_A,STU_A],[SIL_A,SIU_A],[SFL_A,SFU_A])$ and $SB=([STL_B,STU_B],[SIL_B,SIU_B],[SFL_B,SFU_B])$, let $SV(SA)=\frac{(2+STL_A-SIL_A-SFL_A)+(2+STU_A-SIU_A-SFU_A)}{6}$ and $SV(SB)=\frac{(2+STL_B-SIL_B-SFL_B)+(2+STU_B-SIU_B-SFU_B)}{6}$, and let $AV(SA)=\frac{(STL_A+STU_A)-(SFL_A+SFU_A)}{2}$ and $AV(SB)=\frac{(STL_B+STU_B)-(SFL_B+SFU_B)}{2}$, respectively, then if $SV(SA)<SV(SB)$, $SA<SB$; if $SV(SA)=SV(SB)$, (1)if $AV(SA)=AV(SB)$, then $SA=SB$; (2) if $AV(SA)<AV(SB)$, $SA<SB$.

2.6 Definition [17]. Let $SA=([STL_A,STU_A],[SIL_A,SIU_A],[SFL_A,SFU_A])$ and $SB=([STL_B,STU_B],[SIL_B,SIU_B],[SFL_B,SFU_B])$, the operations are established:

$$(1)\ SA \oplus SB = \begin{pmatrix} \left(STL_A + STL_B - STL_A STL_B, STU_A + STU_B - STU_A STU_B\right), \\ \left[SIL_A SIL_B, SIU_A SIU_B\right], \left[SFL_A SFL_B, SFU_A SFU_B\right] \end{pmatrix};$$

$$(2)\ SA \otimes SB = \begin{pmatrix} \left[STL_A STL_B, STU_A STU_B\right], \\ \left[SIL_A + SIL_B - SIL_A SIL_B, SIU_A + SIU_B - SIU_A SIU_B\right], \\ \left[SFL_A + SFL_B - SFL_A SFL_B, SFU_A + SFU_B - SFU_A SFU_B\right] \end{pmatrix};$$

$$(3)\ \delta SA = \begin{pmatrix} \left[1-\left(1-STL_A\right)^{\delta}, 1-\left(1-STU_A\right)^{\delta}\right], \\ \left[\left(SIL_A\right)^{\delta}, \left(SIU_A\right)^{\delta}\right], \left[\left(SFL_A\right)^{\delta}, \left(SFU_A\right)^{\delta}\right] \end{pmatrix}, \delta > 0;$$

$$(4)\ \left(SA\right)^{\delta} = \begin{pmatrix} \left[\left(STL_A\right)^{\delta}, \left(STU_A\right)^{\delta}\right], \left[\left(SIL_A\right)^{\delta}, \left(SIU_A\right)^{\delta}\right], \\ \left[1-\left(1-SFL_A\right)^{\delta}, 1-\left(1-SFU_A\right)^{\delta}\right] \end{pmatrix}, \delta > 0.$$

2.7 Definition [18]. Let $SA = \left(\left[STL_A, STU_A\right], \left[SIL_A, SIU_A\right], \left[SFL_A, SFU_A\right]\right)$ and $SB = \left(\left[STL_B, STU_B\right], \left[SIL_B, SIU_B\right], \left[SFL_B, SFU_B\right]\right)$, the INN Euclidean distance (INNED) between $SA = \left(\left[STL_A, STU_A\right], \left[SIL_A, SIU_A\right], \left[SFL_A, SFU_A\right]\right)$ and $SB = \left(\left[STL_B, STU_B\right], \left[SIL_B, SIU_B\right], \left[SFL_B, SFU_B\right]\right)$ is established:

$$INNED\left(SA, SB\right) = \sqrt{\frac{1}{6}\left(\begin{array}{l} \left|STL_A - SL_B\right|^2 + \left|STU_A - STU_B\right|^2 + \left|SIL_A - SIL_B\right|^2 + \\ \left|SIU_A - SIU_B\right|^2 + \left|SFL_A - SFL_B\right|^2 + \left|SFU_A - SFU_B\right|^2 \end{array}\right)} \quad (4)$$

2.1 The KEMIRA Method: Definition and Importance

The KEmeny Median Indicator Ranks Accordance (KEMIRA) method is a compensatory decision-making approach introduced by Krylovas et al. in 2014. This method is designed to rank various alternatives based on a set of criteria that are divided into two distinct groups within a decision matrix. The criteria are classified and prioritized by professionals, with their respective weights determined to ensure an accurate evaluation. KEMIRA's structured framework and adaptability make it an effective tool for solving complex decision-making problems.

One of KEMIRA's key features is its ability to handle criteria of different types, characteristics, and specialties by grouping them. For example, in real-world applications, criteria can be categorized into two or more groups to better reflect the priorities of decision-makers. This grouping allows for a more tailored and nuanced assessment, making the method particularly effective in addressing

MADM problems.

KEMIRA has gained significant attention in various fields due to its versatility and robustness. Researchers have applied and improved the method in numerous decision-making scenarios, including construction management, resource allocation, and educational quality evaluation. Its ability to incorporate professional expertise and classify criteria based on type and importance makes it a preferred choice in both academic and practical contexts.

In summary, the KEMIRA method provides a structured, flexible, and reliable approach to decision-making. By dividing criteria into meaningful groups and prioritizing them based on professional judgment, KEMIRA ensures a comprehensive and accurate evaluation process, making it indispensable for addressing complex decision-making challenges in various domains.

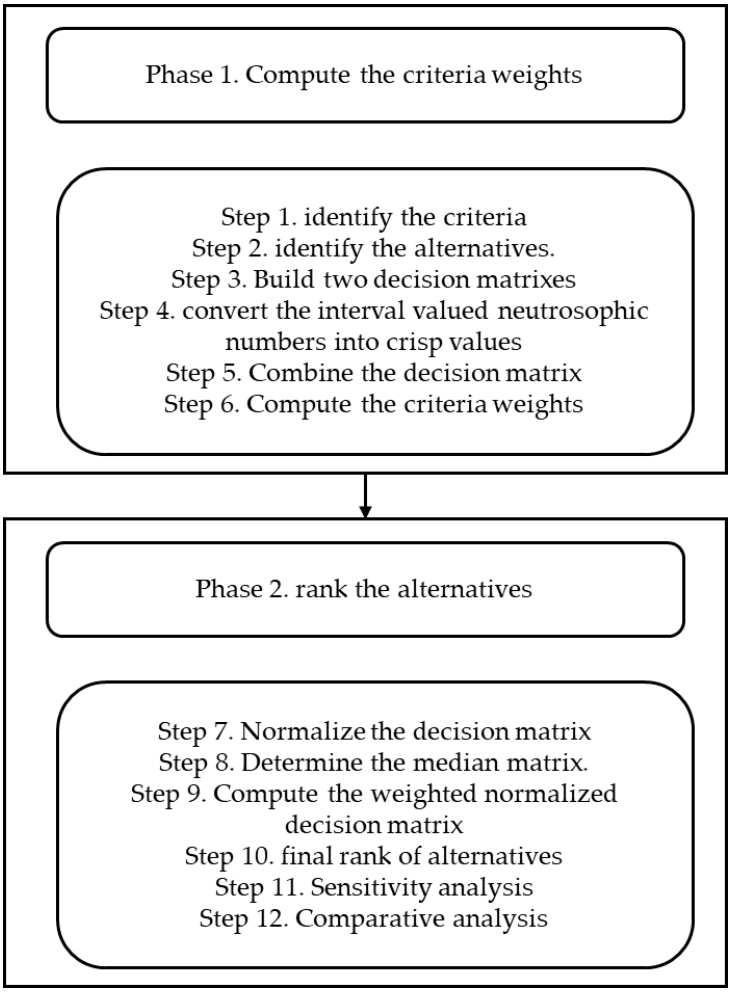
Phase 1. Compute the criteria weights
Step 1. identify the criteria
Step 2. identify the alternatives.
Step 3. Build two decision matrixes
Step 4. convert the interval valued neutrosophic numbers into crisp values
Step 5. Combine the decision matrix
Step 6. Compute the criteria weights
Phase 2. rank the alternatives
Step 7. Normalize the decision matrix
Step 8. Determine the median matrix.
Step 9. Compute the weighted normalized decision matrix
Step 10. final rank of alternatives
Step 11. Sensitivity analysis
Step 12. Comparative analysis

Figure 1. The steps of the proposed method.

3. Optimized KEMIRA Technique for Multi-Attribute Decision-Making Analysis

This section outlines the steps of the proposed method, as illustrated in Figure 1.

A. Building Two Decision Matrices

The decision matrices are constructed using two distinct groups of criteria, which are designed to evaluate the alternatives. The information is gathered from experts and decision-makers to ensure a comprehensive assessment of the criteria.

B. Normalizing the Decision Matrices

The normalization process is applied to both decision matrices. This includes adjustments for both positive and negative criteria to ensure consistency and comparability across all evaluation dimensions.

$$A_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}} \tag{5}$$

$$B_{ij} = \frac{y_{ij} - \min y_{ij}}{\max y_{ij} - \min y_{ij}} \tag{6}$$

C. Determine the median matrix

The medium matrix is computed by the optimal value obtained from the different between the different matrices.

$$R_A = argmin \sum_{j=1}^{n} p\left(R_j, R_{j'}\right) \tag{7}$$

Where $p\left(R_j, R_{j'}\right)$ can be computed as:

$$p\left(R_j, R_{j'}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij}^{j} - a_{ij}^{j'} \right| \tag{8}$$

Where

$$a_{ij} = \begin{cases} 0 & if\ A_j < A_{j'} \\ 1 & if\ A_j > A_{j'} \end{cases} \tag{9}$$

D. Compute the criteria weights.

E. Compute the weighted normalized decision matrix

$$E_i = \sum_{j=1}^{n} A_{ij} w_{x_j} \tag{10}$$

$$F_i = \sum_{j=1}^{n} B_{ij} w_{y_j} \tag{11}$$

$$T(E, F) = \min \sum |E_i - F_i| \tag{12}$$

F. The final rank of alternatives

$$E_i + F_i = \sum_{j=1}^{n} A_{ij} w_{x_j} + \sum_{j=1}^{n} B_{ij} w_{y_j} \tag{13}$$

4. Implementation

This section presents the results of the proposed method. The evaluation of university English

teaching quality is approached as a MADM problem. Ten local comprehensive and applied universities are assessed based on seven key attributes:

- Ability of instructors to deliver content
- Academic standards
- Tangible results
- Educational background
- Availability and quality of learning materials
- Student feedback
- Student performance

Table 1. Expert Opinions on Evaluation Criteria

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_2	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])
A_3	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.3],[0.1,0.2],[0.8,0.9])
A_4	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_5	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_6	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_7	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
A_8	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
A_9	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_{10}	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_2	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])
A_3	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.3],[0.1,0.2],[0.8,0.9])
A_4	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_5	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_6	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.8,0.9],[0.1,0.2],[0.1,0.2])
A_7	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.1,0.3],[0.1,0.2],[0.8,0.9])
A_8	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_9	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_{10}	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_2	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.8,0.9],[0.1,0.2],[0.1,0.2])
A_3	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.3],[0.1,0.2],[0.8,0.9])
A_4	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_5	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_6	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
A_7	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
A_8	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_9	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.3],[0.1,0.2],[0.8,0.9])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_{10}	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])

A. We build the two-decision matrix with seven criteria and ten alternatives and put then into one matrix using the neutrosophic numbers as shown in Table 1. These numbers are converted to the crisp values and combined into one matrix.

B. Eqs. (5 and 6) are used to normalize the decision matrix as shown in Table 2. All criteria in this study are positive criteria. Then we subtract each value in the decision matrix by the minimum value and then divide it by difference between the maximum and minimum value.

Table 2. Normalized Decision Matrix Representation

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.407407	0.121951	0.159091	0.537594	0.456311	0.582278	0.699187
A_2	0.562963	0.463415	0.363636	0.590226	0.834951	0.827004	1
A_3	0	0.987805	1	0.992481	0	0.932489	0
A_4	0.248148	0.695122	0.318182	0.406015	0.375405	0.915612	0.314363
A_5	1	0.670732	0.375	0.556391	0.770227	0.181435	0.249322
A_6	0.381481	0.788618	0.44697	1	0.339806	0.476793	0.539295
A_7	0.318519	0.174797	0.473485	0.511278	0.352751	0	0.233062
A_8	0.214815	1	0.382576	0.447368	0.38835	0.063291	0.246612
A_9	0.225926	0	0.386364	0.432331	0.495146	0.156118	0.376694

A_{10}	0.248148	0.292683	0	0	1	1	0.506775

C. Eqs. (7,8, and 9) are used to compute the median matrix.

D. The criteria weights are computed as shown in Table 3.

Table 3. Criteria Weights Representation

	WEIGHT	RANK
C_1	0.142156	5
C_2	0.135014	1
C_3	0.137295	2
C_4	0.148696	6
C_5	0.14022	4
C_6	0.156914	7
C_7	0.139704	3

E. The weighted normalized decision matrix is computed as shown in Table 4.

Table 4. Weighted normalized decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.057916	0.016465	0.021842	0.079938	0.063984	0.091368	0.097679
A_2	0.080029	0.062568	0.049925	0.087764	0.117077	0.129769	0.139704
A_3	0	0.133368	0.137295	0.147578	0	0.146321	0
A_4	0.035276	0.093851	0.043685	0.060373	0.052639	0.143673	0.043918
A_5	0.142156	0.090558	0.051485	0.082733	0.108001	0.02847	0.034831
A_6	0.05423	0.106475	0.061367	0.148696	0.047648	0.074816	0.075342
A_7	0.045279	0.0236	0.065007	0.076025	0.049463	0	0.03256
A_8	0.030537	0.135014	0.052526	0.066522	0.054454	0.009931	0.034453
A_9	0.032117	0	0.053046	0.064286	0.069429	0.024497	0.052626
A_{10}	0.035276	0.039516	0	0	0.14022	0.156914	0.070798

F. Then we ranked the alternatives as shown in Table 5.

Table 5. Ranking of Alternatives

	RANK
A_1	4
A_2	10
A_3	8
A_4	6
A_5	7
A_6	9
A_7	1
A_8	3
A_9	2
A_{10}	5

4.1 Analysis of Results and Comparative Discussion

The evaluation of university English teaching quality is a critical aspect of ensuring effective learning outcomes for students. English, as a global language, plays a pivotal role in academic, professional, and social environments. Therefore, assessing the quality of English education in universities is essential to ensure students acquire the necessary language skills to meet real-world demands. This evaluation involves analyzing various factors, including teaching methods, curriculum design, teacher performance, and student engagement. One of the most important

indicators of teaching quality is the effectiveness of teaching methods. Innovative and interactive teaching techniques, such as group discussions, role-playing, and project-based learning, allow students to actively participate and practice their language skills.

A departure from traditional lecture-based approaches to more student-centered methods can significantly improve learning outcomes. Additionally, incorporating technology, such as language learning apps, multimedia tools, and online resources, can enhance the teaching process. Curriculum design is another key factor in evaluating teaching quality. A well-structured curriculum should balance the four core language skills: reading, writing, listening, and speaking. It should also integrate cultural and practical elements to help students understand and use English in real-life scenarios.

Regular updates to the curriculum based on global trends and industry needs are essential to keep it relevant and effective. Teacher performance also greatly influences teaching quality. Effective English teachers possess not only subject knowledge but also the ability to motivate and inspire students. Their communication skills, teaching style, and ability to provide constructive feedback are crucial in fostering a positive learning environment.

Continuous teacher training and professional development programs can help educators stay updated with modern teaching strategies and methodologies. Student engagement is another critical component of the evaluation process. Active participation in class activities, completion of assignments, and willingness to communicate in English are indicators of successful teaching. Feedback from students through surveys and discussions can provide valuable insights into areas that require improvement.

Finally, evaluating university English teaching quality involves a comprehensive analysis of teaching methods, curriculum design, teacher performance, and student engagement. By addressing these aspects, universities can ensure that their English programs equip students with the necessary language skills to excel in their academic and professional pursuits. Continuous evaluation and improvement of teaching quality are vital for meeting the evolving demands of English education in a globalized world.

4.2 Sensitivity Analysis

This section shows the Sensity analysis under different values. In this study, we proposed eight cases by changing the criteria weights. First, we proposed all criteria have the same weight. Then we proposed one criterion that has 20% weights and others have the same weight. Table 6 shows the different criteria for weights.

Then we applied the proposed method under different weights. Then we rank the alternatives. We show the proposed methos rank is stable under different weights. Table 7 shows the rank of alternatives under different weights.

Table 6. The different criteria weights.

	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6	CASE 7	CASE 8
C_1	0.142857	0.2	0.133333	0.133333	0.133333	0.133333	0.133333	0.133333
C_2	0.142857	0.133333	0.2	0.133333	0.133333	0.133333	0.133333	0.133333
C_3	0.142857	0.133333	0.133333	0.2	0.133333	0.133333	0.133333	0.133333
C_4	0.142857	0.133333	0.133333	0.133333	0.2	0.133333	0.133333	0.133333
C_5	0.142857	0.133333	0.133333	0.133333	0.133333	0.2	0.133333	0.133333
C_6	0.142857	0.133333	0.133333	0.133333	0.133333	0.133333	0.2	0.133333
C_7	0.142857	0.133333	0.133333	0.133333	0.133333	0.133333	0.133333	0.2

Table 7. The different rank of alternatives.

	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6	CASE 7	CASE 8
A_1	4	4	3	4	5	4	4	5
A_2	10	10	10	10	10	10	10	10
A_3	8	7	9	9	8	7	9	7
A_4	6	6	6	6	6	5	6	6
A_5	7	9	7	7	7	9	7	8
A_6	9	8	8	8	9	8	8	9
A_7	1	2	2	2	2	1	1	1
A_8	3	3	5	3	3	3	3	3
A_9	2	1	1	1	1	2	2	2
A_{10}	5	5	4	5	4	6	5	4

4.3. Comparative analysis

This section shows the comparative analysis between the proposed method and other MADM methods. We compared the proposed method with three different MAD methods under different frameworks such as fuzzy-CODAS, picture fuzzy-TOPSIS, triangular fuzzy-VIKOR. We show the proposed method is strong compared with other methods. Table 8 shows the comparative analysis results.

Table 8. The results of comparative analysis

	PROPOSE METHOD	FUZZY-CODAS	PICTURE FIZZY-TOPSIS	TRIANGULAR FUZZY-VIKOR
A_1	4	5	4	4
A_2	10	10	10	10
A_3	8	7	8	6
A_4	6	6	7	8
A_5	7	8	6	7
A_6	9	9	9	9
A_7	1	1	1	1
A_8	3	3	3	3
A_9	2	2	2	2
A_{10}	5	4	5	5

Table 9. Features of comparative results

	PROPOSE METHOD	FUZZY-CODAS	PICTURE FIZZY-TOPSIS	TRIANGULAR FUZZY-VIKOR
FRAMEWORK	Interval valued neutrosophic sets	Fuzzy sets	Picture fuzzy sets	Triangular fuzzy sets
TRUTHINESS	Yes	Yes	Yes	Yes
INDETERMINACY	Yes	No	No	No
FALSITY	Yes	Yes	Yes	Yes

Table 9 shows the features of the proposed method compared with other MADM methods. We show our method has an advantage to deal with uncertainty and value information compared with other methods.

From Table 8, it is evident that the ranking results of the three approaches show slight differences. However, all the models identify the same local comprehensive and applied university as the optimal choice and the same one as the worst. This consistency demonstrates that the INN-KEMIRA approach is both reasonable and effective in evaluating university English teaching quality.

The INN-KEMIRA approach possesses several valuable characteristics that make it a robust tool for decision-making. First, it can effectively capture both individual uncertainty and overall uncertainty within INNs. This allows the model to handle complex and uncertain information more accurately. Second, the INN- KEMIRA approach is characterized by a simple structure, clear physical meaning, and strong differentiation ability, which enhances its practical applicability.

Overall, the INN- KEMIRA approach offers a novel and effective solution for addressing multi-attribute decision-making problems under uncertainty. Its ability to balance simplicity, clarity, and differentiation makes it a powerful tool for evaluating and improving education quality management in higher education. By adopting the INN- KEMIRA approach, higher education institutions can ensure a more objective and accurate evaluation process, ultimately contributing to the enhancement of teaching quality and overall institutional performance.

5. Conclusion

The university English teaching quality evaluation is crucial for enhancing educational outcomes and ensuring students are equipped with essential language skills. English, as a global language, plays a vital role in academic, professional, and social contexts. A well-structured evaluation process helps identify strengths and weaknesses in teaching methods, curriculum design, and teacher performance, ensuring continuous improvement. By assessing teaching quality, universities can ensure that students develop proficiency in reading, writing, listening, and speaking, which are critical for their future careers and global communication. It also helps institutions align

their teaching strategies with international standards and industry demands, making graduates more competitive. Moreover, evaluating teaching quality fosters accountability among educators and encourages the adoption of innovative, student-centered teaching methods. It also provides valuable feedback for curriculum updates, ensuring relevance to real-world applications. Ultimately, high-quality English education contributes to students' personal and professional development, benefiting both individuals and society. The university English teaching quality evaluation is viewed as the MADM. In this paper, the INN- KEMIRA approach is developed by integrating the KEMIRA method with INSs for handling MADM problems under uncertainty. This novel approach leverages the characteristics of INSs to effectively model uncertainty and imprecision in decision-making scenarios. Finally, a numerical example focusing on university English teaching quality evaluation is provided to demonstrate the practical application of the INN- KEMIRA approach. Additionally, comparative analyses are conducted to highlight the advantages and rationality of the proposed method.

Overall, this study offers a new framework for addressing MADM problems in uncertain environments, particularly in the context of education quality management. By extending the KEMIRA approach to INSs and combining it with the average method, the proposed INN- KEMIRA model provides a systematic, accurate, and reliable solution for complex decision-making tasks. The demonstrated application in university English teaching quality evaluation highlights its potential for broader use in similar fields.

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Neutrosophic Soft Sets in One And Two-Dimensions Using Iteration Method

M.N.Bharathi [1] and G.Jayalalitha [2]

[1] Research scholar Department of Mathematics Vels Institute of Science Technology and Advanced Studies; e-mail: bharathimeenakshi70922@gmail.com

[2] Professor Department of Mathematics Vels Institute of Science Technology and Advanced Studies; e-mail: g.jayalalithamaths.sbs@velsuniv.ac.in

Abstract

This paper introduces a different perspective of *Neutrosophic Fractals* and *Neutrosophic Soft Fractals*, merging the principles of Neutrosophic Logic, Soft set theory, and Fractal Geometry to address indeterminacy in complex, self-similar structures specifically the Von Koch curve and the Sierpinski triangle. It sightsees the complex qualities of Neutrosophic soft sets by incorporating attributes of falsification, indefiniteness, and truth into union and intersection operations. The research elucidates the interplay between Neutrosophic Logic and fractal geometry, leading to more precise modeling of complex systems. Proving theorems and providing examples examine the intricate interactions between membership characteristics in these fractal structures, demonstrating self-similarity. Fractal geometry is applied innovatively to improve the representation of uncertainty, indeterminacy, and falsity in Neutrosophic Logic, enhancing mathematical modeling techniques. Results show that the Sierpinski triangle provides a better representation than the Koch curve.

Keywords: Fuzzy set, Neutrosophic soft set, Fractals, Von Koch curve, Sierpinski triangle.

1. Introduction

Lotfi Zadeh is obtainable to us with a fuzzy set. The membership value of fuzzy sets can occasionally be difficult to ascertain [4, 13]. Interval-valued fuzzy sets were therefore proposed as a way to take into consideration the degree of uncertainty in membership values. From the point

of view of philosophy, real standard or unusual subsets of] −0, 1 + [define the amount for the Neutrosophic set [1, 6]. Neutrosophic soft sets have three categories truth, indeterminacy, and falsity [2, 8]. The truth and the falsity of memberships must be considered in certain real-world scenarios, such as expert systems, belief systems, information fusion, and so on, for it to be appropriately defined in an unclear, ambiguous environment [3].

Florentin Smarandache introduced Neutrosophic logic in 1995 as an extension of Neutrosophic logic, encompassing fuzzy logic, intuitionistic logic, and three other logics incorporating an indeterminate value. A Neutrosophic set of logical variables (x) is represented in triple order as $X= (t, i, f)$, where t denotes the truth value of the membership function, i represents the indefiniteness value of membership operation, and f signifies falsification value for the function defining the membership [1].

The maps in Fractals are likeness transforms for n = m, and the dimension of R is log r/log n. But if n is greater than m, then a new strategy is needed, mainly because under the iteration of the mappingsf_i, squares are stretched into narrow rectangles [4, 11]. This frame of mind has evolved in the past few decades [9]. The mathematics of non-smooth things is a subject about which a lot is said, and worth saying. Additionally, irregular sets offer a far superior depiction of several natural events compared to conventional geometry figures [10]. Studying such irregular collections may be done broadly with the help of fractal geometry. One of the most well-known and simple-to-make fractals is the middle third Cantor set; yet, it exhibits many typical fractal characteristics [12].

Such recursive processes may be used to generate many more sets. For instance, equilateral triangles of unit side lengths can be repeatedly removed from an original equilateral triangle to get the Sierpinski triangle or gasket [2]. It is preferable to conceptualize this process as repeatedly substituting three half-height triangles for an equilateral triangle for numerous purposes. A Cantor dust, a plane equivalent of the Cantor set. Every square that is left gets split into 16 smaller squares at each step [5].

Section 2 defines the Neutrosophic soft set with an example, Neutrosophic value class, and Neutrosophic two sets of pairs for union and intersection. Section 3 explains the methods, detailing the Neutrosophic soft set and its examples, followed by the Neutrosophic union and intersection

of two pairs with a theorem. Neutrosophic logic is used in the Von Koch curve and Sierpinski triangle explained with iterations, and demonstrated using Neutrosophic logic with figures.

2. Preliminaries

2.1 Neutrosophic Soft Set

Consider the scenario wherever G is a set, a subset of G is A, and P (U) is said to be a collection of all Neutrosophic sets defined by the universal set U. In this context, pairs (F, A) are termed soft Neutrosophic sets, where F represents a mapping of the subset A to the set P (U) [8].

Example

Let U be the usual cars below process and G be the set. Every parameter consists of a word or phrase that uses a Neutrosophic term. Examine G = {expensive, repair, non-repair, models, types of engines, water wash}. In this instance, defining a Neutrosophic soft set entails highlighting expensive, repairs, models, types of engines, and so on. Let's state that there are five cars in the universe U, denoted by U = {c1, c2, c3, c4, c5} and that the set of parameters A = {g1, g2, g3, g4}, where g1 denotes the parameter *costly*, g2 denotes *repair*, g3 denotes *models*, and g4 denotes *types of engines [8]*.

Table 1-Neutrosophicset of Car Engines

Universe U	C1	C2	C3	C4	C5
COSTLY	0.43,0.74,0.89	0.57,0.64,0.47	0.15,0.51,0.42	0.65,0.25,0.41	0.67,0.51,0.42
REPAIR	0.54,0.22,0.47	0.86,0.23,0.42	0.71,0.17,0.15	0.63,0.14,0.28	0.78,0.12,0.95
MODELS	0.77,0.38,0.12	0.65,0.46,0.91	0.56,0.71,0.42	0.43,0.11,0.25	0.46,0.52,0.61
TYPES OF MACHINES	0.67,0.75,0.83	0.96,0.18,0.55	0.16,0.75,0.83	0.85,0.17,0.98	0.28,0.54,0.66

2.2 Neutrosophic Set in Value Class

Neutrosophic soft sets (F, G) are characterized by the numerical value class denoted by H (F, G), encompassing a set of all values corresponding to the Neutrosophic set. For instance, to provided example (2.1), H (F, G) is defined as $\{v_1, v_2, \cdots v_{10}\}$. H (F, G) is a subset of P (U).

2.3 Neutrosophic Soft Set in Two Pairs

Consider two pairs of Neutrosophic soft sets (NSs) F and B defined across the unified Universe. If$A \subset B$ for all $x \in U, g \in A$, then the following conditions hold: TF (g)(x) ≤ TE(g)(x), similarly IF ≤ IG and FF ≥ FE, then the pair (F, A) termed as of (E, B) denoted by(F, A) ⊆ (E, B). Conversely, if (E, B) is an NS of (F, A).

2.4 Neutrosophic Union with Two Sets

Let (C, A) and (E, B) be the two pairs of neutrosophic Soft sets in the same Universe. The (C, A) ∪ (E, B) can be expressed (K, H), where H= A ∪ B.

a) TK(g)(n) = TC(g)(n) if g ∈ A − B, TE(g)(n) if g ∈ B − A, and max (TC(g)(n), TE(g)(n)) if g ∈ A ∩ B.
b) IK(g)(n) = IC(g)(n) if g ∈ A - B, IE(g)(n) if g ∈ B - A, and IC(g)(n) + IE(g)(n)/2 if g ∈ A ∩ B.
c) FK(g)(n) = FC(g)(n) for g ∈ A − B, FE(g)(n) for g ∈ B − A, and min (FC(g)(n), FE(g)(n)) for g ∈ A ∩ B.

2.5 Neutrosophic Intersection of Two Sets

Let us consider the two pairs (C, A) and (E, B) to be the Neutrosophic sets spanning identical universes [1, 2]. The intersection of the two pairs is indicated as (C, A)∩ (E, B) and it is also defined by (K, H), where H=A intersection B. The three membership functions of (K, H) are as follows:

a) Truth membership of K(g)(n) = min (TH(g)(n), TE(g)(n))
b) Indefiniteness membership of K(g)(n) = IC(g)(m) + IE(g)(n)/2
c) Falsification membership of K (g) (n) = max (F C (g)(n), FE(g)(n)), for g ∈ H.

3. Methods

The methods section demonstrates the Neutrosophic soft set of union and intersection with theorems and how the Sierpinski triangle can represent Neutrosophic soft sets. The union of parings with T, I, and F values is a Neutrosophic set with a numerical value given by H (F, G). These variables are divided into subcomponents like $(T_1, T_2, \ldots T_A)$ for truth, $(I_1, I_2, \ldots I_B)$ for indefiniteness, and $(F_1, F_2, \ldots F_C)$ for falsification, where [A+B+C = n ≥1]. Iterations 1, 2, 3, and 4 of this quadratic Sierpinski triangle are discussed. Koch curve and Sierpinski triangle for these iterations, as shown in the figure and tables.

Theorem 3.1

a) For any two neutrosophic soft sets (C, A) and (E, B), the distributive property between union and intersection is satisfied [6],

i. $(C, A) \cup (C, A) = (C, A)$.

ii. $(C, A) \cup (E, B) = (E, B) \cup (C, A)$

iii. $(C, A) \cap (C, A) = C, A$.

iv. $(C, A) \cap (E, B) = (E, B) \cap (C, A)$.

v. $(C, A) \cup \Phi = (C, A)$.

vi. $(C, A) \cap \Phi = \Phi. 2r$

vii. $[(C, A)^c]^c = (C, A)$

b) For any neutrosophic soft sets (E, B), (K, H), and (C, A), the following properties hold:

i. Property of distribution of a union across an intersection

$$[(E, B) \cup (K, H)] \, U \, (C, A) = [(K, H) \cup (E, B)] \cup (C, A).$$

ii. An intersection's associative attribute:

$$[(E, B) \cap (K, H)] \cap (C, A) = [(K, H) \cap [(C, A) \cap (E, B)].$$

iii. The union's associative property:

$$[(E, B) \cap (K, H)] \cap (C, A) = [(C, A) \cup (E, B)] \cap [(K, H) \cup (C, A)].$$

iv. The property of the distribution of intersection over union

$$[(E, B) \cup (K, H)] \cap (C, A) = [(C, A) \cap (E, B)] \, U \, [(C, A) \cap (K, H)].$$

Proof

A pair of sets unions gives the same set; the intersection of the two pairs has a variable in pair one; one pair of sets union of the null set gives the pair of the same set; one pair of set interests with null set gives null set; and the complement of one pair of sets having another complement gives the same set. These examples illustrate the apparent nature of the proof. Hypotheses 1 through 4 are evident in part b, as well as the distributive property of the union across the intersection, an intersection between two pairs, property with the associative union, and distributive property of intersection over the union.

3.1von Koch Curve

The techniques of standard geometry and calculus are not adequate for humans to investigate fractals; instead, they require alternative approaches. In fractals, geometry has to form a dimension, which is a fundamental tool. The idea is that a (smooth) curve has one dimension while a surface has two. The Cantor set should have the size log 2/log 3 = 0.63 and the von Koch curve has a dimension log 4/ log 3 = 1.26. If the Koch curve is *higher than 1-dimensional* and *smaller than 2-dimensional* or has an unlimited length, then this latter estimate is at least consistent with that. An analysis of the sense that the *dimensions* validate and stand for self-similarity clambering properties.

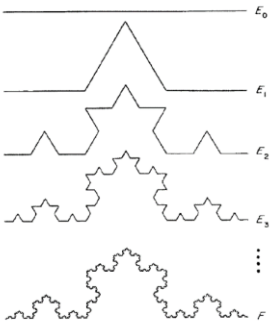

Figure 1 Von Koch curve

The von Koch curve has features in many ways similar to those listed for the middle third cantor set. It is made up of four quarters each similar to the whole, but scaled by a factor 1/3. The fine structure is reflected in the irregularies at all scales nevertheless, this intricate structure stems from a simple construction. Whilst it is reasonable to call F a curve, it is much too irregular to have

tangents in the classical sense. A simple calculation shows that E_K is of length $(\frac{4}{3})^k$ letting k tend to infinity implies that F has infinite length. On the other hand, F occupies zero area in the plane, so neither length nor area provides a handy description of the size of F [4].

$$dim_H F = dim_B F = \frac{log4}{log3}$$

3.1.1 Iteration for Koch Curve

The von Koch curve method to explore the fractals in the Sierpinski triangle with the baseline was eliminated.

i) Let an equilateral triangle, a generate (zig-zag curve) be created by replacing every line segment with four copies of the original line portion, each of which is shortened to one-third of its original length. D is equal to minus log N/log S. The fractal size for any strictly identical fractal may be found using this method.

T= 0.6= $1/3^6$=0.001

I=0.5= $1/3^5$=0.004

F=0.2=$1/3^2$=0.111

$$0 \leq T + I + F \leq 3 = 0.001 + 0.004 + 0.111 = 0.116$$

$$= \frac{\log 4}{\log 0.001} + \frac{\log 4}{\log 0.004} + \frac{\log 4}{\log 0.111}$$

$$= \frac{0.602059}{3} - \frac{0.602059}{2.3979} - \frac{0.602059}{0.9546}$$

= 0.2006+0.2510+0.2510=0.7026

3.2 Neutrosophic Logic

In Neutrosophic sets the indeterminate value Iinto Paradox (true and false) and Uncertainty (real or untrue), Belnap's four-valued logic expands. Furthermore, dividing Iinto inconsistency, unpredictability, and ambiguity yields logic. In advanced Neutrosophic logic, truth is subdivided into sub-components $(T_1, T_2, \dots T_A)$, indefiniteness$(I_1, I_2, \dots I_B)$, and falsification$(F_1, F_2, \dots F_C)$,

where [A+B+C = n ≥1]. Their subcomponents T_A, I_B, F_Ccan comprise numerous sets, possibly countable or unbounded. In the context of implementing software engineering concepts, the conventional unit interval *[0, 1]* may be employed. For instance, an assertion can range from *[0.4, 0.6]* true, *[0.5]* indeterminate, or anywhere from (0.15, 0.25) ambiguous, corresponding to 0.4 or 0.6 false.

3.3 Sierpinski Triangle

The Sierpinski triangle or gasket is obtained by repeatedly removing (inverted) equilateral triangles from an initial equilateral triangle of unit side length. For many purposes, this procedure is better understood as repeatedly replacing an equilateral triangle with three triangles of half the height.

Example

The Sierpinski triangle or gasket F is constructed from an equilateral triangle by repeatedly removing inverted equilateral triangles. Then

$$dim_H F = dim_B F = \frac{log3}{log2}$$

Solution: The set F is the attractor of the three obvious similarities of ratios ½, which maps from the triangle E_0 onto the triangles of E_1. The open set condition holds, taking V as the interior of E_0. Thus, the open set condition $V \supset \bigcup_{i=1}^{m} S_i(V)$ holds for the similarities $S_i$on $\mathbb{R}^n$ with ratios $0 < V_i < 1$ for$1 \le i \le m$. If F is the attractor of the IFS$\{S_1, \dots, S_m\}$, that is $F = \bigcup_{i=1}^{m} S_i(F)$, then $dim_H F = dim_B F = S$, where S is given by $\sum_{i=1}^{m} r_i^s = 1$. Moreover, for this value of S, $0 < \mathcal{H}^s(F) < \infty$.

3.3.1 Iteration 1

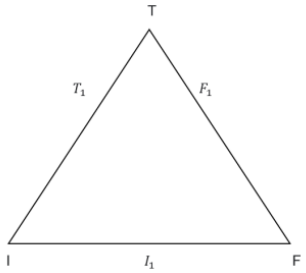

Figure 2 **Sierpinski equilateral Triangle with membership functions**

Assume that the triangle has 1 ratio with truth (T) = 0.6, indefiniteness (I) = 0.5, and falsification (F) = 0.2, which is 0.6+0.5+0.2 = 1.3≤ 3, satisfying the condition of the Neutrosophic set value of T+I+F≤3. This iteration 1 states that the triangle contains three sides of the same length, width, and height identified in the NSs as truth, indeterminacy, and falsity. It could be represented by a membership function between*]-0 to 1+ [*.

3.3.2 Iteration 2

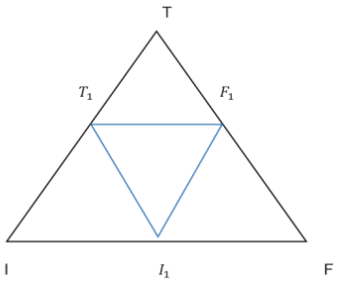

Figure 3 **Sierpinski Triangle iteration 2 with membership functions**

In this second iteration, the triangle's three sides with the same length, blue color, width, and height are referred to as truth, truth, indefiniteness, and falsification in the Neutrosophic set. Its membership function may be represented by a range from*]-0 to 1+ [*. Assuming truth (T_1) = 0.8, indeterminacy (I_1) = 0.5, and falsehood (F_1) = 0.1, the iteration is satisfied by 0.8+0.5+0.1= 1.4≤3.

3.3.3 Iteration 3

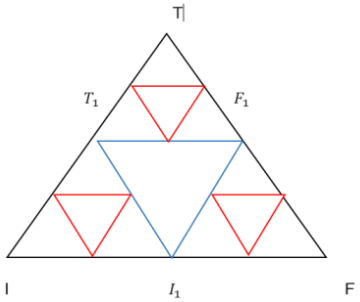

Figure 4 **Sierpinski Triangle iteration 3 with membership functions**

The triangle in iteration 3 indicates that there is a difference between the first and second iterations with a 3 sides red color triangle. This difference is found in iteration 2, where another quadrilateral triangle with the same width and height separates the two iterations. Indicates a Neutrosophic membership value in a similar manner, provided that the requirement is satisfied [5].

3.3.4 Iteration 4

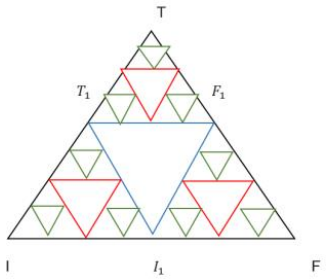

Figure 5 **Sierpinski Triangle iteration 4 with membership functions**

And so on. Eventually, it will be possible to state that the Neutrosophic Sierpinski triangle, which is the result of the combination of the self-similarity in the Sierpinski triangle and the criteria in the Neutrosophic set, fulfills the Sierpinski triangle. The Sierpinski triangle has an equilateral triangle and it is also dented by the Neutrosophic set notations like the truth of $T_A(x)$, the indeterminacy of $I_B(x)$, the falsity of$F_C(x)$.

4. Results

The Von Koch curve proposes the elimination of both self-similarity and adjacent lines when applied, offering unique characteristics for various applications. In the antenna response, there are

sudden spikes at Iteration 1 =1 and Figure *1*, Iteration 2 =1 /2 and Figure *2*, Iteration 3 =1 /4 and *Figure 3*, Iteration 4 =1/8 and *Figure 4*, and goes on this frequency. Sierpinski triangle must be represented as a Neutrosophic soft set with three membership values using an example like five cars in the universe, denoted by U = {c1, c2, c3, c4, c5} and that parameters A = {g1, g2, g3, g4}, where g1 denotes the parameter *costly*, g2 denotes *repair*, g3 denotes *models*, and g4 denotes *types of engines.*

5. Conclusion

The *Neutrosophic Fractals* and *Neutrosophic Soft Fractals* cover the boundaries of Fractal analysis by incorporating the dimensions of truth, indeterminacy, and falsity to the Von Koch curve and Sierpiński triangle demonstrate that memberships representing truth, indefiniteness, and falsification alignment can effectively the geometric patterns observed. This approach bridges mathematical concepts from different domains, underscoring the relevance of Neutrosophic soft sets in fractal analysis, supported by theorems and empirical evidence. The study ranks the Sierpiński Triangle higher than the Koch Curve, with relative closeness scores of 0.625 and 0.375, respectively, indicating that the Sierpiński Triangle better satisfies the combined fractal dimension, length, and area criteria. It highlights the potential of Neutrosophic soft sets for innovative applications in mathematics and computational sciences, demonstrating the interconnectedness between mathematical frameworks and geometric patterns.

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Enhanced EXPROM Methodology Using SuperHyperSoft Sets for Evaluating the Quality of Cultural Industry Development in the Digital Economy with Single-Valued Neutrosophic Sets

Yue Zhao*
Henan Vocational College of Tuina, Luoyang, 471023, Henan, China,
*Corresponding author, E-mail: Zhaoyue1107@126.com; zhaoy55@mail3.sysu.edu.cn

Abstract: With the advent of the digital economy era, the high-quality development of cultural industry has received people's attention. How to take advantage of the favorable opportunities of rapid development of the digital economy to further promote the innovative development of cultural industry is an important problem that needs to be solved in development process of Chinese cultural industries. The quality evaluation of cultural industry development in the context of the digital economy is MAGDM. The single-valued neutrosophic sets (SVNSs) is a useful tool to cope with uncertain information during the quality evaluation of cultural industry development in the context of the digital economy. In this paper, the single-valued neutrosophic number EXPROM (SVNN-EXPROM) model is formed to cope with the MAGDM. The SuperHyperSoft set (SHSS), which is an extension of HyperSoft set (HSS) is applied in this study to treat various criteria and sub criteria values. The average model is utilized to obtain weight numbers in light under SVNSs. Finally, numerical examples and comparative analysis for quality evaluation of cultural industry development in the context of the digital economy is utilized to validate SHSS-SVNN-EXPROM model. The main contributions of this study are formed: (1) the average model is formed to obtain the weight numbers; (2) the SHSS-SVNN-EXPROM model is formed under SVNNs with SHSS to deal with various criteria values; (3) Finally, numerical example and comparative analysis for quality evaluation of cultural industry development in the context of the digital economy is utilized to validate SHSS-SVNN-EXPROM model.

Keywords: MAGDM; SVNSs; EXPROM model; quality evaluation; SuperHyperSoft Set.

1. Introduction

The concept of "digital economy" was born in an atmosphere where the new economy is widely accepted and recognized. During the Clinton administration in the 1990s, driven directly by knowledge and technological innovation, the US economy continued to grow for nearly a decade, and people referred to it as the "new economy" [1]. Among them, the application of modern technology represented by electronic computers, the Internet and e-commerce has played a key role. The connotation of the digital economy has undergone a continuous expansion process [2]. The market-oriented application of information and communication technology, such as the birth of telecommunications, software and Internet industries, marks the arrival of the initial stage of the digital economy-the digital industrialization stage. Subsequently, information technology and traditional industries are increasingly merging, leading to a series of new economic styles [3]. These new economic styles are different from the new forms of

digital industrialization, and their prominent features are reflected in the disruptive impact of digital technology on economic efficiency and industrial chain after deep application in traditional industries[4, 5]. This not only greatly improves production efficiency and quality, but also reconstructs traditional industries, enabling them to leap from industrialization to informatization. The so-called "Industry 4.0", intelligent logistics, modern agriculture, "technology finance", etc. all belong to this scope, which is the stage of industrial digitization. As two major development patterns of digital economy, digital industrialization and industrial digitization constitute the basic content of digital economy practice. Whether it is the G20 Hangzhou Summit's signing of the G20 Digital Economy Development and Cooperation Initiative, the 14th Five-Year Plan for Digital Economy Development issued through Chinese government or Statistical Classification of Digital Economy and Its Core Industries. The natural closeness between cultural industry and the digital economy enables the two to integrate quickly and deeply, resulting in a new form of industry-the digital cultural industry [6]. According to industry classification standards of China's national economy, the digital cultural industry is divided into four categories: digital cultural technology equipment manufacturing industry, digital creative content industry, digital cultural entertainment industry, and digital cultural service industry [7]. In China's economic practice, the digital cultural industry covers two major areas: the digital industrialization of digital cultural technology equipment manufacturing and the digitalization of digital cultural service supply. The scope of digital cultural industry is not limited to peripheral industries of digital economy, but also involves the core industries of digital economy. The digital cultural industry is no longer an optional form of industry, but an inevitable direction for development of cultural industries in digital economy era. In the era of digital economy, promoting the digital transformation of the cultural industry has become a necessary way to improve the cultural industry development quality [8]. Efficiency improvement brought by digitalization to the allocation of production resources and the creation of new cultural demands requires the emphasis on digital construction and continuous improvement of development quality in the development of the cultural industry. Against the backdrop of high-quality development becoming a development goal in various fields in China, promoting high-quality development of the digital cultural industry has become a necessary choice for the development of the cultural industry. So, what is high-quality development of digital cultural industry and what are its standards? According to current scholars, only the development of a good industrial ecosystem can achieve high-quality development. Specifically, in the digital cultural industry, it is necessary to build the high-level cultural industry digital ecosystem, achieve the integration of culture and technology, and match cultural supply with cultural demand in terms of technological innovation, demand feedback, and consumption upgrading. Whether it is the construction of a digital ecosystem or promoting the matching of cultural supply and demand, digitization is the core element. Only by grasping the core of digital construction can we truly achieve high-quality development of digital cultural industry. The foundation for empowering spiritual life and common prosperity through the digital cultural industry is digital technology facilities. We need to comprehensively and systematically promote the construction of digital infrastructure, narrow the spiritual gap brought about by the digital divide, and achieve a balance between the supply and demand of spiritual products. On the one hand, we need to comprehensively layout the construction of digital infrastructure from bottom to top. Building digital infrastructure such as data centers and cloud platforms, improving the cloud, network, and end

infrastructure of the cultural industry, connecting the digital collection networked transmission intelligent computing digital chain, and providing technical support for promoting the free circulation of spiritual product resources across regions, levels, and populations [9]; Develop guidelines and regulations for digital governance, and strengthen the protection of intellectual property rights for intelligent spiritual products. On the other hand, we need to comprehensively promote construction of digital technology support systems and strengthen the integrated application and innovation of 5G, big data, cloud computing, Internet of Things, blockchain and other technologies in the cultural industry, ensure the effective supply of digital spiritual products, and meet the spiritual and cultural needs of people [10].

The quality evaluation of cultural industry development in the context of the digital economy is the real-life MAGDM [11]. The SVNSs [12] is useful technique to cope with uncertain information during the quality evaluation of cultural industry development in the context of the digital economy. Unfortunately, few valuable existing works were managed the EXPROM [13, 14,15] under SVNSs [12].

The main objective of this study is to cope with MAGDM through employing the SHSS-SVNN-EXPROM model to rank the alternatives. Finally, numerical examples and comparative analysis for quality evaluation of cultural industry development in the context of the digital economy is utilized to validate SHSS-SVNN-EXPROM model.

The main contributions of this study are formed: (1) the average model is formed to obtain the weight numbers; (2) the SHSS-SVNN-EXPROM model is formed under SVNNs; (3) the SHSS is employed to treat various sub criteria values in the decision making process (4) Finally, numerical example and comparative analysis for quality evaluation of cultural industry development in the context of the digital economy is utilized to validate the SHSS-SVNN-EXPROM model.

The remaining sections are organized as follows: Section 2 introduces the SVNSs. Section 3 develops the SHSS-SVNN-EXPROM model tailored for multi-attribute group decision-making (MAGDM). Section 4 applies the model to evaluate the quality of cultural industry development within the context of the digital economy, supplemented by comparative analyses to validate the effectiveness of the SHSS-SVNN-EXPROM model. Finally, Section 5 presents the conclusion.

2. Preliminaries

This section presents some definitions of the SHSS and SVNSs.

SuperHyperSoft Set (SHSS)

SHSS is an extension of the HyperSoft set and has various HyperSoft Sets. SHSS is used in this work for representing the computed criteria and sub criteria for selecting the best alternatives under SVNNs depending on the selected criteria and sub criteria [17,18,19].

Suppose the universe set $U = \{C_1, \dots, C_n\}$, $P(U)$ is a power set of U. As well, C_1, C_2, C_3 are the criteria and the alternatives are assessed under these criteria

$P(C_1) \times P(C_2) \times P(C_3) \to P(U)$

$$P(C_1) \times P(C_2) \times P(C_3) =$$

$$\left\{ \begin{array}{c} \{\{C_{11}\},\{C_{12}\},\{C_{11},C_{12}\}\} \times \\ \{\{C_{21}\},\{C_{22}\},\{C_{21},C_{22}\}\} \times \\ \{\{C_{31}\},\{C_{32}\},\{C_{33}\},\{C_{31},C_{32}\},\{C_{31},C_{33}\},\{C_{33},C_{33}\},\{C_{31},C_{32}\}\} \end{array} \right\}$$

Definition 1 [12]. The SVNSs is outlined:

$$SA = \left\{ \left(\vartheta, ST_A(\vartheta), SI_A(\vartheta), SF_A(\vartheta) \right) \middle| \vartheta \in \Theta \right\} \tag{1}$$

where $ST_A(\vartheta), SI_A(\vartheta), SF_A(\vartheta)$ depicts truth membership, indeterminacy membership and falsity membership, $ST_A(\vartheta), SI_A(\vartheta), SF_A(\vartheta) \in [0,1]$, $0 \le ST_A(\vartheta) + SI_A(\vartheta) + SF_A(\vartheta) \le 3$.

Definition 2 [16]. The score value function (SVF) of $SA = (ST_A, SI_A, SF_A)$ is outlined:

$$SVF(SA) = \frac{(2 + ST_A - SI_A - SF_A)}{3}, SVF(SA) \in [0,1]. \tag{2}$$

Definition 3 [16]. The accuracy value function (AVF) of $SA = (ST_A, SI_A, SF_A)$ is outlined:

$$AVF(SA) = ST_A - SF_A, \ AVF(SA) \in [-1,1] \ . \tag{3}$$

Peng et al. [16] outlined the order framework between two SVNNs.

Definition 4[16]. Let $SA = (ST_A, SI_A, SF_A)$ and $SB = (ST_B, SI_B, SF_B)$, let $SVF(SA) = \frac{(2 + ST_A - SI_A - SF_A)}{3}$ and $SVF(SB) = \frac{(2 + ST_B - SI_B - SF_B)}{3}$, and let $AVF(SA) = ST_A - SF_A$ and $AVF(SB) = ST_B - SF_B$, if $SVF(SA) < SVF(SB)$, then: $QA < QB$; if $SVF(SA) = SVF(SB)$, then: (1)if $AVF(SA) = AVF(SB)$, then $SA = SB$; (2) if $AVF(SA) > AVF(SB)$, then: $SA < SB$.

Definition 6. Let $SA = (ST_A, SI_A, SF_A)$ and $SB = (ST_B, SI_B, SF_B)$, then SVNN cosine similarity measure (SVNNCSM) based on the $SA = (ST_A, SI_A, SF_A)$ and $SB = (ST_B, SI_B, SF_B)$ is outlined:

$$SVNNCSM(SA, SB) = \frac{ST_A \times ST_B + SI_A \times SI_B + (1 - SF_A) \times (1 - SF_B)}{\left(\begin{array}{l} \sqrt{(ST_A)^2 + (SI_A)^2 + (1 - SF_A)^2} \\ \cdot \sqrt{(ST_B)^2 + (SI_B)^2 + (1 - SF_B)^2} \end{array} \right)},$$

$$SVNNCSM(SA, SB) \in [0,1], \tag{4}$$

Definition 6. Let $SA=\left(ST_A, SI_A, SF_A\right)$ and $SB=\left(ST_B, SI_B, SF_B\right)$, then SVNN cosine function similarity measure (SVNNCFSM) based on the $SA=\left(ST_A, SI_A, SF_A\right)$ and $SB=\left(ST_B, SI_B, SF_B\right)$ is outlined:

$$SVNNCFSM\left(SA, SB\right)=\frac{1}{2}\left(\cos\left[\begin{array}{l}\frac{\pi}{12}\left(\left|ST_A-ST_B\right|+\left|SI_A-SI_B\right|+\left|SF_A-SF_B\right|\right)\\ +\frac{\pi}{4}\max\left(\left(\left|ST_A-ST_B\right|,\left|SI_A-SI_B\right|,\left|SF_A-SF_B\right|\right)\right)\end{array}\right]\right)$$

$$SVNNCFSM\left(SA, SB\right)\in\left[0,1\right], \quad (5)$$

Definition 7 [12]. Let $SA=\left(ST_A, SI_A, SF_A\right)$ and $SB=\left(ST_B, SI_B, SF_B\right)$, the operations laws are outlined:

(1) $SA\oplus SB=\left(ST_A+ST_B-ST_A\cdot ST_B, SI_A\cdot SI_B, SF_A\cdot SF_B\right)$;

(2) $SA\otimes SB=\left(ST_A\cdot ST_B, SI_A+SI_B-SI_A\cdot SI_B, SF_A+SF_B-SF_A\cdot SF_B\right)$;

(3) $\gamma SA=\left(1-\left(1-ST_A\right)^{\gamma},\left(SI_A\right)^{\gamma},\left(SF_A\right)^{\gamma}\right), \gamma>0$;

(4) $\left(SA\right)^{\gamma}=\left(\left(ST_A\right)^{\gamma},\left(SI_A\right)^{\gamma},1-\left(1-SF_A\right)^{\gamma}\right), \gamma>0.$

The SVNNWA and SVNNWG approach are outlined.

Definition 8 [16]. If $SA_j = \left(ST_j, SI_j, SF_j\right)$, the SVNNWA operator is outlined:

$$\begin{aligned} \text{SVNNWA}_{sw}\left(SA_1, SA_2, \cdots, SA_n\right) &= \overset{n}{\underset{j=1}{\oplus}}\left(sw_j SA_j\right) \\ &= \left(1-\prod_{j=1}^{n}\left(1-ST_j\right)^{sw_j}, \prod_{j=1}^{n}\left(SI_j\right)^{sw_j}, \prod_{j=1}^{n}\left(SF_j\right)^{sw_j}\right) \end{aligned} \tag{6}$$

with weight $sw=\left(sw_1, sw_2, ..., sw_n\right)^T$, $\sum_{j=1}^{n} sw_j = 1.$

Definition 9 [16]. If $SA_j = \left(ST_j, SI_j, SF_j\right)$, the SVNNWG approach is outlined:

$$\begin{aligned} \text{SVNNWG}_{sw}\left(SA_1, SA_2, \cdots, SA_n\right) &= \overset{n}{\underset{j=1}{\otimes}}\left(SA_j\right)^{sw_j} \\ &= \left(\prod_{j=1}^{n}\left(ST_j\right)^{sw_j}, 1-\prod_{j=1}^{n}\left(1-SI_j\right)^{sw_j}, 1-\prod_{j=1}^{n}\left(1-SF_j\right)^{sw_j}\right) \end{aligned} \tag{7}$$

with weight $sw=\left(sw_1, sw_2, ..., sw_n\right)^T$, $\sum_{j=1}^{n} sw_j = 1.$

3. Methodology

This part shows the steps of the proposed methodology as shown in Figure 1.

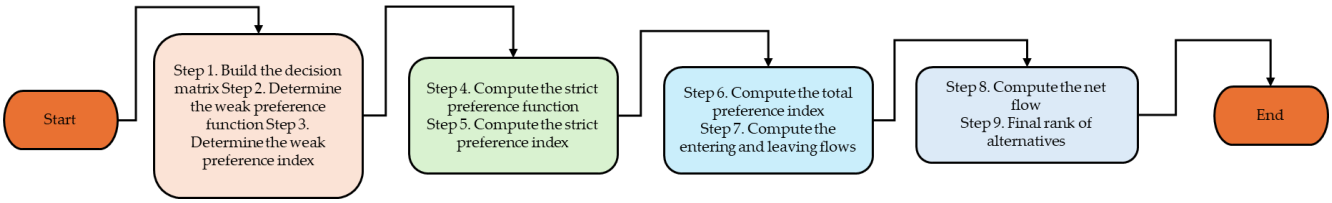
Start
Step 1. Build the decision matrix Step 2. Determine the weak preference function Step 3. Determine the weak preference index
Step 4. Compute the strict preference function Step 5. Compute the strict preference index
Step 6. Compute the total preference index Step 7. Compute the entering and leaving flows
Step 8. Compute the net flow Step 9. Final rank of alternatives
End

Figure 1. The steps of the proposed methodology.

Step 1. Build the decision matrix

$$Q = \begin{bmatrix} q_1(A_1) & \cdots & q_n(A_1) \\ \vdots & \ddots & \vdots \\ q_1(A_m) & \cdots & q_n(A_m) \end{bmatrix}; i = 1, \dots, m; j = 1, \dots, n \tag{8}$$

Step 2. Determine the weak preference function

To compute the weak preference function, we compute the pair difference between alternatives in the decision matrix as:

$$d_j(A_i, A_{i'}) = q_j(A_i) - q_j(A_{i'}) \tag{9}$$

Then we compute the value of preference function as:

$$P_j(A_i, A_{i'}) = q_j\left[d_j(A_i, A_{i'})\right] \tag{10}$$

$$0 \le P_j(A_i, A_{i'}) \le 1 \tag{11}$$

Step 3. Determine the weak preference index

The weak preference index can be computed as:

$$WP\left(P_j(A_i, A_{i'})\right) = \sum_{j=1}^{n} P_j(A_i, A_{i'}).\, w_j \,/\, \sum_{j=1}^{n} w_j \quad (12)$$

Step 4. Compute the strict preference function

The strict preference function can be computed by the positive and negative criteria as:

$$q_j(x) = \max\{q_j(x_1), q_j(x_2), \dots, q_j(x_n)\} \quad (13)$$

$$q_j(x) = \min\{q_j(x_1), q_j(x_2), \dots, q_j(x_n)\} \quad (14)$$

Step 5. Compute the strict preference index

$$SP(A_i, A_{i'}) = \sum_{j=1}^{n} P_j(A_i, A_{i'}).\, w_j \,/\, \sum_{j=1}^{n} w_j \quad (15)$$

Step 6. Compute the total preference index

$$TP(A_i, A_{i'}) = \min\{1,\}WP\left(P_j(A_i, A_{i'})\right) + SP\left(P_j(A_i, A_{i'})\right) \quad (16)$$

Step 7. Compute the entering and leaving flows

$$E^{+}(A_i) = \frac{1}{m-1}\sum_{A_i \in A} TP(A_i, A_{i'}) \quad (17)$$

$$E^{-}(A_i) = \frac{1}{m-1}\sum_{A_i \in A} TP(A_{i'}, A_i\,) \quad (18)$$

Step 8. Compute the net flow

$$E(A_i) = E^{+}(A_i) + E^{-}(A_{i'}) \quad (19)$$

Step 9. Final rank of alternatives

The alternative A_i can be better than $A_{i'}$ if

$$if\ E(A_i) > E(A_{i'}) \quad (20)$$

4. Illustrative Example

In this study, the SVNN-EXPROM technique is employed to evaluate the quality of cultural industry development in the context of the digital economy. Five high-tech development zones are assessed based on eight attributes. Three experts and decision-makers participate in the evaluation process, using SVNNs to assess the criteria and alternatives. The evaluations are then converted into crisp values and aggregated into a single decision matrix.

The criteria weights calculated using the average method, are as follows: W1 is 0.1264, W2 is 0.1426, W3 is 0.1530, W4 is 0.1124, W5 is 0.1332, W6 is 0.1046, W7 is 0.1072, and W8 is 0.1207.

The criteria used in this study are:

1. Digital Infrastructure and Technology Integration

Assesses the availability and quality of digital infrastructure (e.g., internet access, data storage, and cloud computing) and the integration of advanced technologies (e.g., AI, AR/VR, blockchain) into cultural industry operations. Internet Penetration and Speed, Adoption of Emerging Technologies, Digitalization of Cultural Content,

2. Innovation and Creative Output

Evaluates the ability of the cultural industry to generate new and innovative cultural products, content, and services that appeal to diverse audiences in the digital economy. Cultural Product Diversity, Originality of Content, Involvement in Global Cultural Trends.

3. Market Competitiveness

Measures the cultural industry's ability to compete in domestic and international markets, including its branding, export potential, and market share in the digital economy. Global Market Presence, Branding and Marketing Efficiency, Revenue Growth in Digital Channels.

4. Talent and Skills Development

Focus on the availability of skilled personnel in the cultural industry, including technical, creative, and managerial talent, and the efforts to develop these skills through training and education. Availability of Skilled Workforce, Investment in Skill Development Programs, Collaboration with Educational Institutions.

5. Cultural Heritage Preservation and Digital Adaptation

Evaluates efforts to preserve cultural heritage while adapting it for digital platforms to ensure relevance in the digital economy. Digital Archiving of Heritage, Use of Technology in Cultural Conservation, Engagement with Younger Audiences

6. Policy Support and Regulation

Assesses the role of government policies and regulations in supporting the cultural industry, such as tax incentives, copyright protection, and funding for digital initiatives. Government Funding and Subsidies, Intellectual Property Rights Protection, Policy Alignment with Digital Economy Goals.

7. Audience Engagement and Satisfaction

Measures the level of audience interaction and satisfaction with cultural products and services, focusing on user experience in digital platforms and consumer feedback mechanisms. User Experience and Accessibility, Customer Feedback and Reviews, Audience Retention Rates

8. Sustainability and Social Responsibility

Examine the cultural industry's commitment to environmental sustainability and social responsibility, including green practices and community-oriented projects. Adoption of Green Technologies, Community Engagement Initiatives, Sustainability Reporting and Compliance

The criteria and their values for SHSS

A. Digital Infrastructure and Technology Integration {sufficient, insufficient}
B. Innovation and Creative Output {sufficient, insufficient}
C. Market Competitiveness {low, medium, high}
D. Talent and Skills Development {penetrable, impenetrable}
E. Cultural Heritage Preservation and Digital Adaptation {adjustable, non-adjustable}
F. Policy Support and Regulation {appropriate, inappropriate}
G. Audience Engagement and Satisfaction {sufficient, insufficient}
H. Sustainability and Social Responsibility {bad, good}

Based on the SHSS can be suggest a set of groups for sub criteria to rank the alternatives as:

{sufficient, insufficient}, {sufficient, insufficient}, {low, medium, high}, {penetrable, impenetrable}, {adjustable, non-adjustable}, {appropriate, inappropriate}, {sufficient, insufficient}, {bad, good}

We choose the values as:

{sufficient}, {sufficient}, {medium}, {penetrable}, {non-adjustable}, {appropriate}, {sufficient, insufficient}, {bad, good}

There are four groups for SHSS or four HyperSoft sets as:

Group 1: {sufficient}, {sufficient}, {medium}, {penetrable}, {non-adjustable}, {appropriate}, {sufficient}, {bad}

Group 2: {sufficient}, {sufficient}, {medium}, {penetrable}, {non-adjustable}, {appropriate}, {sufficient}, {good}

Group 3: {sufficient}, {sufficient}, {medium}, {penetrable}, {non-adjustable}, {appropriate}, {insufficient}, {bad}

Group 4: {sufficient}, {sufficient}, {medium}, {penetrable}, {non-adjustable}, {appropriate}, {insufficient}, { good}

We can implement the proposed method under four groups as:

Based on the Group 1 Eq. (8) is used to build the decision matrix.

The opinions of the first expert as:

(0.2,0.7,0.8) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9)

(0.3,0.6,0.7) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.9,0.1,0.2)

(0.5,0.5,0.5) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.3,0.6,0.7) (0.8,0.2,0.3)

(0.6,0.4,0.5) (0.6,0.4,0.5) (0.7,0.3,0.4) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.2,0.7,0.8) (0.7,0.3,0.4)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.2,0.7,0.8) (0.3,0.6,0.7) (0.5,0.5,0.5) (0.6,0.4,0.5)

The opinions of the second expert as:

(0.6,0.4,0.5) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.6,0.4,0.5) (0.6,0.4,0.5) (0.1,0.8,0.9)

(0.9,0.1,0.2) (0.6,0.4,0.5) (0.9,0.1,0.2) (0.6,0.4,0.5) (0.7,0.3,0.4) (0.9,0.1,0.2) (0.9,0.1,0.2) (0.6,0.4,0.5)

(0.1,0.8,0.9) (0.9,0.1,0.2) (0.6,0.4,0.5) (0.9,0.1,0.2) (0.6,0.4,0.5) (0.1,0.8,0.9) (0.1,0.8,0.9) (0.9,0.1,0.2)

(0.6,0.4,0.5) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.1,0.8,0.9) (0.1,0.8,0.9) (0.1,0.8,0.9) (0.3,0.6,0.7) (0.1,0.8,0.9) (0.6,0.4,0.5)

The opinions of the third expert as:

(0.1,0.8,0.9) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.1,0.8,0.9) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9)

(0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.5,0.5,0.5) (0.1,0.8,0.9)

(0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.9,0.1,0.2)

(0.6,0.4,0.5) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.8,0.2,0.3)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.8,0.2,0.3) (0.1,0.8,0.9) (0.8,0.2,0.3) (0.3,0.6,0.7) (0.8,0.2,0.3) (0.6,0.4,0.5)

Then we determine the weak preference function for the first alternative as:

0 0 0 0 0 0 0 0

-0.377777778 0.488888889 -0.2 0.077777778 -0.355555556 -0.111111111 -0.277777778 -0.388888889

-0.155555556 0.022222222 0.322222223 -0.088888889 -0.111111111 -0.211111111 0.144444445 -0.700000001

-0.255555556 0.277777778 -0.133333333 0.011111111 -0.488888889 0.066666667 -0.311111111 -0.388888889

-0.355555556 0 0.077777778 0.433333334 0 0.077777778 -0.122222222 -0.433333334

We determined the weak preference index using Eq. (12), followed by the computation of the strict preference function using Eq. (13). Next, we calculated the strict preference index using Eq. (15) and the total preference index using Eq. (16). Subsequently, we computed the entering and leaving flows and, finally, the net flow using Eq. (19). Based on these computations, the alternatives were ranked as follows: A1 > A5 > A3 > A4 > A2.

For Group 2, the same steps were applied as in Group 1, resulting in the final rank of alternatives:
A1 > A5 > A3 > A4 > A2.
For Group 3, the process was repeated using the same methodology, yielding the same final rank:
A1 > A5 > A3 > A4 > A2.
For Group 4, the steps were again applied as in Group 1, resulting in the identical rank:
A1 > A5 > A3 > A4 > A2.
The final consolidated rank across all four groups is:
A1 > A5 > A3 > A4 > A2.
This analysis indicates that Alternative 1 (A1) is the best option, while Alternative 2 (A2) is the least favorable across all four groups.

4.1 Results and Discussion

The world is currently undergoing a profound technological revolution, with new-generation information intelligent technologies such as 5G, Internet of Things, blockchain, cloud computing, artificial intelligence, etc., bringing opportunities for economic digitization. The digital industry is emerging like mushrooms after rain, and the digital economy era has arrived. As the core driving force of economic growth in the new era, the digital economy has penetrated various fields of human production and become the commanding heights of a new round of competition in various industries. The growth of digital economy in major countries around the world is significantly higher than the growth of their domestic gross domestic product, and governments are continuing to promote development of key areas of digital economy.

The digital economy can integrate well with traditional industries and play an important role in connecting, driving and balancing the development of three major industries of the national economy. From the natural evolution of technology, digital technology will first spread from the digital industry to the industry with the highest correlation with it. The symbolic characteristics of the cultural industry have a natural commonality with digital technology.

Compared to other industries, the cultural industry can more conveniently obtain digital empowerment and integrate with it. From the development process of China's digital economy, the cultural industry not only bears the brunt, but also achieves a dual increase in economic structure and scale through digital transformation and the emergence of new formats. The natural affinity between cultural industry and the digital economy makes it the main area for countries to develop their digital economy.

The digital transformation of the cultural industry has taken solid steps in practice, but the weak theoretical research cannot be ignored, and there are obvious deficiencies in some basic cognition and internal law exploration. The quality evaluation of cultural industry development in the context of the digital economy is MAGDM.

We obtained the criteria weights using the average method. We show criterion 3 has the highest importance and criterion 6 has the lowest importance. Then we applied the SHSS with the proposed method to deal with the various values of sub criteria. Then we ranked the alternatives. We show the alternative 1 is the best and alternative 2 is the worst in four groups.

4.2. Comparative Analysis

The outlined SHSS-SVNN-EXPROM approach is always compared with SVNN-VIKOR technique, SVNN-COPRAS technique, SVNN-WASPAS approach, SVNN-TODIM technique , SVNN-TOPSIS technique and SVNN-CODAS technique. The sufficient comparative results are verified in Table 1.

Table 1. Order for different approaches

METHOD	ORDER
SVNN-VIKOR TECHNIQUE	A1>A5>A4>A3>A2
SVNN-COPRAS TECHNIQUE	A1>A3>A5>A4>A2
SVNN-WASPAS TECHNIQUE	A1>A5>A3>A4>A2
SVNN-TODIM TECHNIQUE	A1>A5>A3>A4>A2
SVNN-TOPSIS TECHNIQUE	A1>A3>A5>A4>A2
SVNN-CODAS TECHNIQUE	A1>A5>A3>A4>A2

The similarity coefficients between SVNN-VIKOR technique , SVNN-COPRAS technique, SVNN-WASPAS approach, SVNN-TODIM approach, SVNN-TOPSIS approach, SVNN-CODAS approach and SHSS-SVNN-EXPROM approach was obtained in light with WS coefficients, the derived results are outlined in Table 2.

Table 2. The WS coefficient

	SHSS-SVNN-EXPROM	*SVNN-VIKOR technique*	*SVNN-COPRAS technique*	*SVNN-WASPAS technique*	*SVNN-TODIM technique*	*SVNN-TOPSIS technique*	*SVNN-CODAS technique*
A_1	5	5	5	5	5	5	5
A_1	1	1	1	1	1	1	1
A_1	3	2	4	3	3	4	3
A_1	2	3	2	2	2	2	2
A_1	4	4	3	4	4	3	4
	-	0.9	0.9	1	1	0.9	1

The WS coefficient information corroborates the order of SHSS-SVNN-EXPROM approach is same to the order of SVNN-WASPAS technique, SVNN-TODIM, and SVNN-CODAS. And it differs from the SVNN-VIKOR, SVNN-COPRAS, SVNN-TOPSIS.

Thus, the main advantages of SHSS-SVNN-EXPROM approach are outlined: (1) the outlined SHSS-SVNN-EXPROM not only corroborated the uncertainty for MAGDM, (2) the outlined SHSS-SVNN-EXPROM conducted different behavior approach as MAGDM when these approaches are combined. (3) it can deal with various values of sub criteria.

5. Conclusion

Contemporary industrial development has the attribute of globalization, thanks in large part to the universality and uniformity of industrial technical standards. The unification of technical standards is crucial to promoting the cultural industry to realize in-depth development and a wider range of product circulation. The unification of technical standards can not only effectively reduce transaction costs and generate more network externalities, but also help build a more open and unified cultural products trading market, so that all parties are involved in the main body to obtain greater benefits. In the era of digital economy, technology is updated and iterated more rapidly. Specifically in the field of cultural industry,

there is an urgent need to build and update the production technology standards, product quality standards and safety standards of cultural products. The market scale of China's cultural industry is huge, and there is a huge demand for industrial standards. At the same time, the development of China's cultural industry is facing fierce competition with international industry giants. Only by creating various types of industrial standards suitable for the needs of industrial development and future trends, it is possible to maintain the safety of China's cultural industry and thus enhance China's cultural construction. New, effective and universal technical standards occupy the core position among all kinds of standards related to the industry, and they are also the industrial commanding heights for which all countries are competing. The quality evaluation of cultural industry development in the context of the digital economy is MAGDM. In this paper, the SHSS-SVNN-EXPROM model is formed to cope with the MAGDM. The average method is utilized to obtain the weight numbers under SVNSs. Finally, numerical examples for quality evaluation of cultural industry development in the context of the digital economy and comparative analysis is utilized to validate SHSS-SVNN-EXPROM model.

The main contributions of this study are formed: (1) the average model is formed to obtain the weight numbers; (2) the SHSS-SVNN-EXPROM model is formed SHSS to deal with various criteria under SVNNs; (3) Finally, numerical example and comparative analysis for quality evaluation of cultural industry development in the context of the digital economy is utilized to validate SHSS-SVNN-EXPROM model.

There may be some possible research limitations for quality evaluation of cultural industry development in the context of the digital economy, which could be further conducted in our future research directions: (1) It is the worthwhile research content to conduct prospect theory model for quality evaluation of cultural industry development in the context of the digital economy under SVNSs; (2) It is also worthwhile research content to conduct regret theory for quality evaluation of cultural industry development in the context of the digital economy under SVNSs environment; (3) In subsequent research contents, the application of SVNSs needs to be formed with consensus issues for quality evaluation of cultural industry development in the context of the digital economy.

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Some harmonic aggregation operators for N-valued neutrosophic trapezoidal numbers and their application to multi-criteria decision-making

İrfan DELİ[1] and Vakkas ULUÇAY[2,*]

[1]Kilis 7 Aralık University, 79000 Kilis, Turkey; irfandeli@kilis.edu.tr

[2]Kilis 7 Aralık University, 79000 Kilis, Turkey; vakkas.ulucay@kilis.edu.tr

* Correspondence: vakkas.ulucay@kilis.edu.tr

Abstract: As an extension of the both trapezoidal fuzzy numbers and neutrosophic trapezoidal numbers, the N-valued neutrosophic trapezoidal numbers, which are special neutrosophic multi-sets on subset of real numbers. Harmonic mean is a conservative average, which is widely used to aggregate central tendency data. In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data information. In this paper, we investigate a method for the situations in which the input data are expressed in neutrosophic values. Therefore, we propose two aggregations are called harmonic aggregation operators and weighted harmonic mean operators on N-valued neutrosophic trapezoidal numbers. We also proved some desired properties such as idempotency, monotonitiy, commutativity and boundedness of the developed operators. Moreover, we developed an algorithm by defining a score function under N-valued neutrosophic trapezoidal numbers to compare the N-valued neutrosophic trapezoidal numbers. Finally, we gave an illustrative example, using the proposed aggregation operators to rank the alternatives with N-valued neutrosophic trapezoidal numbers.

Keywords: Neutrosophic sets; neutrosophic multi-sets; N-valued neutrosophic trapezoidal numbers; score degrees; accuracy degrees; multi-criteria decision-making.

1. Introduction

Many different theories put forward to deal with problems involving uncertainty such as are interval mathematics, probability theory, fuzzy set theory, intuitionistic fuzzy set theory, neutrosophic set theory in our daily lives. Among these theories, current and widely applied one is the fuzzy set theory, developed by Zadeh [1], which is constructed with the help of a membership function in [0,1]. Then, Atanassov [2] proposed intuitionistic fuzzy set theory by adding a non-membership membership function to the membership function such that the sum of the values of the membership function and the non-membership function for each element of the universal set X always falls within

the range [0,1]. This limitation of the membership function and non-membership function creates modeling many difficulties for problems which contain uncertain informations. To overcome this situation Smarandache [3] presented a new set theory called neutrosophic set theory, which generalization of fuzzy set and intuitionistic fuzzy set theory which have independent truth membership function, indeterminacy membership function and membership on the interval [0,1]. We can find more study examples given about the theory in Wang et al. [4], Karaaslan [5], Khatter [6], Tamilarasi and Paulraj [7], Pramanik et al. [8].

Since multi-sets constructed by Yager [9] are an important theory in classical mathematics, Miyamoto [10] defined with help of membership series to use in uncertain structures. In addition, intuitionistic fuzzy multi-sets were defined by Rajarajeswari and Uma [11] to model some different problems by using the non-membership degree series. Finally, for some situations that could not model with multi-sets and intuitionistic fuzzy multi-sets, neutrosophic multi-sets were proposed by Ye and Ye [12] and Smarandache [13] based on the indeterminacy degree series in addition to the membership degree and non-membership degree series in intuitionistic fuzzy multi-sets. These theories have been applied to various fields over time, for example, Rajarajeswari and Uma [14] intuitionistic fuzzy multi similarity measure based on cotangent function, Rajarajeswari and Uma [15] correlation measure for intuitionistic fuzzy multi sets, Shinoj and John [16-18] on intuitionistic fuzzy multisets , Ejegwa and Awolola [19] some algebraic structures of intuitionistic fuzzy multisets, Ejegwa [20] on new operations on intuitionistic fuzzy multisets, Sebastian and Ramakrishnan [21-23] on multi-fuzzy sets, Syropoulos [24] on generalized fuzzy multisets, Broumi and Deli [25] correlation measure for neutrosophic refined sets, Chatterjee et al. [26] son single valued neutrosophic multisets, Deli [27] refined neutrosophic sets and refined neutrosophic soft sets, Deli et al. [28] on neutrosophic refined sets, Mondal et al. [29] on linguistic refined neutrosophic strategy, Sahin et al. [30,31] on neutrosophic multi-sets, Ye and Smarandache [32] on refined single-valued neutrosophic sets, Ulucay [33] are some of the studies.

Ulucay et al. [34] developed trapezoidal fuzzy multi-numbers on real numbers set R based on the repeated occurrences of any element trapezoidal fuzzy multi-numbers have studied many authors in Deli and Keleş [35] on N-valued fuzzy numbers, Ulucay et al. [36] worked a new hybrid distance-based similarity measure for refined neutrosophic sets, Ulucay et al. [37] MCDM based on intuitionistic trapezoidal fuzzy multi-numbers, Kesen and Deli [38] developed trapezoidal fuzzy multi aggregation operator based on archimedean norms, Deli and Kesen [39] defined bonferroni geometric mean operator of trapezoidal fuzzy multi numbers. To generalization the concept of neutrosophic multi-sets. Deli et al. [40] put forward the concept of N-valued neutrosophic trapezoidal numbers (NVNT-numbers). Then, Kesen and Deli [41], developed a method for NVNT-numbers under multi-criteria decision-making problems based on centroid point of NVNT-numbers. In decision- making process, aggregation operators are the most effective way to make inferences by obtaining a value from many values. Therefore, Harmonic aggregation operator studied different structures which contain uncertain except for NVNT-numbers in Shit et al. [42], Zhao et al. [43], Xu [44] Aydin et al. [45], Ulucay and Sahin [46] and Bakbak and Ulucay [47], Başer and Uluçay [48] developed energy of a neutrosophic soft set, Mohamed et al. [49] on neutrosophic approach for evaluating possible industry 5.0 enablers in consumer electronics, Mohamed et al.[50] developed MCDM approaches integrated with MEREC and RAM on single-valued neutrosophic, Uluçay and Şahin [51] on intuitionistic fuzzy soft expert graphs, Kané et al. [52] developed a new algorithm for fuzzy transportation problems with trapezoidal fuzzy numbers under fuzzy circumstances, Sezgin and Yavuz [53] defined a new type of operation for soft sets, Ghaforiyan et al. [54] developed a method on identifying and prioritizing antifragile tourism strategies in a neutrosophic environment, Adak et al. [55] developed some new operations on pythagorean fuzzy sets.

The originality of this paper is that it is the first study related to some harmonic aggregation operators for N-valued neutrosophic trapezoidal numbers. Because of the ability of harmonic operators in catching the average value of data and gives less importance to outliers we proposed harmonic mean operators for N-valued neutrosophic trapezoidal numbers. Hence, integrating harmonic mean operators with N-valued neutrosophic trapezoidal numbers provides many advantages to the solutions of multi-criteria decision making (MCDM) problems. For this, in this study we give harmonic aggregation operators and weighted harmonic mean operators on N-valued neutrosophic trapezoidal numbers with some desired properties such as idempotency, monotonicity, commutativity and boundedness of the developed operators. Finally, we gave an algorithm and illustrative example, using the proposed aggregation operators in multi-criteria decision-making problems.

2. Preliminaries

This section firstly introduces several the known definitions and propositions that would be helpful for better study of this paper.

Definition 2.1 [48] A t-norm is a function $t: [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

i. $t(0,0) = 0$ and $t(\mu_{X_1}(x),1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x),\ x \in E$

ii. If $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then

$t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}(x), \mu_{X_4}(x))$

iii. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$

iv. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x))$

Definition 2.2 [48] An s-norm is a function $s: [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

i. $s(1,1) = 1$ and $s(\mu_{X_1}(x),0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x), x \in E$

ii. if $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then

$s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$

iii. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$

iv. $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

Definition 2.3 [4] Assume that $\mathcal{H}$ is the universe. Then, a single-valued neutrosophic set (N-set) $\mathcal{L}$ in $\mathcal{H}$ defined as

$$\mathcal{L} = \{< x, \mu_{\mathcal{L}}(x)\ ,\ \eta_{\mathcal{L}}(x)\ ,\ \vartheta_{\mathcal{L}}(x) > : x \in \mathcal{H}\} \tag{1}$$

where $\mu_{\mathcal{L}}(x)\ ,\ \eta_{\mathcal{L}}(x)\ ,\ \vartheta_{\mathcal{L}}(x) \in [0,1]$ for each point x in such that $0 \leq \mu_{\mathcal{L}}(x) + \eta_{\mathcal{L}}(x) + \vartheta_{\mathcal{L}}(x) \leq 3$.

Definition 2.4 [13] Let $\mathcal{H}$ be a universe. $\mathcal{L}$ neutrosophic multi-set set $\mathcal{L}$ on $\mathcal{H}$ can be defined as follows:

$\mathcal{L}$={<x,($\mu_{\mathcal{L}}^1(x), \mu_{\mathcal{L}}^2(x), \dots, \mu_{\mathcal{L}}^P(x)), (\eta_{\mathcal{L}}^1(x), \eta_{\mathcal{L}}^2(x), \dots, \eta_{\mathcal{L}}^P(x)), (\vartheta_{\mathcal{L}}^1(x), \vartheta_{\mathcal{L}}^2(x), \dots, \vartheta_{\mathcal{L}}^P(x)) >: x \in E\}$,

where

$$\mu_{\mathcal{L}}^1(x), \mu_{\mathcal{L}}^2(x), \dots, \mu_{\mathcal{L}}^P(x), \eta_{\mathcal{L}}^1(x), \eta_{\mathcal{L}}^2(x), \dots, \eta_{\mathcal{L}}^P(x), \vartheta_{\mathcal{L}}^1(x), \vartheta_{\mathcal{L}}^2(x), \dots, \vartheta_{\mathcal{L}}^P(x): \mathcal{H} \to [0,1]$$

such that $0 \leq \sup \mu_{\mathcal{L}}^i(x) + \sup \eta_{\mathcal{L}}^i(x) + \sup \vartheta_{\mathcal{L}}^i(x) \leq 3$ (i=1,2,…,P) for any $x \in \mathcal{H}$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively.

Definition 2.5 [40] Let

$A_1 = \langle (a_1, b_1, c_1, d_1); \left(\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}\right), \left(\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}\right), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1})\rangle$ be a NVNT-number. If A_1 is not normalized NVTN-number $(a_1, b_1, c_1, d_1 \notin [0,1])$, the normalized NVTN-number of A_1, denoted by $\overline{A}_1$ is given by;

$$\begin{aligned}\overline{A}_1 = \ & \langle \left[\frac{a_1}{a_1+b_1+c_1+d_1}, \frac{b_1}{a_1+b_1+c_1+d_1}, \frac{c_1}{a_1+b_1+c_1+d_1}, \frac{d_1}{a_1+b_1+c_1+d_1}\right]; \\ & \left(\eta^1_{\overline{A}_1}, \eta^2_{\overline{A}_1}, \dots, \eta^P_{\overline{A}_1}\right), \left(\vartheta^1_{\overline{A}_1}, \vartheta^2_{\overline{A}_1}, \dots, \vartheta^P_{\overline{A}_1}\right), (\theta^1_{\overline{A}_1}, \theta^2_{\overline{A}_1}, \dots, \theta^P_{\overline{A}_1})\rangle.\end{aligned} \tag{2}$$

Definition 2.6 [40] Let

$A_1 = \langle [a_1, b_1, c_1, d_1]; \left(\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}\right), \left(\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}\right), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1})\rangle$, $A_2 = \langle [a_2, b_2, c_2, d_2]; \left(\eta^1_{A_2}, \eta^2_{A_2}, \dots, \eta^P_{A_2}\right), \left(\vartheta^1_{A_2}, \vartheta^2_{A_2}, \dots, \vartheta^P_{A_2}\right), \left(\theta^1_{A_2}, \theta^2_{A_2}, \dots, \theta^P_{A_2}\right)\rangle \in \Lambda$ and $\gamma \neq 0$ be any real number. Then,

i. $$\begin{aligned}A_1 + A_2 = \ & \langle [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \left(s\left(\eta^1_{A_1}, \eta^1_{A_2}\right), , \dots, s\left(\eta^p_{A_1}, \eta^p_{A_2}\right)\right) \\ & \left(t\left(\vartheta^1_{A_1}, \vartheta^1_{A_2}\right), \dots, t\left(\vartheta^p_{A_1}, \vartheta^p_{A_2}\right)\right), (t(\theta^1_{A_1}, \theta^1_{A_2}), \dots, t(\theta^p_{A_1}, \theta^p_{A_2}))\rangle\end{aligned} \tag{3}$$

ii. $$A_1.A_2 = \begin{cases} \langle [a_1a_2, b_1b_2, c_1c_2, d_1d_2]; \left(t\left(\eta^1_{A_1}, \eta^1_{A_2}\right), \dots, t\left(\eta^p_{A_1}, \eta^p_{A_2}\right)\right), \\ \left(s\left(\vartheta^1_{A_1}, \vartheta^1_{A_2}\right), \dots, s\left(\vartheta^p_{A_1}, \vartheta^p_{A_2}\right)\right), (s(\theta^1_{A_1}, \theta^1_{A_2}), \dots, s(\theta^p_{A_1}, \theta^p_{A_2}))\rangle, (d_1 > 0, d_2 > 0) \\ \langle [a_1d_2, b_1c_2, c_1b_2, d_1a_2]; \left(t\left(\eta^1_{A_1}, \eta^1_{A_2}\right), \dots, t\left(\eta^p_{A_1}, \eta^p_{A_2}\right)\right), \\ \left(s\left(\vartheta^1_{A_1}, \vartheta^1_{A_2}\right), \dots, s\left(\vartheta^p_{A_1}, \vartheta^p_{A_2}\right)\right), (s(\theta^1_{A_1}, \theta^1_{A_2}), \dots, s(\theta^p_{A_1}, \theta^p_{A_2}))\rangle, (d_1 < 0, d_2 > 0) \\ \langle [d_1d_2, c_1c_2, b_1b_2, a_1a_2]; \left(t\left(\eta^1_{A_1}, \eta^1_{A_2}\right), \dots, t\left(\eta^p_{A_1}, \eta^p_{A_2}\right)\right), \\ \left(s\left(\vartheta^1_{A_1}, \vartheta^1_{A_2}\right), \dots, s\left(\vartheta^p_{A_1}, \vartheta^p_{A_2}\right)\right), (s(\theta^1_{A_1}, \theta^1_{A_2}), \dots, s(\theta^p_{A_1}, \theta^p_{A_2}))\rangle, (d_1 < 0, d_2 < 0) \end{cases} \tag{4}$$

iii. $$\gamma A_1 = \langle[\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1]; (1-(1-\eta_{A_1}^1)^\gamma, \dots, 1-(1-\eta_{A_1}^p)^\gamma), \quad (5)$$

$$((\vartheta_{A_1}^1)^\gamma, \dots, (\vartheta_{A_1}^p)^\gamma)\rangle \ (\gamma \geq 0).$$

iv. $$A_1{}^\gamma = \langle[a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma]; ((\eta_{A_1}^1)^\gamma, \dots, (\eta_{A_1}^p)^\gamma), (1-(1-\vartheta_{A_1}^1)^\gamma, \dots, 1-(1-\vartheta_{A_1}^p)^\gamma)\rangle,$$

$$(\gamma \geq 0). \quad (6)$$

v. $$A_1 \leq A_2 \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \eta_{A_1}^1 \leq \eta_{A_2}^1, \eta_{A_1}^2 \leq \eta_{A_2}^2, \dots, \eta_{A_1}^P \leq \eta_{A_2}^P,$$

$$\vartheta_{A_1}^1 \geq \vartheta_{A_2}^1, \vartheta_{A_1}^2 \geq \vartheta_{A_2}^2, \dots, \vartheta_{A_1}^P \geq \vartheta_{A_2}^P, \theta_{A_1}^1 \geq \theta_{A_2}^1, \theta_{A_1}^2 \geq \theta_{A_2}^2, \dots, \theta_{A_1}^P \geq \theta_{A_2}^P. \quad (7)$$

Definition 2.7 [44] Let $x_1, x_2, x_3, \dots, x_n$ be n real numbers. Then, harmonic mean operator

$$M_{harmonic}(x_1, x_2, x_3, \dots, x_n) = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\dots+\frac{1}{x_n}} \quad (8)$$

$$= \frac{n}{\sum_{j=1}^n \frac{1}{x_j}}$$

Definition 2.8 [44] Let $x_1, x_2, x_3, \dots, x_n$ be n real numbers. Then, weighted harmonic mean operator

$$M_{weighted\ harmonic}(x_1, x_2, x_3, \dots, x_n) = \frac{n}{\frac{w_1}{x_1}+\frac{w_2}{x_2}+\frac{w_3}{x_3}+\dots+\frac{w_n}{x_n}} \quad (9)$$

$$= \frac{\boldsymbol{n}}{\sum_{\boldsymbol{j=1}}^{\boldsymbol{n}} \frac{\boldsymbol{w_j}}{\boldsymbol{x_j}}}$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is a weight vector of x_j $(j = 1,2,3, \dots, n), w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$.

3. Some weight harmonic mean operators for NVNT-numbers

Definition 3.1 Let $\mathcal{L}_r = \langle[a_r, b_r, c_r, d_r]; (\mu_{\mathcal{L}_r}^1, \mu_{\mathcal{L}_r}^2, \dots, \mu_{\mathcal{L}_r}^P), (\eta_{\mathcal{L}_r}^1, \eta_{\mathcal{L}_r}^2, \dots, \eta_{\mathcal{L}_r}^P), (\vartheta_{\mathcal{L}_r}^1, \vartheta_{\mathcal{L}_r}^2, \dots, \vartheta_{\mathcal{L}_r}^P)\rangle$ be a collection of NVNT-numbers for $(r = 1,2,3, \dots, n)$. A mapping $f_{\text{NVNTNWHM}}^w: \mathcal{L}_r{}^n \to \mathcal{L}$ is called N-

valued neutrosophic trapezoidal numbers weighted harmonic mean (NVNTNWHM) operator if it satisfies:

$$\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathcal{L}_r}} \tag{10}$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is the associated weight vector of $\mathcal{L}_r$ for $r = 1,2,3, \dots, n$ and

$$\sum_{r=1}^{n} w_r = 1.$$

Theorem 3.2 Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}]; \left(\mu_{\mathcal{L}_r}^1, \mu_{\mathcal{L}_r}^2, \dots, \mu_{\mathcal{L}_r}^{\mathrm{P}}\right), \left(\eta_{\mathcal{L}_r}^1, \eta_{\mathcal{L}_r}^2, \dots, \eta_{\mathcal{L}_r}^{\mathrm{P}}\right), \left(\vartheta_{\mathcal{L}_r}^1, \vartheta_{\mathcal{L}_r}^2, \dots, \vartheta_{\mathcal{L}_r}^{\mathrm{P}}\right) \rangle$ be a collection of NVNT-numbers for $r = 1,2,3, \dots, n$, $k = 1,2.3, \dots p$ and the associated weight vector of $\mathcal{L}_r$ is $w = (w_1, w_2, w_3, \dots, w_n)^T$ for $\sum_{r=1}^{n} w_r = 1$ then

$$\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \dots + \frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}} \right.$$
$$\left. \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right)$$
$$\left(\frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}, \frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}, \dots, \right.$$
$$\left. \frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}} \right), \left(\frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \vartheta_{\mathcal{L}_r}^1)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}, \right.$$
$$\left. \frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \vartheta_{\mathcal{L}_r}^2)^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}, \dots, \frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right) \right\rangle \tag{11}$$

Proof When n=2, then $\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2)$ is calculated as follows:

$$\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathcal{L}_r}} = \qquad \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2}}$$

$$= \frac{1}{\frac{w_1}{\langle [a_1, b_1, c_1, d_1]; (\mu_{\mathcal{L}_1}^1, \mu_{\mathcal{L}_1}^2, \dots, \mu_{\mathcal{L}_1}^P), (\eta_{\mathcal{L}_1}^1, \eta_{\mathcal{L}_1}^2, \dots, \eta_{\mathcal{L}_1}^P), (\vartheta_{\mathcal{L}_1}^1, \vartheta_{\mathcal{L}_1}^2, \dots, \vartheta_{\mathcal{L}_1}^P)\rangle} + \frac{w_2}{\langle [a_2, b_2, c_2, d_2]; (\mu_{\mathcal{L}_2}^1, \mu_{\mathcal{L}_2}^2, \dots, \mu_{\mathcal{L}_2}^P), (\eta_{\mathcal{L}_2}^1, \eta_{\mathcal{L}_2}^2, \dots, \eta_{\mathcal{L}_2}^P), (\vartheta_{\mathcal{L}_2}^1, \vartheta_{\mathcal{L}_2}^2, \dots, \vartheta_{\mathcal{L}_2}^P)\rangle}}$$

$$= \frac{1}{w_1 \frac{1}{\left\langle \left[\frac{1}{d_1}, \frac{1}{c_1}, \frac{1}{b_1}, \frac{1}{a_1}\right]; (\mu_{\mathcal{L}_1}^1, \mu_{\mathcal{L}_1}^2, \dots, \mu_{\mathcal{L}_1}^P), (\eta_{\mathcal{L}_1}^1, \eta_{\mathcal{L}_1}^2, \dots, \eta_{\mathcal{L}_1}^P), (\vartheta_{\mathcal{L}_1}^1, \vartheta_{\mathcal{L}_1}^2, \dots, \vartheta_{\mathcal{L}_1}^P)\right\rangle} + w_2 \frac{1}{\left\langle \left[\frac{1}{d_2}, \frac{1}{c_2}, \frac{1}{b_2}, \frac{1}{a_2}\right]; (\mu_{\mathcal{L}_2}^1, \mu_{\mathcal{L}_2}^2, \dots, \mu_{\mathcal{L}_2}^P), (\eta_{\mathcal{L}_2}^1, \eta_{\mathcal{L}_2}^2, \dots, \eta_{\mathcal{L}_2}^P), (\vartheta_{\mathcal{L}_2}^1, \vartheta_{\mathcal{L}_2}^2, \dots, \vartheta_{\mathcal{L}_2}^P)\right\rangle}}$$

$$= \frac{1}{\frac{1}{\left\langle \left[\frac{w_1}{d_1}, \frac{w_1}{c_1}, \frac{w_1}{b_1}, \frac{w_1}{a_1}\right]; \left(\frac{(1+\mu_{\mathcal{L}_1}^1)^{w_1} - (1-\mu_{\mathcal{L}_1}^1)^{w_1}}{(1+\mu_{\mathcal{L}_1}^1)^{w_1} + (1-\mu_{\mathcal{L}_1}^1)^{w_1}}, \frac{(1+\mu_{\mathcal{L}_1}^2)^{w_1} - (1-\mu_{\mathcal{L}_1}^2)^{w_1}}{(1+\mu_{\mathcal{L}_1}^2)^{w_1} + (1-\mu_{\mathcal{L}_1}^2)^{w_1}}\right), \left(\frac{2(\eta_{\mathcal{L}_1}^1)^{w_1}}{(2-\eta_{\mathcal{L}_1}^1)^{w_1} + (\eta_{\mathcal{L}_1}^1)^{w_1}}, \frac{2(\eta_{\mathcal{L}_1}^2)^{w_2}}{(2-\eta_{\mathcal{L}_1}^2)^{w_2} + (\eta_{\mathcal{L}_1}^2)^{w_2}}\right), \left(\frac{2(\vartheta_{\mathcal{L}_1}^1)^{w_1}}{(2-\vartheta_{\mathcal{L}_1}^1)^{w_1} + (\vartheta_{\mathcal{L}_1}^1)^{w_1}}, \frac{2(\vartheta_{\mathcal{L}_1}^2)^{w_2}}{(2-\vartheta_{\mathcal{L}_1}^2)^{w_2} + (\vartheta_{\mathcal{L}_1}^2)^{w_2}}\right)\right\rangle} + \frac{1}{\left\langle \left[\frac{w_2}{d_2}, \frac{w_2}{c_2}, \frac{w_2}{b_2}, \frac{w_2}{a_2}\right]; \left(\frac{(1+\mu_{\mathcal{L}_1}^1)^{w_1} - (1-\mu_{\mathcal{L}_1}^1)^{w_1}}{(1+\mu_{\mathcal{L}_1}^1)^{w_1} + (1-\mu_{\mathcal{L}_1}^1)^{w_1}}, \frac{(1+\mu_{\mathcal{L}_1}^2)^{w_1} - (1-\mu_{\mathcal{L}_1}^2)^{w_1}}{(1+\mu_{\mathcal{L}_1}^2)^{w_1} + (1-\mu_{\mathcal{L}_1}^2)^{w_1}}\right), \left(\frac{2(\eta_{\mathcal{L}_1}^1)^{w_1}}{(2-\eta_{\mathcal{L}_1}^1)^{w_1} + (\eta_{\mathcal{L}_1}^1)^{w_1}}, \frac{2(\eta_{\mathcal{L}_1}^2)^{w_2}}{(2-\eta_{\mathcal{L}_1}^2)^{w_2} + (\eta_{\mathcal{L}_1}^2)^{w_2}}\right), \left(\frac{2(\vartheta_{\mathcal{L}_1}^1)^{w_1}}{(2-\vartheta_{\mathcal{L}_1}^1)^{w_1} + (\vartheta_{\mathcal{L}_1}^1)^{w_1}}, \frac{2(\vartheta_{\mathcal{L}_1}^2)^{w_2}}{(2-\vartheta_{\mathcal{L}_1}^2)^{w_2} + (\vartheta_{\mathcal{L}_1}^2)^{w_2}}\right)\right\rangle}}$$

$$= 1 \Big/ \left\langle \left[\sum_{r=1}^{2} \frac{w_r}{d_r}, \sum_{r=1}^{2} \frac{w_r}{c_r}, \sum_{r=1}^{2} \frac{w_r}{b_r}, \sum_{r=1}^{2} \frac{w_r}{a_r}\right]; \left(\frac{\prod_{r=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{w_r} - \prod_{r=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{w_r}}\right.\right.$$
$$\left. \frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^2)^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}\right), \left(\frac{2\prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{2}(2-\eta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^1)^{w_r}}\right.$$
$$\left. \frac{2\prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{2}(2-\eta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^2)^{w_r}}\right), \left(\frac{2\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{2}(2-\vartheta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^1)^{w_r}},\right.$$
$$\left.\left. \frac{2\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{2}(2-\vartheta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}\right)\right\rangle$$

$$= 1 \Big/ \left\langle \left[\frac{1}{\sum_{r=1}^{2} \frac{w_r}{a_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{b_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{c_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{d_r}}\right]; \left(\frac{\prod_{r=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{w_r} - \prod_{r=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{w_r}}\right.\right.$$
$$\left. \frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^2)^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}\right), \left(\frac{2\prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{2}(2-\eta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^1)^{w_r}}\right.$$
$$\left. \frac{2\prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{2}(2-\eta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{2}(\eta_{\mathcal{L}_r}^2)^{w_r}}\right), \left(\frac{2\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{2}(2-\vartheta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^1)^{w_r}},\right.$$
$$\left.\left. \frac{2\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{2}(2-\vartheta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}\right)\right\rangle$$

Suppose that Equation 12 holds for $n = k$, i.e.,

$$\text{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_k) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \dots + \frac{w_k}{\mathcal{L}_k}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{k} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}} \right.$$
$$\left. \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \ldots, \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right)$$
$$\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}, \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \ldots, \right.$$
$$\left. \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}} \right), \left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}, \right.$$
$$\left. \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \ldots, \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right) \right\rangle$$

For $n = k + 1$, using above expression and operational laws, we have

$$\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_k, \mathcal{L}_{k+1}) =$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{k} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}} \right.$$
$$\left. \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \ldots, \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right)$$
$$\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}, \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \ldots, \right.$$
$$\left. \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}} \right), \left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}, \right.$$
$$\left. \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \ldots, \frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right) \right\rangle +$$

$$= \left\langle \left[\frac{1}{\frac{w_{k+1}}{\mathrm{a}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{b}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{c}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{d}_{k+1}}} \right]; \left(\frac{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}} - (1-\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}} + (1-\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}} \right.$$
$$\left. \frac{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}} - (1-\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}} + (1-\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}, \ldots, \frac{(1+\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}} - (1-\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}{(1+\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}} + (1-\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}} \right)$$
$$\left(\frac{2(\eta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}{(2-\eta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}} + (\eta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}, \frac{2(\eta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}{(2-\eta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}} + (\eta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}, \ldots, \right.$$
$$\left. \frac{2(\eta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}{(2-\eta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}} + (\eta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}} \right), \left(\frac{2(\vartheta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}} + (\vartheta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}, \right.$$
$$\left. \frac{2(\vartheta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}} + (\vartheta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}, \ldots, \frac{2(\vartheta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}} + (\vartheta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}} \right) \right\rangle$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{a_r}}, \frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{b_r}}, \frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{c_r}}, \frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{d_r}} \right]; \left(\frac{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^1)^{w_r} - \prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^1)^{w_r}} \right.$$

$$\left. \frac{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^2)^{w_r} - \prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}, \ldots, \frac{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^p)^{w_r} - \prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^p)^{w_r}}{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^p)^{w_r} + \prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^p)^{w_r}} \right)$$

$$\left(\frac{2\prod_{r=1}^{k+1}(\eta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{k+1}(2-\eta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{k+1}(\eta_{\mathcal{L}_r}^1)^{w_r}}, \frac{2\prod_{r=1}^{k+1}(\eta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{k+1}(2-\eta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{k+1}(\eta_{\mathcal{L}_r}^2)^{w_r}}, \ldots, \right.$$

$$\left. \frac{2\prod_{r=1}^{k+1}(\eta_{\mathcal{L}_r}^p)^{w_r}}{\prod_{r=1}^{k+1}(2-\eta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{k+1}(\eta_{\mathcal{L}_r}^2)^{w_r}} \right), \left(\frac{2\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{k+1}(2-\vartheta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^1)^{w_r}}, \right.$$

$$\left. \frac{2\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{k+1}(2-\vartheta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}, \ldots, \frac{2\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^p)^{w_r}}{\prod_{r=1}^{k+1}(2-\vartheta_{\mathcal{L}_r}^p)^{w_r} + \prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^p)^{w_r}} \right) \right\rangle$$

So, the proof is complete.

Theorem 3.3 (Idempotency)

Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; (\mu_{\mathcal{L}_r}^1, \mu_{\mathcal{L}_r}^2, \ldots, \mu_{\mathcal{L}_r}^p), (\eta_{\mathcal{L}_r}^1, \eta_{\mathcal{L}_r}^2, \ldots, \eta_{\mathcal{L}_r}^p), (\vartheta_{\mathcal{L}_r}^1, \vartheta_{\mathcal{L}_r}^2, \ldots, \vartheta_{\mathcal{L}_r}^p) \rangle$ be a collection of NVNT-numbers for $r = 1,2,3, \ldots, n$. If $\mathcal{L}_n = \mathcal{L}$ for all r that is all are identical then,

$$\text{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_n) = \mathcal{L}. \tag{12}$$

Proof We know that

$$\text{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \cdots + \frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{a_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{b_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{c_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{d_r}} \right]; \left(\frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^1)^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^1)^{w_r}} \right.$$

$$\left. \frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^2)^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^2)^{w_r}}, \ldots, \frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^p)^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^p)^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^p)^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^p)^{w_r}} \right)$$

$$\left(\frac{2\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{n}(2-\eta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^1)^{w_r}}, \frac{2\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{n}(2-\eta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^2)^{w_r}}, \ldots, \right.$$

$$\left. \frac{2\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^p)^{w_r}}{\prod_{r=1}^{n}(2-\eta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^2)^{w_r}} \right), \left(\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^1)^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^1)^{w_r} + \prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^1)^{w_r}}, \right.$$

$$\left. \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^2)^{w_r} + \prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^2)^{w_r}}, \ldots, \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^p)^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^p)^{w_r} + \prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^p)^{w_r}} \right) \right\rangle$$

$$= \left\langle \left[\frac{1}{\frac{\sum_{r=1}^{n} w_r}{a}}, \frac{1}{\frac{\sum_{r=1}^{n} w_r}{b}}, \frac{1}{\frac{\sum_{r=1}^{n} w_r}{c}}, \frac{1}{\frac{\sum_{r=1}^{n} w_r}{d}} \right]; \left(\frac{(1+\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n} w_r} - (1-\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n} w_r}}{(1+\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n} w_r} + (1-\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n} w_r}} \right.$$

$$\left. \frac{(1+\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n} w_r} - (1-\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n} w_r}}{(1+\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n} w_r} + (1-\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n} w_r}}, \ldots, \frac{(1+\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n} w_r} - (1-\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n} w_r}}{(1+\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n} w_r} + (1-\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n} w_r}} \right)$$

$$\left(\frac{2(\eta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}}{(2-\eta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}+(\eta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}},\frac{2(\eta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}}{(2-\eta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}+(\eta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}},\ldots,\right.$$
$$\left.\frac{2(\eta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}{(2-\eta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}+(\eta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}\right),\left(\frac{2(\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}}{(2-\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}+(\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}},\right.$$
$$\left.\frac{2(\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}}{(2-\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}+(\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}},\ldots,\frac{2(\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}{(2-\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}+(\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}\right)\Bigg\rangle$$

$$=\left\langle\left[\frac{1}{\frac{1}{a}},\frac{1}{\frac{1}{b}},\frac{1}{\frac{1}{c}},\frac{1}{\frac{1}{d}}\right];\left(\frac{(1+\mu_{\mathcal{L}}^{1})-(1-\mu_{\mathcal{L}}^{1})}{(1+\mu_{\mathcal{L}}^{1})+(1-\mu_{\mathcal{L}}^{1})},\frac{(1+\mu_{\mathcal{L}}^{2})-(1-\mu_{\mathcal{L}}^{2})}{(1+\mu_{\mathcal{L}}^{2})+(1-\mu_{\mathcal{L}}^{2})},\cdots,\frac{(1+\mu_{\mathcal{L}}^{p})-(1-\mu_{\mathcal{L}}^{p})}{(1+\mu_{\mathcal{L}}^{p})+(1-\mu_{\mathcal{L}}^{p})}\right.\right.$$
$$\left(\frac{2(\eta_{\mathcal{L}}^{1})}{(2-\eta_{\mathcal{L}}^{1})+(\eta_{\mathcal{L}}^{1})},\frac{2(\eta_{\mathcal{L}}^{2})}{(2-\eta_{\mathcal{L}}^{2})+(\eta_{\mathcal{L}}^{2})},\ldots,\frac{2(\eta_{\mathcal{L}}^{p})}{(2-\eta_{\mathcal{L}}^{p})+(\eta_{\mathcal{L}}^{p})}\right.$$
$$\left.\left.\left.\left(\frac{2(\vartheta_{\mathcal{L}}^{1})}{(2-\vartheta_{\mathcal{L}}^{1})+(\vartheta_{\mathcal{L}}^{1})},\frac{2(\vartheta_{\mathcal{L}}^{2})}{(2-\vartheta_{\mathcal{L}}^{2})+(\vartheta_{\mathcal{L}}^{2})},\ldots,\frac{2(\vartheta_{\mathcal{L}}^{p})}{(2-\vartheta_{\mathcal{L}}^{p})+(\vartheta_{\mathcal{L}}^{p})}\right)\right\rangle=\mathcal{L}.\right.$$

Theorem 3.4 (Monotoniticy property):

Let $\mathcal{L}_r=\langle[a_r,b_r,c_r,d_r];(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\ldots,\mu_{\mathcal{L}_r}^{p}),(\eta_{\mathcal{L}_r}^{1},\eta_{\mathcal{L}_r}^{2},\ldots,\eta_{\mathcal{L}_r}^{p}),(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\ldots,\vartheta_{\mathcal{L}_r}^{p})\rangle$ and $\mathcal{L}_r'=\langle[a_r',b_r',c_r',d_r'];\left((\mu_{\mathcal{L}_r'})^{1},(\mu_{\mathcal{L}_r'})^{2},\ldots,(\mu_{\mathcal{L}_r'})^{p}\right),\left((\eta_{\mathcal{L}_r'})^{1},(\eta_{\mathcal{L}_r'})^{2},\ldots,(\eta_{\mathcal{L}_r'})^{p}\right),\left((\vartheta_{\mathcal{L}_r'})^{1},(\vartheta_{\mathcal{L}_r'})^{2},\ldots,(\vartheta_{\mathcal{L}_r'})^{p}\right)\rangle$ be two collections of NVNT-numbers. If $a_r\le a_r'$, $b_r\le b_r'$, $c_r\le c_r'$, $d_r\le d_r'$, $\mu_{\mathcal{L}_r}^{1}\le(\mu_{\mathcal{L}_r'})^{1}$, $\mu_{\mathcal{L}_r}^{2}\le(\mu_{\mathcal{L}_r'})^{2},\ldots,\mu_{\mathcal{L}_r}^{p}\le(\mu_{\mathcal{L}_r'})^{p},\eta_{\mathcal{L}_r}^{1}\ge(\eta_{\mathcal{L}_r'})^{1},\eta_{\mathcal{L}_r}^{2}\ge(\eta_{\mathcal{L}_r'})^{2},\ldots,\eta_{\mathcal{L}_r}^{p}\ge(\eta_{\mathcal{L}_r'})^{p}$ and $\vartheta_{\mathcal{L}_r}^{1}\ge(\vartheta_{\mathcal{L}_r'})^{1},\vartheta_{\mathcal{L}_r}^{2}\ge(\vartheta_{\mathcal{L}_r'})^{2},\ldots,\vartheta_{\mathcal{L}_r}^{p}\ge(\vartheta_{\mathcal{L}_r'})^{p}$ then

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\ldots,\mathcal{L}_n)\le\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\ldots,\mathcal{L}_n'). \tag{13}$$

Proof. Since $a_r\le a_r'$ and $\varphi_r\ge 0$ for all r.

$$\frac{1}{a_r}\ge\frac{1}{a_r'}\Rightarrow\frac{\varphi_r}{a_r}\ge\frac{\varphi_r}{a_r'}\Rightarrow\sum_{r=1}^{n}\frac{\varphi_r}{a_r}\ge\sum_{r=1}^{n}\frac{\varphi_r}{a_r'}\Rightarrow\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{a_r}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{a_r'}}$$

the other calculations are calculated as follows:

$$\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{b_r}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{b_r'}},\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{c_r}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{c_r'}},\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{d_r}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_r}{d_r'}}$$

$$\mu_{\mathcal{L}_r}^{1}\le(\mu_{\mathcal{L}_r'})^{1}\Rightarrow-\mu_{\mathcal{L}_r}^{1}\ge-(\mu_{\mathcal{L}_r'})^{1}\Rightarrow(1-\mu_{\mathcal{L}_r}^{1})\ge\left(1-(\mu_{\mathcal{L}_r'})^{1}\right)\Rightarrow(1-\mu_{\mathcal{L}_r}^{1})^{\varphi_r}\ge\left(1-(\mu_{\mathcal{L}_r'})^{1}\right)^{\varphi_r}$$

$$\Rightarrow \prod_{r=1}^{n}\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}} \geq \prod_{r=1}^{n}\left(1-\left(\mu_{\mathcal{L}_r}^{\prime}\right)^{1}\right)^{\varphi_{\mathrm{r}}} \Rightarrow 1-\prod_{r=1}^{n}\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}} \leq 1-\prod_{r=1}^{n}\left(1-\left(\mu_{\mathcal{L}_r}^{\prime}\right)^{1}\right)^{\varphi_{\mathrm{r}}},$$

similarly

$$\eta_{\mathcal{L}_r}^{1} \geq \left(\eta_{\mathcal{L}_r}^{\prime}\right)^{1} \Rightarrow \left(\eta_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}} \geq \left(\left(\eta_{\mathcal{L}_r}^{\prime}\right)^{1}\right)^{\varphi_{\mathrm{r}}} \Rightarrow \prod_{r=1}^{n}\left(\eta_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}} \geq \prod_{r=1}^{n}\left(\left(\eta_{\mathcal{L}_r}^{\prime}\right)^{1}\right)^{\varphi_{\mathrm{r}}}$$

and

$$\vartheta_{\mathcal{L}_r}^{1} \geq \left(\vartheta_{\mathcal{L}_r}^{\prime}\right)^{1} \Rightarrow \left(\vartheta_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}} \geq \left(\left(\vartheta_{\mathcal{L}_r}^{\prime}\right)^{1}\right)^{\varphi_{\mathrm{r}}} \Rightarrow \prod_{r=1}^{n}\left(\vartheta_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}} \geq \prod_{r=1}^{n}\left(\left(\vartheta_{\mathcal{L}_r}^{\prime}\right)^{1}\right)^{\varphi_{\mathrm{r}}}$$

therefore

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\ldots,\mathcal{L}_n) \leq \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\ldots,\mathcal{L}_n').$$

Theorem 3.5 (Commutativity Property):

Let $\mathcal{L}_r = \langle [\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];\left(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\ldots,\mu_{\mathcal{L}_r}^{\mathrm{P}}\right),\left(\eta_{\mathcal{L}_r}^{1},\eta_{\mathcal{L}_r}^{2},\ldots,\eta_{\mathcal{L}_r}^{\mathrm{P}}\right),\left(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\ldots,\vartheta_{\mathcal{L}_r}^{\mathrm{P}}\right)\rangle$

be a collection of positive NVNT-numbers and $w=(w_1,w_2,w_3,\ldots,w_n)^T$ be an associated weight vector where $w_r \in [0,1]$, $\sum_{r=1}^{n} w_r = 1$.

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\ldots,\mathcal{L}_n) = \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\ldots,\mathcal{L}_n'). \tag{14}$$

where $\mathcal{L}_n'$ is any permutation of $\mathcal{L}_n$ for $r=1,2,3,\ldots,n$.

Proof. We get from Equation (11). Since $\mathcal{L}_n'$ is any permutation of $\mathcal{L}_n$. Therefore

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\ldots,\mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\cdots+\frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}},\ldots,\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}}\right)$$

$$\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\eta_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}},\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}},\ldots,\right.$$

$$\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}}\right),\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w}_\mathrm{r}}},\right.$$

$$\left.\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w}_\mathrm{r}}},\ldots,\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_\mathrm{r}}}\right)\right\rangle$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{a_r'}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{b_r'}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{c_r'}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{d_r'}} \right]; \left(\frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{1})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{1})^{w_r}} \right.$$

$$\left. \frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{2})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{2})^{w_r}}, \ldots, \frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{p})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{p})^{w_r}} \right)$$

$$\left(\frac{2\prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-(\eta')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{1})^{w_r}}, \frac{2\prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-(\eta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{2})^{w_r}}, \ldots, \right.$$

$$\left. \frac{2\prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-(\eta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{2})^{w_r}} \right), \left(\frac{2\prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{1})^{w_r}}, \right.$$

$$\left. \frac{2\prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{2})^{w_r}}, \ldots, \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{p})^{w_r}} \right) \right\rangle$$

$$= \frac{1}{\frac{w_1}{\mathcal{L}_1'} + \frac{w_2}{\mathcal{L}_2'} + \cdots + \frac{w_n}{\mathcal{L}_n'}}$$

$$= \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1', \mathcal{L}_2', \mathcal{L}_3', \ldots, \mathcal{L}_n').$$

Hence the proof completed.

Theorem 3.6 (Boundedness Property):

Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; (\mu_{\mathcal{L}_r}^{1}, \mu_{\mathcal{L}_r}^{2}, \ldots, \mu_{\mathcal{L}_r}^{p}), (\eta_{\mathcal{L}_r}^{1}, \eta_{\mathcal{L}_r}^{2}, \ldots, \eta_{\mathcal{L}_r}^{p}), (\vartheta_{\mathcal{L}_r}^{1}, \vartheta_{\mathcal{L}_r}^{2}, \ldots, \vartheta_{\mathcal{L}_r}^{p}) \rangle$

be a collection of positive NVNT-numbers and let,

$$\mathcal{L}_r^{+} = \left\langle \left[\max_r\{a_r\}, \max_r\{b_r\}, \max_r\{c_r\}, \max_r\{d_r\} \right]; \left(\max_r\{\mu_{\mathcal{L}_r}^{1}\}, \max_r\{\mu_{\mathcal{L}_r}^{2}\}, \ldots, \max_r\{\mu_{\mathcal{L}_r}^{p}\} \right), \right.$$

$$\left. \left(\min_r\{\eta_{\mathcal{L}_r}^{1}\}, \min_r\{\eta_{\mathcal{L}_r}^{2}\}, \ldots, \min_r\{\eta_{\mathcal{L}_r}^{p}\} \right), \left(\min_r\{\vartheta_{\mathcal{L}_r}^{1}\}, \min_r\{\vartheta_{\mathcal{L}_r}^{2}\}, \ldots, \min_r\{\vartheta_{\mathcal{L}_r}^{p}\} \right) \right\rangle$$

$$\mathcal{L}_r^{-} = \left\langle \left[\min_r\{a_r\}, \min_r\{b_r\}, \min_r\{c_r\}, \min_r\{d_r\} \right]; \left(\min_r\{\mu_{\mathcal{L}_r}^{1}\}, \min_r\{\mu_{\mathcal{L}_r}^{2}\}, \ldots, \min_r\{\mu_{\mathcal{L}_r}^{p}\} \right), \right.$$

$$\left. \left(\max_r\{\eta_{\mathcal{L}_r}^{1}\}, \max_r\{\eta_{\mathcal{L}_r}^{2}\}, \ldots, \max_r\{\eta_{\mathcal{L}_r}^{p}\} \right), \left(\max_r\{\vartheta_{\mathcal{L}_r}^{1}\}, \max_r\{\vartheta_{\mathcal{L}_r}^{2}\}, \ldots, \max_r\{\vartheta_{\mathcal{L}_r}^{p}\} \right) \right\rangle$$

Then,

$$\mathcal{L}_r^{-} \leq \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_n) \leq \mathcal{L}_r^{+}. \tag{15}$$

Proof. Since $\min_r\{a_r\} \leq a_r \leq \max_r\{a_r\}, \forall r$

$$\frac{\varphi_r}{\min_r\{a_r\}} \geq \frac{\varphi_r}{a_r} \geq \frac{\varphi_r}{\max_r\{a_r\}} \Rightarrow \sum_{r=1}^{n} \frac{\varphi_r}{\min_r\{a_r\}} \geq \sum_{r=1}^{n} \frac{\varphi_r}{a_r} \geq \sum_{r=1}^{n} \frac{\varphi_r}{\max_r\{a_r\}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\min_r\{a_r\}}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{a_r}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\max_r\{a_r\}}}$$

In the same way,

$$\Rightarrow \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\min_r\{b_r\}}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{b_r}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\max_r\{b_r\}}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\min_r\{c_r\}}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{c_r}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\max_r\{c_r\}}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\min_r\{d_r\}}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{d_r}} \leq \frac{1}{\sum_{r=1}^{n} \frac{\varphi_r}{\max_r\{d_r\}}}$$

and,

$$\min_r\{\mu_r\} \leq \mu_r \leq \max_r\{\mu_r\} \Rightarrow \left(1 - \min_r\{\mu_r\}\right) \geq (1 - \mu_r) \geq \left(1 - \max_r\{\mu_r\}\right)$$

$$\Rightarrow \left(1 - \min_r\{\mu_r\}\right)^{\varphi_r} \geq (1 - \mu_r)^{\varphi_r} \geq \left(1 - \max_r\{\mu_r\}\right)^{\varphi_r}, \varphi_r \geq 0, \forall r.$$

$$= \prod_{r=1}^{n} \left(1 - \min_r\{\mu_r\}\right)^{\varphi_r} \geq \prod_{r=1}^{n} (1 - \mu_r)^{\varphi_r} \geq \prod_{r=1}^{n} \left(1 - \max_r\{\mu_r\}\right)^{\varphi_r}$$

$$= 1 - \prod_{r=1}^{n} \left(1 - \min_r\{\mu_r\}\right)^{\varphi_r} \leq 1 - \prod_{r=1}^{n} (1 - \mu_r)^{\varphi_r} \leq 1 - \prod_{r=1}^{n} \left(1 - \max_r\{\mu_r\}\right)^{\varphi_r}$$

Again,

$$\min_r\{\eta_r\} \leq \eta_r \leq \max_r\{\eta_r\} \Rightarrow \left(\min_r\{\eta_r\}\right)^{\varphi_r} \leq (\eta_r)^{\varphi_r} \leq \left(\max_r\{\eta_r\}\right)^{\varphi_r}, \varphi_r \geq 0, \forall r$$

$$= \prod_{r=1}^{n} \left(\min_r\{\eta_r\}\right)^{\varphi_r} \leq \prod_{r=1}^{n} (\eta_r)^{\varphi_r} \leq \prod_{r=1}^{n} \left(\max_r\{\eta_r\}\right)^{\varphi_r}$$

In the same way,

$$\min_r\{\vartheta_r\} \leq \vartheta_r \leq \max_r\{\vartheta_r\} \Rightarrow \left(\min_r\{\vartheta_r\}\right)^{\varphi_r} \leq (\vartheta_r)^{\varphi_r} \leq \left(\max_r\{\vartheta_r\}\right)^{\varphi_r}, \varphi_r \geq 0, \forall r$$

$$= \prod_{r=1}^{n} \left(\min_{r}\{\vartheta_r\}\right)^{\varphi_r} \leq \prod_{r=1}^{n} (\vartheta_r)^{\varphi_r} \leq \prod_{r=1}^{n} \left(\max_{r}\{\vartheta_r\}\right)^{\varphi_r}.$$

Definition 3.2 Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; (\mu_{\mathcal{L}_r}^1, \mu_{\mathcal{L}_r}^2, \dots, \mu_{\mathcal{L}_r}^P), (\eta_{\mathcal{L}_r}^1, \eta_{\mathcal{L}_r}^2, \dots, \eta_{\mathcal{L}_r}^P), (\vartheta_{\mathcal{L}_r}^1, \vartheta_{\mathcal{L}_r}^2, \dots, \vartheta_{\mathcal{L}_r}^P) \rangle$ be a collection of positive NVNT-number, then

$$S(\mathcal{L}_r) = \frac{1}{4p}[a+b+c+d] \times \left(2p + \sum_{r=1}^{p} \mu_{\mathcal{L}_r}^P - \sum_{r=1}^{p} \eta_{\mathcal{L}_r}^P - \sum_{r=1}^{p} \vartheta_{\mathcal{L}_r}^P\right)$$

and

$$A(\mathcal{L}_r) = \frac{1}{4p}[a+b+c+d] \times \left(2p + \sum_{r=1}^{p} \mu_{\mathcal{L}_r}^P - \sum_{r=1}^{p} \eta_{\mathcal{L}_r}^P + \sum_{r=1}^{p} \vartheta_{\mathcal{L}_r}^P\right)$$

is called the score and accuracy degrees of $\mathcal{L}_r$, respectively.

Example 3.2 : Let $\mathcal{L} = \langle [2,3,5,7]; (0.2,0.5,0.6,0.8), (0.3,0.4,0.5,0.1), (0.2,0.3,0.4,0.8) \rangle$ be NVNT-number then,

$$S(\mathcal{L}) = \frac{1}{4.4}[2+3+5+7] \times \big(8 + (0.2+0.5+0.6+0.8) - (1.3) - (1.7)\big) = 7.54$$

$$A(\mathcal{L}) = \frac{1}{4.4}[2+3+5+7] \times \big(8 + (0.2+0.5+0.6+0.8) - (1.3) + (1.7)\big) = 11.16.$$

Definition 3.4 Let $\mathcal{L}_r^1$ and $\mathcal{L}_r^2$ be two NVNT-numbers;

a. If $S(\mathcal{L}_r^1) < S(\mathcal{L}_r^2)$, then $\mathcal{L}_r^1$ is smaller than $\mathcal{L}_r^2$, denoted by $\mathcal{L}_r^1 < \mathcal{L}_r^2$.

b. If $S(\mathcal{L}_r^1) = S(\mathcal{L}_r^2)$;

   i. If $A(\mathcal{L}_r^1) < A(\mathcal{L}_r^2)$, then $\mathcal{L}_r^1$ is smaller than $\mathcal{L}_r^2$, denoted by $\mathcal{L}_r^1 < \mathcal{L}_r^2$.

   ii. If $A(\mathcal{L}_r^1) = A(\mathcal{L}_r^2)$, then $\mathcal{L}_r^1$ and $\mathcal{L}_r^2$ are the same, denoted by $\mathcal{L}_r^1 = \mathcal{L}_r^2$.

4. An algorithm for proposed work

In this section, we shall present a multi-criteria decision-making problem with normalized NVNT-numbers under neutrosophic information using NVNTNWHM operator.

Assume that $K = \{K_1, K_2, \dots, K_m\}$ be the set of altenatives and $G = \{g_1, g_2, \dots, g_n\}$ be the set of criterias. In Deli et al. (2021), the normalized NVNT-numbers decision matrix is given as;

$$(K_{kj})_{mxn} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ K_{m1} & K_{m2} & \cdots & K_{mn} \end{pmatrix}$$

such that

$K_{kj} = \langle [a_{kj}, b_{kj}, c_{kj}, d_{kj}], \left(\eta_{kj}^1, \eta_{kj}^2, \eta_{kj}^3, \dots, \eta_{kj}^p\right), \left(\vartheta_{kj}^1, \vartheta_{kj}^2, \vartheta_{kj}^3, \dots, \vartheta_{kj}^p\right), \left(\theta_{kj}^1, \theta_{kj}^2, \theta_{kj}^3, \dots, \theta_{kj}^p\right) \rangle$, (k=1,2,…,m) and (j=1,2,…,n).

It is carried out the following algorithm to get best choice:

Algorithm:

Step 1: Identify and determine the criterias and alternatives and then construct decision matrices,

$$(K_{kj})_{mxn}, \text{ (k=1,2,…,m; j=1,2,…,n).}$$

Step 2: Get preferable for $K_1, K_2, \dots, K_m$ based on V_i $(i = 1,2,3, \dots, m)$ to aggregate the normalized NVNT-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ as;

$$V_i = \text{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n).$$

Step 3: Calculate score value whose formula is given in Definition 3.2 for each V_i to rank alternatives.

Step 4: Rank all score value of V_i according to descending order.

For more convenience, we have illustrated the proposed algorithm by Flowcard given in Figure-1.

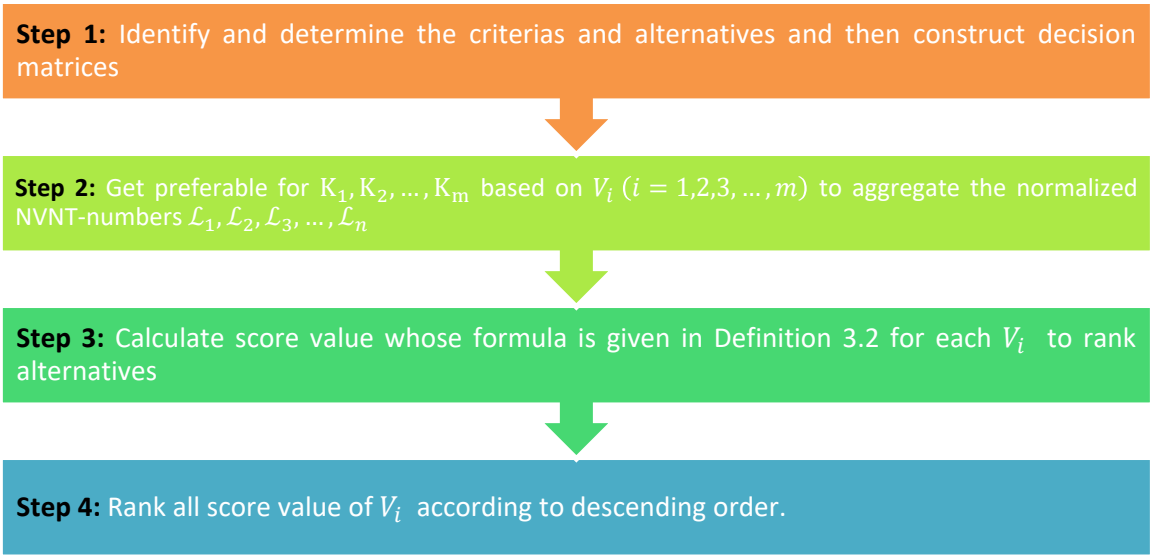
Step 1: Identify and determine the criterias and alternatives and then construct decision matrices
Step 2: Get preferable for K1, K2, …, Km based on Vi (i = 1,2,3, …, m) to aggregate the normalized NVNT-numbers ℒ1, ℒ2, ℒ3, …, ℒn
Step 3: Calculate score value whose formula is given in Definition 3.2 for each Vi to rank alternatives
Step 4: Rank all score value of Vi according to descending order.

Figure-1: Flowcard for Proposed Algorithm

5. Application of the proposed method

In this section, an explanatory example is given to view the strength of the presented work. A firewall is a computer network security system that restricts internet traffic into, out of, or inside a private network. This software or dedicated hardware-software unit works by selectively blocking or allowing data packets. It generally aims to help prevent malicious activity and prevent anyone inside or outside a private network from getting involved in unauthorized web activities.

Example 5.1 Suppose Kilis 7 Aralık University wants to select the best firewall software for the security and safety of their information. Let there are initially three firewall software that is to be considered and set of these firewall software is given as: $K = \{k_1 = \text{Software Firewall Systems}, k_2 = \text{Hardware Firewall Systems}, k_3 = \text{Hybrid Firewalls}, k_4 = \text{Circuit Level Firewalls}, k_5 = \text{Static Packet Filter Firewalls}\}$ and according to three criteria determined $G = \{g_1 = \text{full protection}, g_2 = \text{price}, g_3 = \text{usefulness}\}$. Thent we try to choose and rank all alternatives K_k for all k=1,2,…,5 by using the following algorithm.

Algorithm:

Step 1: The evaluation matrix$(K_{kj})_{5x3}$ is given by an expert as;

$$(K_{kj})_{5x3} = \begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{matrix} \begin{pmatrix} \langle[0.32,0.44,0.51,0.69]; (0.3,0.5,0.7,0.8), (0.1,0.3,0.2,0.3), (0.6,0.3,0.5,0.6)\rangle \\ \langle[0.23,0.25,0.41,0.45]; (0.4,0.2,0.3,0.5), (0.2,0.5,0.7,0.6), (0.7,0.5,0.6,0.8)\rangle \\ \langle[0.66,0.72,0.79,0.83]; (0.7,0.6,0.4,0.8), (0.5,0.5,0.5,0.6), (0.4,0.3,0.4,0.5)\rangle \\ \langle[0.52,0.63,0.76,0.91]; (0.8,0.7,0.5,0.6), (0.4,0.3,0.7,0.5), (0.1,0.5,0.7,0.7)\rangle \\ \langle[0.13,0.35,0.41,0.58]; (0.7,0.8,0.9,0.9), (0.2,0.7,0.5,0.6), (0.1,0.7,0.8,0.4)\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle[0.28,0.32,0.38,0.43]; (0.5,0.3,0.4,0.6), (0.2,0.1,0.5,0.4), (0.4,0.6,0.5,0.7)\rangle \\ \langle[0.12,0.15,0.18,0.23]; (0.3,0.7,0.9,0.9), (0.1,0.2,0.3,0.7), (0.3,0.4,0.7,0.5)\rangle \\ \langle[0.65,0.66,0.72,0.75]; (0.6,0.8,0.9,0.8), (0.2,0.5,0.4,0.3), (0.2,0.3,0.6,0.6)\rangle \\ \langle[0.08,0.15,0.27,0.37]; (0.3,0.9,0.8,0.4), (0.8,0.7,0.6,0.5), (0.1,0.1,0.4,0.3)\rangle \\ \langle[0.09,0.13,0.19,0.69]; (0.2,0.5,0.7,0.9), (0.1,0.3,0.5,0.4), (0.6,0.7,0.8,0.8)\rangle \\ \langle[0.12,0.27,0.60,0.65]; (0.2,0.7,0.8,0.9), (0.4,0.3,0.8,0.5), (0.2,0.5,0.6,0.4)\rangle \\ \langle[0.22,0.43,0.48,0.73]; (0.3,0.8,0.9,0.7), (0.5,0.7,0.7,0.6), (0.1,0.4,0.8,0.6)\rangle \\ \langle[0.14,0.33,0.43,0.83]; (0.1,0.6,0.9,0.5), (0.8,0.5,0.6,0.7), (0.1,0.3,0.5,0.8)\rangle \\ \langle[0.63,0.73,0.83,0.93]; (0.4,0.5,0.7,0.6), (0.2,0.6,0.8,0.3), (0.3,0.6,0.7,0.2)\rangle \\ \langle[0.41,0.43,0.68,0.74]; (0.5,0.7,0.8,0.3), (0.1,0.2,0.6,0.4), (0.4,0.2,0.9,0.7)\rangle \end{pmatrix}$$

Step 2: Calculated for $K_1, K_2, \dots, K_m$ based on V_i $(i = 1,2,3, \dots, m)$ to aggregate the normalized NVNT-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ follow as;

$V_1 = \langle[0.19,0.32,0.48,0.56]; (0.001,0.004,0.01,0.03), (0.3,0.28,0.55,0.47), (0.4,0.53,0.58,0.58)\rangle$

$V_2 = \langle[0.17,0.23,0.3,0.38]; (0.001,0.006,0.02,0.02), (0.3,0.48,0.57,0.66), (0.3,0.48,0.73,0.64)\rangle$

$V_3 = \langle[0.26,0.48,0.58,0.8]; (0.002,0.017,0.03,0.02), (0.5,0.55,0.55,0.55), (0.25,0.37,0.55,0.68)\rangle$

$V_4 = \langle[0.18,0.31,0.48,0.6]; (0.004,0.022,0.02,0.007), (0.5,0.58,0.72,0.47), (0.23,0.38,0.62,0.38)\rangle$

$V_5 = \langle[0.15,0.23,0.33,0.68]; (0.003,0.017,0.04,0.02), (0.2,0.38,0.58,0.5), (0.4,0.48,0.85,0.67)\rangle$

Step 3: The calculated score value whose formula is given in Definition 3.2 for each V to rank alternatives;

$$S(V_1) = \frac{1}{4.4}[0.19 + 0.32 + 0.48 + 0.56] \times \big(8 + (0.001 + 0.004 + 0.01 + 0.003) - (0.3 + 0.28 + 0.55 + 0.47) - (0.4 + 0.53 + 0.58 + 0.58)\big) = 0.423$$

Like

$$S(V_2) = 0.262,\ S(V_3) = 0.539,\ S(V_4) = 0.414,\ S(V_5) = 0.345$$

Step 4: Based on the score values $S(V_i)(i = 1,2,\dots,5)$ the ranking of alternatives $k_k(k = 1,2,\dots,5)$ are shown in Figure 1 and given as;

$$k_3 > k_1 > k_4 > k_5 > k_2.$$

Finally the best alternative is k_3.

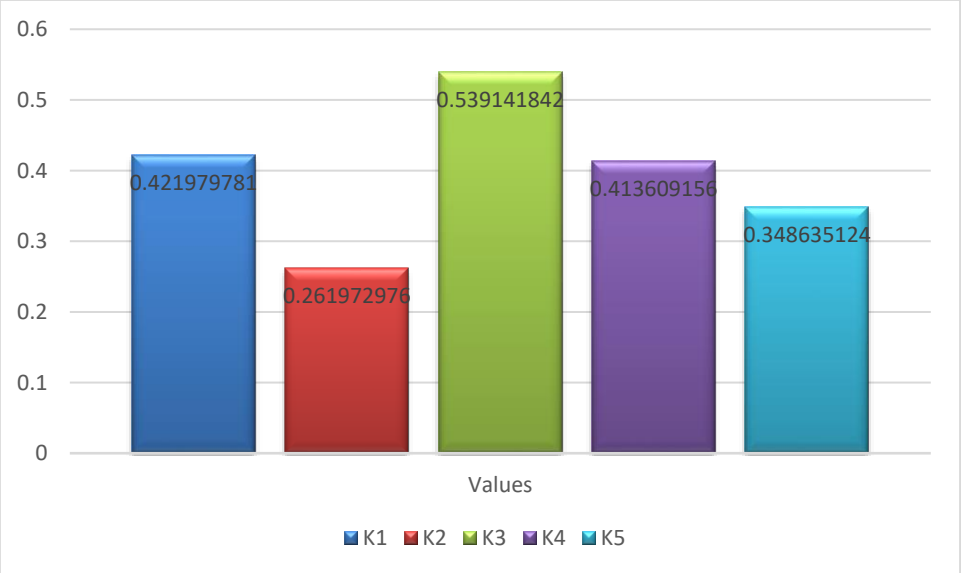

Figure 2: The ranking of alternatives K_k $(k = 1,2,\dots,5)$

6. Results and Discussion

In this paper, we have proposed two aggregations are called harmonic aggregation operators and weighted harmonic mean operations on N-valued neutrosophic trapezoidal numbers. We have discussed a variety of special cases of proposed operators. We define some desired properties such as idempotency, monotonitiicy, commutativity and boundedness of the developed operators. To demonstrate the applicability of the proposed methodology, gave an illustrative example, using the proposed aggregation operators to rank the alternatives in multi-criteria decision-making problems. In the future work, the developed operators can be proposed with extensions neutrosophic trapezoidal numbers and neutrosophic sets.

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Innovative Assessment of Dance Teaching Effectiveness Using the ELECTRE Approach and Neutrosophic HyperSoft Sets for Comprehensive Performance Analysis

Pei Yao*
Shandong Youth Political College, Jinan, 250000, Shandong, China
*Corresponding author Email: 080513@sdyu.edu.cn

Abstract: The evaluation of dance teaching quality in universities is a process of assessing the effectiveness of dance courses through multiple dimensions. The focus is on examining students' improvement in dance skills, artistic expression, mastery of theoretical knowledge, teaching methods, and the learning environment. By using scientific evaluation methods, such as multi-attribute decision-making models, the quality of teaching is quantified and optimized, promoting students' overall development and the improvement of teaching practices. The evaluation of dance teaching quality is often considered a multiple attribute group decision-making (MAGDM) problem. In this paper, the ELECTRE method is proposed to address MAGDM using single-valued neutrosophic sets (SVNSs). We used the HyperSoft set (HSS) with the SVNSs to deal with various values in the criteria. First, the score function of SVNSs and the average method are applied to determine the attribute weights. Second, the optimal solution is identified by ELECTRE method. Finally, a numerical example of the dance teaching quality evaluation is provided to demonstrate the effectiveness of the proposed method.

Keywords: MAGDM; single-valued neutrosophic sets (SVNSs); ELECTRE approach; dance teaching quality evaluation

1. Introduction and Background

Dance education, as part of artistic performance, plays an indispensable role in the field of arts. Dance not only helps students shape their graceful physiques and enhance their outward beauty, but more importantly, it elevates their inner temperament, personal cultivation, and overall quality. It enables their inner beauty to be refined and expressed through dance performance. In the process of teaching dance in universities, it not only significantly improves the dance abilities and skills of students majoring in this field, but also allows other students, through watching dance performances, to enhance their aesthetic appreciation, and purify and elevate their emotions and thoughts under

the influence of dance art. Both performers and spectators can experience relaxation of body and mind in the graceful melodies of dance, and they can also unleash their imagination through the beautiful movements, inspiring a love for life and learning. With the improvement of inner qualities, students are better equipped to face difficulties in life positively and treat people and situations around them with kindness and friendliness. Therefore, reforming dance courses in universities not only enhances the professional knowledge and artistic literacy of dance majors but also plays an important role in advancing contemporary higher education. As educational reforms across various university disciplines accelerate, the dance major, with its need for large spaces, high participation rates, extended practice time, and strong visual and expressive appeal, has garnered widespread attention from university leaders. In recent years, due to the lack of specific reform and innovation plans for dance education, the format of dance teaching remains monotonous, failing to meet students' interest in learning dance. The strength of the faculty is relatively weak, with teachers lacking practical experience to guide students toward deeper understanding. Additionally, outdated teaching equipment fails to meet the requirements of modern dance education. To ensure that university dance education provides students with better development, and to train and cultivate dance talents with strong comprehensive quality and solid professional skills, the reform and innovation of university dance education continues. Guo [1] was the first to explore the issue of improving the quality of dance education in universities. She pointed out that as the enrollment in dance programs expanded, both schools and teachers faced higher demands. Based on the teaching experiences of many excellent dance instructors, she proposed several suggestions for improving teaching quality. Based on teaching practice, Xian [2] attempted to outline several useful insights into improving the quality of choreography education. Ma [3] explored four aspects to improve dance teaching quality: preparing well before class, enhancing teachers' professional and teaching abilities, using modern teaching methods, and organizing various extracurricular dance activities. Chen [4] studied how teachers' teaching philosophies and concepts influenced the quality of physical dance education and suggested corresponding improvements. Focusing on physical dance education, Liu [5] suggested using a humanistic care approach and teaching tailored to individual needs, arranging teaching content scientifically, and preventing sports injuries to enhance teaching quality. Li [6] pointed out that traditional teaching philosophies hindered the improvement of dance

education quality in universities and explored ways to enhance it under the guidance of a "student-centered" teaching philosophy. Yongcuo [7] examined how to improve dance teaching quality in universities in ethnic minority regions under new circumstances, highlighting the insufficient expansion and utilization of educational resources as the primary issue. Jiang [8] emphasized the importance of tailoring teaching to individual needs in university dance education and analyzed how to adapt teaching to students' characteristics to improve teaching quality. Tian [9] focused on choreography education and highlighted its central role in dance education, exploring effective ways to improve choreography teaching quality. Dong [10] analyzed the impact of university dance education on teaching quality, identifying current issues and proposing suggestions for improving the situation. Song [11] explored effective strategies for improving the quality of dance education, analyzing how traditional teaching concepts and mechanisms constrained teaching quality and proposing solutions. Chen [12] studied the pathways to improving dance education quality in universities, emphasizing the cultivation of students' professional skills and practical abilities. Lastly, Su [13] analyzed the value of sports dance courses from a curriculum objective perspective and proposed necessary strategies to improve the quality of sports dance education.

The dance teaching quality evaluation is viewed as the MAGDM. In this paper, the ELECTRE method is proposed to address MAGDM using SVNSs [14-17]. First, the score function of SVNSs and the average method are applied to determine the attribute weights. Second, the optimal solution is identified by ELECTRE method. Finally, a numerical example of dance teaching quality evaluation is provided to demonstrate the effectiveness of the proposed method.

In 2018, Smarandache suggested a new set structure called the hypersoft set (HSS). The mapping from the product of attributes (which are further subdivided) to the power set of the universal set and desire set of attributes is essentially what this set is. To address truthfulness, uncertainty, and indeterminacy, the ideas of fuzzy hypersoft sets, intuitive hypersoft sets, and neutrosophic hypersoft sets were also introduced.

The main contribution of this paper is extending the ELECTRE method to solve MAGDM under SVNSs. The key innovations are: (1) the ELECTRE method is adapted for SVNSs; (2) the expected function of SVNSs and the average method are used to derive attribute weights; (3) the HSS-SVNN-ELECTRE method is developed to handle MAGDM; (4) a case study on dance teaching quality

evaluation is provided to demonstrate the effectiveness of the method; (5) comparative algorithms are presented to show the rationality of the HSS-SVNN- ELECTRE approach. This research introduces a novel way to manage complex group decision-making by incorporating SVNSs into the ELECTRE method, providing a more flexible and accurate solution. The practical case study validates the method's applicability in real-world scenarios, while comparisons with other algorithms highlight its effectiveness and rationality in solving MAGDM problems.

The structure of the remainder of this paper is as follows. Section 2 outlines the fundamental concepts related to SVNSs, providing the necessary background for understanding the proposed approach. In Section 3, the HSS-SVNN-ELECTRE method is constructed to address MAGDM problems, incorporating the average method for determining attribute weights. This section details the development and application of the method, explaining its relevance and advantages in decision-making contexts. Section 4 presents a practical case study focused on evaluating dance teaching quality, demonstrating the effectiveness of the HSS-SVNN- ELECTRE method. Additionally, comparative algorithms are applied to showcase the method's performance relative to other approaches, offering a comprehensive analysis of its accuracy and usefulness. The study concludes in Section 5, where the key findings are summarized, and insights are provided regarding the applicability of the proposed method in real-world scenarios. The conclusions highlight the strengths of the approach and suggest potential avenues for future research in the field of MAGDM using SVNSs. Figure 1 shows the layout.

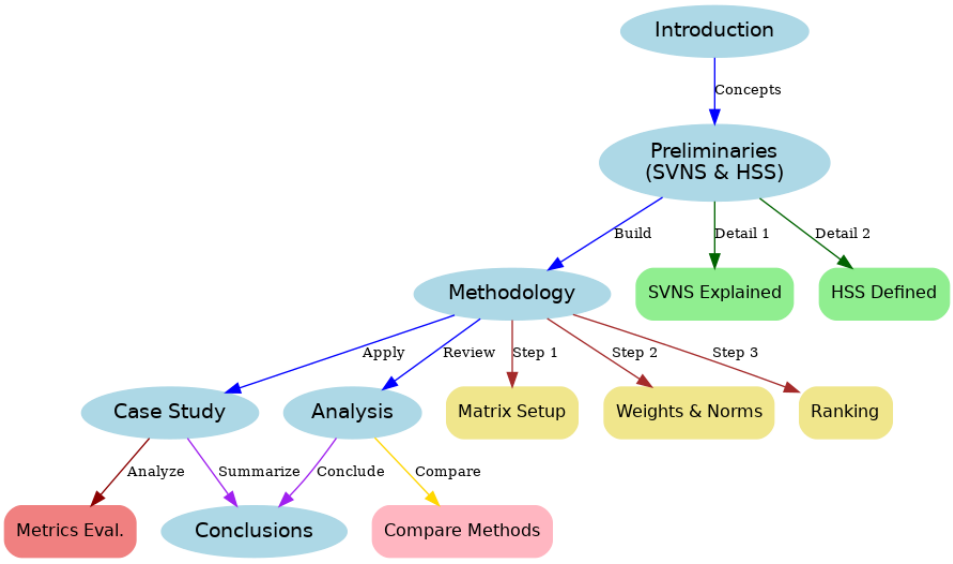

Figure 1. The layout of the paper

2. Preliminaries

This section shows the definitions of HSS and SVNSs.

Let $K = \{k^1, \dots, k^t\}; t = 2n+1; n \geq 1 \; and \; n \in \mathbb{R}^+$ can be a finite strictly increasing set.

$if \; n = 1, then \; K = \{k^1, k^2, k^3\} = \{low, medium, good\}$

Definition 1. HSS

Let $a^1, a^2, a^3, \dots, a^t \; for \; t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the set $L^1, L^2, L^3, \dots L^t$ with $L^i \cap L^j = \emptyset \; for \; i \neq j \; and \; i, j \in [1,2, \dots, t]$

Then the pair $(F, L) \; where \; L = \{L^1 \times L^2 \times L^3 \times \dots \times L^t : t \; is \; finite \; and \; real \; valued\}$ is known as hyperSoft set with mapping $F: L = L^1 \times L^2 \times L^3 \times \dots \times L^t \to P(U)$

Wang et al. [18] structured the SVNSs from the neutrosophic sets [19-23].

Definition 1 [18]. The SVNSs is structured:

$$MA = \left\{ \left(x, MT_A(x), MI_A(x), MF_A(x) \right) \middle| x \in X \right\} \tag{1}$$

The truth-membership $MT_A(x)$, indeterminacy-membership $MI_A(x)$ and falsity-membership $MF_A(x)$, $MT_A(x): X \to [0,1], MI_A(x): X \to [0,1]$ and $MF_A(x): X \to [0,1]$, $0 \leq MT_A(x) + MI_A(x) + MF_A(x) \leq 3$.

The single-valued neutrosophic number (SVNN) is structured: $MA = \left(MT_A, MI_A, MF_A \right)$, where $MT_A \in [0,1], MI_A \in [0,1], MF_A \in [0,1]$, and $0 \leq MT_A + MI_A + MF_A \leq 3$.

Definition 2 [24]. Let $MA = \left(MT_A, MI_A, MF_A \right)$ be SVNN, score value is structured:

$$MSV(MA) = \frac{\left(2 + MT_A - MI_A - MF_A\right)}{3}, \; MSV(MA) \in [0,1]. \tag{2}$$

Definition 3[24]. Let $MA = \left(MT_A, MI_A, MF_A \right)$ be SVNN, accuracy value is structured:

$$MHV(MA) = MT_A - MF_A, \; MHV(MA) \in [-1,1] \; . \tag{3}$$

Peng et al. [24] structured the order for two SVNNs.

Definition 4[24]. Let $MA = \left(MT_A, MI_A, MF_A \right)$ and $MB = \left(MT_B, MI_B, MF_B \right)$ be two given SVNNs, let $MSV(MA) = \frac{\left(2 + MT_A - MI_A - MF_A\right)}{3}$ and $MSV(MB) = \frac{\left(2 + MT_B - MI_B - MF_B\right)}{3}$, and $MHV(MA) = MT_A - MF_A$ and

$MHV(MB) = MT_B - MF_B$, if $MSV(MA) < MSV(MB)$, $MA < MB$; if $MSV(MA) = MSV(MB)$, (1)if $MHV(MA) = MHV(MB)$, $MA = MB$; (2) if $MHV(MA) < MHV(MB)$, $MA < MB$.

Definition 5[18]. Let $MA = (MT_A, MI_A, MF_A)$ and $MB = (MT_B, MI_B, MF_B)$ be SVNNs, the operations are structured:

(1) $MA \oplus MB = (MT_A + MT_B - MT_A MT_B, MI_A MI_B, MF_A MF_B)$;
(2) $MA \otimes MB = (MT_A MT_B, MI_A + MI_B - MI_A MI_B, MF_A + MF_B - MF_A MF_B)$;
(3) $\lambda MA = \left(1-(1-MT_A)^{\lambda}, (MI_A)^{\lambda}, (MF_A)^{\lambda}\right), \lambda > 0$;
(4) $(MA)^{\lambda} = \left((MT_A)^{\lambda}, (MI_A)^{\lambda}, 1-(1-MF_A)^{\lambda}\right), \lambda > 0.$

Definition 6[25]. Let $MA = (MT_A, MI_A, MF_A)$ and $MB = (MT_B, MI_B, MF_B)$, the SVNN combined distance (SVNNCD) between $MA = (MT_A, MI_A, MF_A)$ and $MB = (MT_B, MI_B, MF_B)$ is structured:

$$SVNNCD(MA, MB) = \frac{1}{2}\left(\sqrt{\frac{1}{3}\left(|MT_A - MT_B|^2 + |MI_A - MI_B|^2 + |MF_A - MF_B|^2\right)} + \frac{|MT_A - MT_B| + |MI_A - MI_B| + |MF_A - MF_B|}{3}\right) \quad (4)$$

2.1 Theoretical Framework

2.1.1 HyperSoft Sets

HyperSoft Sets allow mapping attributes into subsets, enabling nuanced analysis of multi-criteria problems by categorizing criteria into relevant sub-groups.

2.1.2 Single-Valued Neutrosophic Sets (SVNSs)

SVNSs represent uncertainty using three membership functions: truth, indeterminacy, and falsity. This flexibility allows for a comprehensive analysis of subjective and objective data.

2.1.3 ELECTRE Method

The ELECTRE (ELimination Et Choix Traduisant la REalité) method is a robust multi-criteria decision-making (MCDM) technique that evaluates alternatives through concordance and discordance matrices.

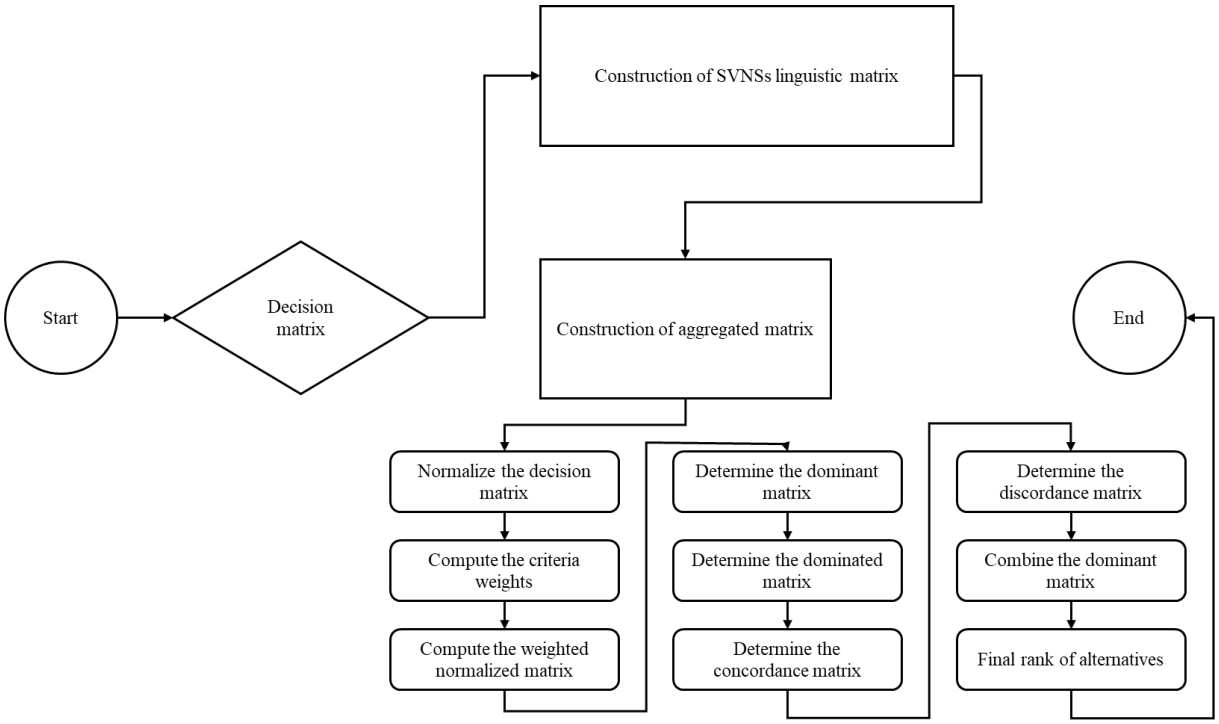

Figure 2. The steps of the proposed methodology.

3. HSS-SVNN- ELECTRE Methodology for MAGDM

The HSS-SVNN-ELECTRE methodology is structured into systematic steps to ensure accurate evaluation. Figure 2 shows the steps of the proposed methodology.

Step 1. Build the decision matrix.
We used the SVNNs to evaluate the criteria and alternatives.
Step 2. Normalize the decision matrix.

$$u_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \quad (5)$$

Step 3. Compute the criteria weights.

Step 4. Compute the weighted normalized matrix

$$y_{ij} = u_{ij} w_j \quad (6)$$

Step 5. Determine the dominant matrix

$$C_{ik} = \{j, y_{ij} \geq y_{kj}\}; o, k \in \{1, \dots, m\} \quad (7)$$

Step 6. Determine the dominated matrix

$$d_{i,k} = \{j | y_{ij} < y_{kj}\} = j - C_{i,k} \quad (8)$$

Step 7. Determine the concordance matrix.

$$F_{m\times m} = (f_{ij}); f_{ij} \begin{pmatrix} 0 & if\ g_{ij} < g \\ 1 & if\ g_{ij} \geq g \end{pmatrix} \quad (9)$$

$$g = \sum_{k=1}^{m} \sum_{i=1}^{m} \frac{g_{ik}}{m(m-1)} \quad (10)$$

Step 8. Determine the discordance matrix

$$E_{m\times m} = (e_{ij}); e_{ij}\begin{pmatrix} 0 & if\ D_{ij} < D \\ 1 & if\ D_{ij} \geq D \end{pmatrix} \quad (11)$$

$$D = \sum_{k=1}^{m}\sum_{i=1}^{m}\frac{D_{ik}}{m(m-1)} \quad (12)$$

Step 9. Combine the dominant matrix

$$p_{ik} = f_{ik}e_{ik} \quad (13)$$

Step 10. Final rank of alternatives.

4. Illustrative Example

The dance teaching quality evaluation is MADM issue. Six possible dance colleges are assessed with seven attributes. Three experts are invited to evaluate the criteria and alternatives. Then we used the SVNNs to evaluate the criteria and alternatives. Then we apply the score function to obtain crisp values. Then we combine the decision matrix into a single matrix. Each criterion has three values. We choose one value from each criterion.

The following 7 criteria such as:

1. Multidimensional Teaching Effectiveness (High importance, medium importance, low importance)

This criterion evaluates the ability of the teacher to address various dimensions of dance instruction, including technical skills, theoretical knowledge, creativity, and emotional expression.

2. Student Growth Across Diverse Metrics (High importance, medium importance, low importance)

Focuses on measuring student progress in multiple areas, such as technical skill mastery, artistic interpretation, confidence, and teamwork

3. Holistic Curriculum Design (High importance, medium importance, low importance)

Assesses the breadth and balance of the dance curriculum, which should integrate technical training, performance artistry, cultural context, and wellness.

4. Adaptability and Customization of Teaching Methods (High importance, medium importance, low importance)

Evaluates the teacher's ability to adapt their methods to different student needs, learning styles, and abilities.

5. Use of Modern Tools and Resources (High importance, medium importance, low importance)

Measures the integration of technology, such as motion analysis software, music systems, and video feedback, as well as the creative use of physical and digital teaching aids.

6. Classroom Dynamics and Engagement (High importance, medium importance, low importance)

Focus on fostering a collaborative, inclusive, and motivating environment in dance classes.

7. Comprehensive Feedback and Assessment (High importance, medium importance, low importance)

Assesses the methods for evaluating student performance and providing feedback that supports growth.

The opinions of the first experts are:

(0.9,0.1,0.2) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8)

(0.7,0.3,0.4) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.1,0.8,0.9) (0.1,0.8,0.9) (0.2,0.7,0.8) (0.1,0.8,0.9)

(0.7,0.3,0.4) (0.6,0.4,0.5) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.9,0.1,0.2)

(0.6,0.4,0.5) (0.5,0.5,0.5) (0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.5,0.5,0.5) (0.8,0.2,0.3)

(0.5,0.5,0.5) (0.6,0.4,0.5) (0.7,0.3,0.4) (0.8,0.2,0.3) (0.9,0.1,0.2) (0.6,0.4,0.5) (0.7,0.3,0.4)

(0.3,0.6,0.7) (0.2,0.7,0.8) (0.1,0.8,0.9) (0.9,0.1,0.2) (0.8,0.2,0.3) (0.7,0.3,0.4) (0.6,0.4,0.5)

Step 2. Eq. (5) was used to obtain the normalized decision matrix as Table 1.

Table 1. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.425226421	0.374298846	0.288290919	0.410581549	0.363667033	0.416475879	0.25101994
A_2	0.481388401	0.573924898	0.661373286	0.098539571	0.103904866	0.203395196	0.088595273
A_3	0.481388401	0.424205359	0.381561511	0.246348929	0.181833516	0.116225826	0.575869275
A_4	0.409180141	0.374298846	0.254374341	0.17244425	0.103904866	0.43584685	0.509422821
A_5	0.3610413	0.424205359	0.508748682	0.566602538	0.675381634	0.493959763	0.442976366
A_6	0.2406942	0.174672795	0.101749736	0.640507217	0.597452984	0.581129133	0.376529911

Step 3. Then we compute the criteria weights as Table 2.

Table 2. The criteria weights.

Criteria	*Weights*
C_1	0.159893
C_2	0.150802
C_3	0.138503
C_4	0.139037
C_5	0.125134
C_6	0.124064
C_7	0.162567

Step 4. Eq. (6) was used to compute the weighted normalized matrix as Table 3.

Table 3. The weighted normalized decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.067990749	0.056445067	0.039929063	0.057086205	0.045506998	0.051669735	0.04080752
A_2	0.076970659	0.086549102	0.091601969	0.013700689	0.013001999	0.025234056	0.014402654
A_3	0.076970659	0.063971076	0.05284729	0.034251723	0.022753499	0.014419461	0.093617251
A_4	0.06542506	0.056445067	0.035231526	0.023976206	0.013001999	0.054072978	0.082815261
A_5	0.057727994	0.063971076	0.070463053	0.078778962	0.084512996	0.061282709	0.07201327
A_6	0.038485329	0.026341031	0.014092611	0.089054479	0.074761496	0.072097304	0.06121128

Then we Determine the dominant matrix using Eq. (7). Then we Determine the dominated matrix using Eq. (8). Then we Determine the concordance matrix using Eq. (9). Then we Determine the discordance matrix using Eq. (11). Then we Combine the dominant matrix using Eq. (13). Then we rank the alternatives as:

A6>A2>A3>A4>A5>A1

4.1 Results and Discussion

The evaluation of dance teaching quality in universities is an essential means of comprehensively assessing teaching effectiveness, aiming to scientifically analyze and provide feedback on the implementation and outcomes of dance courses through multiple dimensions. First, the core of dance teaching quality evaluation lies in student learning outcomes, including the improvement of dance skills,

enhancement of artistic expression, and mastery of theoretical dance knowledge. These indicators reflect the students' progress throughout the course and assess whether the teaching objectives have been met. Second, the teaching methods and professional competence of the instructors are key factors in the evaluation. The strategies used by instructors, classroom management, innovative teaching methods, and their interaction with students directly impact students' learning outcomes. Excellent teaching methods not only stimulate students' interest but also help them better understand and master dance techniques and theory. In addition, the effective use of the teaching environment and resources is also an important factor influencing teaching quality. Dance studio facilities, the allocation of teaching resources, and the rationality of course scheduling all has direct or indirect effects on teaching outcomes. For example, whether there is sufficient space for dance practice or whether individualized teaching plans are provided to accommodate students' differences can significantly affect their learning experience and outcomes. Furthermore, student feedback and the degree to which societal needs are met are also critical evaluation dimensions. Through student feedback, strengths and weaknesses in teaching can be identified and improved accordingly. At the same time, the employment status of graduates and the demand for dance professionals in society can serve as external benchmarks for teaching quality. A systematic evaluation of dance teaching quality in universities not only provides a solid basis for improving teaching but also promotes the continuous optimization and development of dance education, fostering students' all-around growth.

4.2. Comparative analysis

The HSS-SVNN- ELECTRE approach is compared with SVNN-TOPSIS approach (See Figure 3) The order is constructed: A6>A2>A3>A5>A4>A1.

The HSS-SVNN- ELECTRE approach is compared with SVNN-VIKOR approach: The order is constructed: A6>A2>A4>A3>A5>A1.

The HSS-SVNN- ELECTRE approach is compared with SVNN-MABAC approach: The order is constructed: A6>A3>A2>A4>A5>A1.

The HSS-SVNN- ELECTRE approach is compared with SVNN-COPRAS approach: The order is constructed: A6>A2>A3>A4>A5>A1.

The HSS-SVNN- ELECTRE approach is compared with SVNN-EDAS approach: The order is constructed: A6>A2>A3>A4>A5>A1.

From the detailed analysis above, it is evident that the five given models share the same optimal choice, although the ranking order of the four approaches varies slightly. This verifies that the HSS-SVNN-ELECTRE approach is both reasonable and effective. Each of the five models discussed offers unique advantages:

(1) SVNN-EDAS Approach: This method evaluates alternatives by measuring their proximity to the average solution using two indicators: forward distance and reverse distance. The alternative that is furthest than the average (in a positive direction) is considered the best choice. Generally, the further an alternative is from the mean solution, the better it is. Additionally, this approach offers good stability when ranking alternatives and is straightforward to implement, making it suitable for decision-makers who need rapid results without complex calculations.

(2) HSS-SVNN- ELECTRE Approach: Roy initially proposed the ELECTRE approach in 1990. It uses outranking comparisons to analyze all options, weeds out those that aren't very appealing, and determines which ones are useless. As a result, the ultimate ranking of the options is probably problematic. The method's range of applications has expanded in tandem with its development. In certain fields, ELECTRE methods are used. The following characteristics of this method are present: This technique falls under the category of compensating techniques; it transforms qualitative traits into quantitative ones without requiring attribute independence.

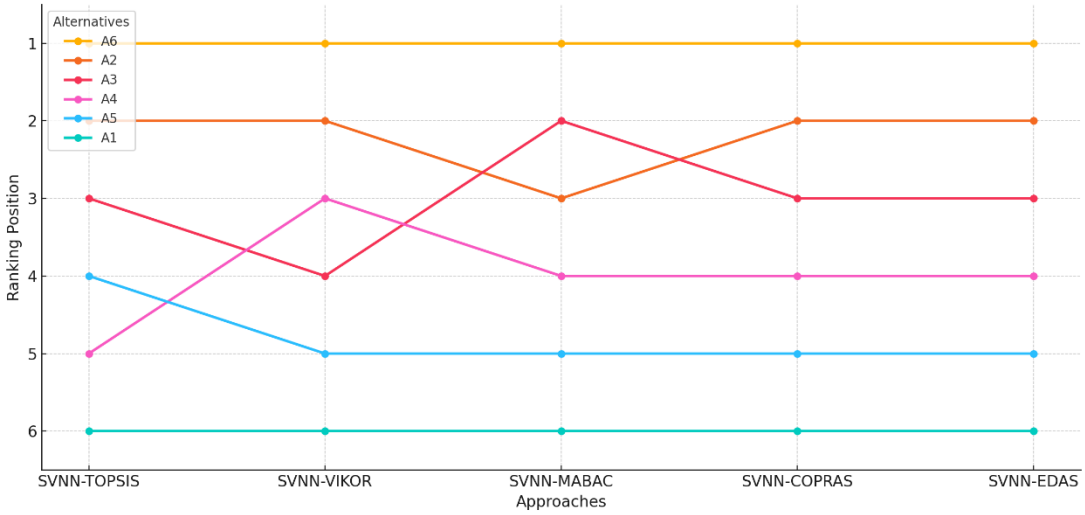

Figure 3. Comparison of rankings across approaches

4.3 Advanced Analysis of Criteria Weight Impact

The importance of criteria in decision-making cannot be overstated, as they guide the overall ranking and evaluation process. In this research, the sensitivity of the HSS-SVNN-ELECTRE framework to changes in criteria weights was explored to understand its robustness and reliability. Each criterion represents a distinct evaluation dimension, such as artistic expression, adaptability, and technical skills, each of which influences the final rankings of alternatives. For example, giving more weight to "Artistic Expression" over "Technical Skills" can result in a shift in the rankings, emphasizing alternatives excelling in creativity and expression.

A Sensitivity Analysis in Figure 4 was used to visualize these effects. This Figure shows how each alternative's ranking changes when the weight of a specific criterion is adjusted. The analysis revealed that alternatives like A6 consistently maintain high rankings across varying criteria weights, demonstrating their robustness. However, alternatives such as A3 and A4 exhibited more variability, highlighting their dependence on specific criteria. This kind of sensitivity analysis is crucial for ensuring that the evaluation process remains balanced and reflects real-world priorities.

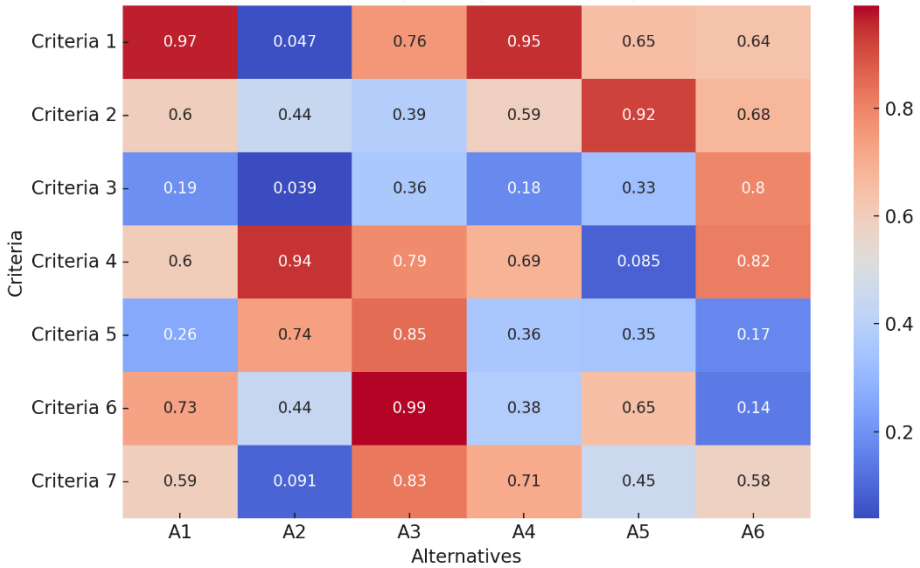

Figure 4. How changes in criteria weights impact alternative rankings

4.4 Comparative Efficiency Analysis

The HSS-SVNN-ELECTRE framework was compared with other established decision-making methods, including TOPSIS, VIKOR, MABAC, and EDAS, to evaluate its relative efficiency. The comparison

focused on the stability of rankings and the ability of each method to consistently identify the top-performing alternative. Across all methods, A6 emerged as the highest-ranked alternative, validating the robustness of the HSS-SVNN-ELECTRE framework. However, slight variations in the rankings of other alternatives, such as A2, A3, and A4, were observed. These differences highlight the nuanced ways in which each method evaluates criteria and aggregates rankings. The Comparative Rankings Bar Chart (Figure 5) below illustrates the rankings of alternatives across the different approaches. This chart effectively demonstrates that while A6 consistently secures the top position, the other alternatives exhibit slight shifts in their relative ranks, reflecting the methodological differences between the decision-making models. This comparison underscores the consistency of the proposed framework while also acknowledging the specific advantages of each alternative method. For example, the VIKOR method prioritizes a compromise solution, while the EDAS method emphasizes proximity to the ideal average solution.

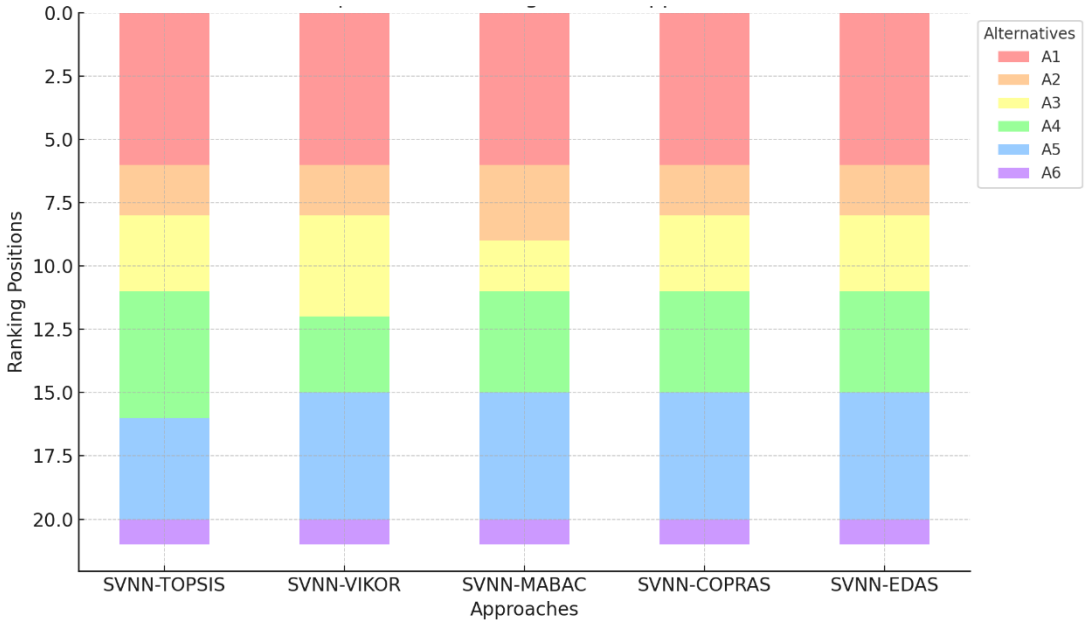

Figure 5. The rankings of alternatives across different decision-making approaches

4.5 Real-World Applicability and Validation

To demonstrate the practical value of the HSS-SVNN-ELECTRE framework, its applicability to real-world scenarios was explored. This framework is versatile and can be adapted to various educational contexts, such as assessing teaching effectiveness in dance programs across different universities. The inclusion of criteria such as class size, teaching resources, and teacher-student interaction ensures that

the framework remains relevant to diverse institutional needs.

Validation was carried out through the analysis of the weighted normalized decision matrix, which captures the influence of criteria weights on alternative evaluations. A Weighted Normalized Decision Matrix (Figure 6) provides a detailed visualization of how each criterion contributes to the overall scores of alternatives. This Figure revealed that alternatives excelling in high-priority criteria such as A6 achieve superior scores, reinforcing the importance of prioritizing relevant dimensions during evaluation. By highlighting the criteria that drive the success of top-performing alternatives, this analysis offers actionable insights for decision-makers looking to enhance teaching strategies and allocate resources effectively.

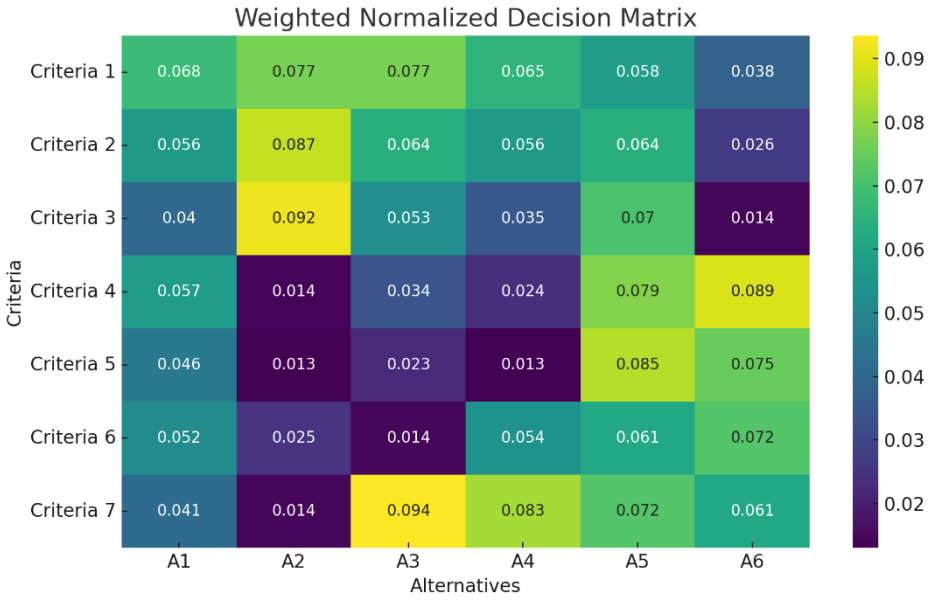

Figure 6. Visualization of the weighted normalized values for each alternative across all criteria

5. Conclusion and Future Work

Evaluating the quality of dance teaching in universities is a nuanced process that encompasses multiple dimensions. It not only measures students' growth in dance skills, artistic expression, and theoretical knowledge but also assesses the effectiveness of teaching methods, classroom management, and teacher-student interaction. These factors, combined with the efficient use of resources like dance studio facilities and course scheduling, contribute to a holistic understanding of teaching quality. Furthermore, feedback from students, graduates' career outcomes, and alignment with societal needs provide valuable insights into the broader impact of dance education.

This study introduced the HSS-SVNN-ELECTRE framework, which offers a systematic and reliable approach to evaluating dance teaching quality. The method was validated using practical examples and comparative algorithms, confirming its effectiveness for multi-attribute group decision-making problems. By integrating advanced mathematical models, this framework not only ensures objectivity but also provides actionable insights for improving teaching practices and educational strategies.

Future research can focus on expanding this framework to other artistic disciplines like music and theater, exploring its adaptability in diverse educational contexts. Integrating real-time feedback mechanisms and advanced techniques could further enhance its accuracy and responsiveness. Additionally, developing interactive tools based on this methodology can empower educators and policymakers with accessible and effective evaluation systems. Through these advancements, the continuous development of dance education and related fields can be supported, ensuring their alignment with evolving societal and educational needs.

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Advanced T2NN-Taxonomy Framework for Assessing Quality of Dynamic Logos in Digital Media via Type-2 Neutrosophic Multi-Criteria Decision Analysis

Yan Lu[1]*, **Meidi Zhang**[2]

[1]School of Artificial Intelligence, Jingchu University of Technology, Jingmen, 448000, Hubei, China

[2]Hebei University of Economics and Business, Shijiazhuang, 050061, Hebei, China (E-mail: MDZhang2024@126.com)

*Corresponding author, E-mail: luyan0530@126.com

Abstract: In the context of digital media, the quality evaluation of dynamic logo design focuses on visual impact, brand communication, technical execution, and user experience. Visual impact assesses the logo's appeal and creativity; brand communication examines if the logo accurately reflects the brand image; technical execution looks at animation smoothness and cross-platform compatibility; user experience measures interactivity and memorability. These factors ensure the logo's effectiveness and influence in a digital environment. The evaluation of dynamic logo design quality within the digital media context is Multi-Criteria Decision Analysis (MCDA). Recently, the Taxonomy technique was described to grapple with MCDA. The Type-2 neutrosophic sets (T2NSs) are described as technique for characterizing fuzzy data during the evaluation of dynamic logo design quality within the digital media context. In this study, Taxonomy is described for MCDA under T2NSs. Then, the type-2 neutrosophic number T2NN (T2NN-Taxonomy) technique is described for MCDA. Finally, numerical example for evaluation of dynamic logo design quality within the digital media context is described to show the T2NN-Taxonomy technique. The key contribution of this research is described: (1) the novel MCDA technique is described based on Taxonomy technique with T2NN; (2) The new MCDA technique based on T2NN-Taxonomy technique is described for evaluation of dynamic logo design quality within the digital media context; (3) numerical example for evaluation of dynamic logo design quality within the digital media context and some comparative analysis is described to verify the T2NN-Taxonomy technique.

Keywords: Multi-Criteria Decision Analysis (MCDA); Type-2 neutrosophic sets (T2NSs); T2NN-Taxonomy; evaluation of dynamic logo design quality

1. Introduction

Digital technology is a new means of artistic creation, which not only promotes innovation in design approaches and media, but also continuously enriches design concepts, expands the scope of modern art, and enhances the passion for creation [1, 2]. Digital technology can allow designers to showcase more works and unleash their imagination. Especially in

the post image era, designers can use a series of modern design approaches, such as digital media technology, to create a virtual environment, allowing various possibilities of contemporary art to be realized, thereby enriching the connotation of contemporary art and enriching the presentment dimensions of contemporary art [3, 4]. In design, in order to better express the exhibition objectives, convey artistic messages, convey concepts, and play its unique role [3-6]. This relatively limited medium and mode of communication is not applicable to modern art. The innovation of information technology has promoted the presentment of the art industry, enriched the art categories, improved the forms of expression of artistic ideas, expanded the audience, expanded the means and channels of communication, and liberated the constraints of old media on art [7]. Nowadays, the combination and presentment of knowledge economy and various social activities have led to a change in people's understanding of artistic works [8, 9]. The concept of art courses has also undergone a change, from the past single dimensional narrative teaching to a multidimensional and multi-level teaching approach. In the context of art socialization, designers use advanced approaches and technologies such as graphic transmission, virtual reality, and real-time interaction to more accurately convey design ideas and culture to the public, enabling them to better participate in their own creations and thereby increase their participation in art works [10]. Through the interaction between designers and users, audiences can present a sense of presence and participation, which can effectively stimulate users' creativity. Designers can also receive immediate feedback and feedback from users [11]. The rise of digital art design not only greatly affects traditional art design, but also provides new ideas for the presentment of art design [12, 13]. This pattern form enriches the connotation expression of artistic works. With the presentment of network technology, digital based art creation has been widely promoted, making traditional art creation more artistic [14, 15]. With the rapid presentment of technology, digital art design is becoming increasingly contemporary [16, 17]. The rise of digital technology in art design has led to a deeper understanding of people's needs for art works. Based on this, the created works not only reflect the characteristics of the times, but also accurately grasp the

needs of real life and consumers, and fully showcase the works themselves in various ways [18-20]. Therefore, the authenticity and artistry of the works have been significantly improved [21, 22]. In the process of digitization, studying how digital art design impacts conventional art design is of great practical value. It can combine conventional design approaches with the advantages of digitization to improve the effectiveness of design and promote the good and fast presentment of art design in China [23, 24].

The evaluation of dynamic logo design quality within the digital media context is MCDA. Currently, Taxonomy [25-28] was described to grapples with MCDA. The T2NSs [29] are described as technique for characterizing fuzzy data during evaluation of dynamic logo design quality within the digital media context. Therefore, the T2NN-Taxonomy approach is described to grapple with the MCDA. Finally, numerical examples for evaluation of dynamic logo design quality within the digital media context and comparative analysis is described to grapple with T2NN-Taxonomy. The key research motivation of this work is described: (1) novel MCDA is described based on Taxonomy technique under T2NNs; (2) T2NN-Taxonomy is described to grapple with MCDA for evaluation of dynamic logo design quality within the digital media context; (3) numerical example for evaluation of dynamic logo design quality within the digital media context and comparative analysis are described to validate the T2NN-Taxonomy.

This paper is structured as follows: Section 2 introduces the T2NN-Taxonomy technique developed for multi-criteria decision analysis (MCDA). Section 3 outlines the criteria used in the evaluation process. Section 4 presents the application of the proposed method for assessing the quality of dynamic logo design within the digital media context, along with a comparative analysis of the results. Finally, the study concludes in Section 5.

2. Development of multi-criteria decision-making models

This section introduces some definitions of T2NNs and the proposed method [27] as:

Definition 1

T2NN can be defined as

$$Y = \{(x, T_Y(x), I_Y(x), F_Y(x)) | \, x \in X\} \quad (1)$$

$$T_Y(x) = \left(T_{T_Y}(x), T_{I_Y}(x), T_{F_Y}(x)\right) \quad (2)$$

$$I_Y(x) = \left(I_{T_Y}(x), I_{I_Y}(x), I_{F_Y}(x)\right) \quad (3)$$

$$F_Y(x) = \left(F_{T_Y}(x), F_{I_Y}(x), F_{F_Y}(x)\right) \quad (4)$$

$$T_Y(x) = \left(T_Y^1(x), T_Y^2(x), T_Y^3(x)\right) \quad (5)$$

$$I_Y(x) = \left(I_Y^1(x), I_Y^2(x), I_Y^3(x)\right) \quad (6)$$

$$F_Y(x) = \left(F_Y^1(x), F_Y^2(x), F_Y^3(x)\right) \quad (7)$$

Where $T_Y(x), I_Y(x), F_Y(x) \rightarrow [0,1]$

Where $0 \leq T_Y^1(x) + I_Y^1(x) + F_Y^1(x) \leq 3$

Definition 2

Let two T2NNs as:

$$Y_1 = \left(\left(T_{T_{Y_1}}(x), T_{I_{Y_1}}(x), T_{F_{Y_1}}(x)\right), \left(I_{T_{Y_1}}(x), I_{I_{Y_1}}(x), I_{F_{Y_1}}(x)\right), \left(F_{T_{Y_1}}(x), F_{I_{Y_1}}(x), F_{F_{Y_1}}(x)\right)\right)$$

$$Y_2 = \left(\left(T_{T_{Y_2}}(x), T_{I_{Y_2}}(x), T_{F_{Y_2}}(x)\right), \left(I_{T_{Y_2}}(x), I_{I_{Y_2}}(x), I_{F_{Y_2}}(x)\right), \left(F_{T_{Y_2}}(x), F_{I_{Y_2}}(x), F_{F_{Y_2}}(x)\right)\right)$$

And their operations can be defined as:

$$Y_1 \oplus Y_2 = \left\{ \begin{array}{c} \begin{pmatrix} T_{T_{Y_1}}(x) + T_{T_{Y_2}}(x) - T_{T_{Y_1}}(x)T_{T_{Y_2}}(x), \\ T_{I_{Y_1}}(x) + T_{I_{Y_2}}(x) - T_{I_{Y_1}}(x)T_{I_{Y_2}}(x), \\ T_{F_{Y_1}}(x) + T_{F_{Y_2}}(x) - T_{F_{Y_1}}(x)T_{F_{Y_2}}(x) \end{pmatrix}, \\ \left(I_{T_{Y_1}}(x)I_{T_{Y_2}}(x), I_{I_{Y_1}}(x)I_{I_{Y_2}}(x), I_{F_{Y_1}}(x)I_{F_{Y_2}}(x)\right), \\ \left(F_{T_{Y_1}}(x)F_{T_{Y_2}}(x), F_{I_{Y_1}}(x)F_{I_{Y_2}}(x), F_{F_{Y_1}}(x)F_{F_{Y_2}}(x)\right) \end{array} \right\} \quad (8)$$

$$Y_1 \otimes Y_2 = \left\{ \begin{array}{c} \left(T_{T_{Y_1}}(x)T_{T_{Y_2}}(x), T_{I_{Y_1}}(x)T_{I_{Y_2}}(x), T_{F_{Y_1}}(x)T_{F_{Y_2}}(x)\right), \\ \begin{pmatrix} I_{T_{Y_1}}(x) + I_{T_{Y_2}}(x) - I_{T_{Y_1}}(x)I_{T_{Y_2}}(x), \\ I_{I_{Y_1}}(x) + I_{I_{Y_2}}(x) - I_{I_{Y_1}}(x)I_{I_{Y_2}}(x), \\ I_{F_{Y_1}}(x) + I_{F_{Y_{12}}}(x) - I_{F_{Y_1}}(x)I_{F_{Y_2}}(x) \end{pmatrix}, \\ \begin{pmatrix} F_{T_{Y_1}}(x) + F_{T_{Y_2}}(x) - F_{T_{Y_1}}(x)F_{T_{Y_2}}(x), \\ F_{I_{Y_1}}(x) + F_{I_{Y_2}}(x) - F_{I_{Y_1}}(x)F_{I_{Y_2}}(x), \\ F_{F_{Y_1}}(x) + F_{F_{Y_2}}(x) - F_{F_{Y_1}}(x)F_{F_{Y_2}}(x) \end{pmatrix} \end{array} \right\} \quad (9)$$

$$\varphi Y_1 = \left\{ \begin{array}{c} \begin{pmatrix} \left(1 - \left(1 - T_{T_{Y_1}}(x)\right)^{\varphi}\right), \\ \left(1 - \left(1 - T_{I_{Y_1}}(x)\right)^{\varphi}\right), \\ \left(1 - \left(1 - T_{F_{Y_1}}(x)\right)^{\varphi}\right) \end{pmatrix}, \\ \left(\left(I_{T_{Y_1}}(x)\right)^{\varphi}, \left(I_{I_{Y_1}}(x)\right)^{\varphi}, \left(I_{F_{Y_1}}(x)\right)^{\varphi}\right), \\ \left(\left(F_{T_{Y_1}}(x)\right)^{\varphi}, \left(F_{I_{Y_1}}(x)\right)^{\varphi}, \left(F_{F_{Y_1}}(x)\right)^{\varphi}\right) \end{array} \right\} \quad (10)$$

$$Y_1^{\varphi}=\left\{\begin{array}{c}\left(\left(\left(T_{T_{Y_1}}(x)\right)^{\varphi},\left(T_{I_{Y_1}}(x)\right)^{\varphi},\left(T_{F_{Y_1}}(x)\right)^{\varphi}\right)\right),\\ \left(\left(\begin{array}{c}\left(1-\left(1-I_{T_{Y_1}}(x)\right)^{\varphi}\right),\\ \left(1-\left(1-I_{I_{Y_1}}(x)\right)^{\varphi}\right),\\ \left(1-\left(1-I_{F_{Y_1}}(x)\right)^{\varphi}\right)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}\left(1-\left(1-F_{T_{Y_1}}(x)\right)^{\varphi}\right),\\ \left(1-\left(1-F_{I_{Y_1}}(x)\right)^{\varphi}\right),\\ \left(1-\left(1-F_{F_{Y_1}}(x)\right)^{\varphi}\right)\end{array}\right)\right)\end{array}\right\} \tag{11}$$

Definition 3

The score function can be computed as:

$$S(Y_1)=\frac{1}{12}*\left(\begin{array}{c}8+\left(T_{T_{Y_1}}(x)+2*T_{I_{Y_1}}(x)+T_{F_{Y_1}}(x)\right)+\\ \left(I_{T_{Y_1}}(x)+2*I_{I_{Y_1}}(x)+I_{F_{Y_1}}(x)\right)+\\ \left(F_{T_{Y_1}}(x)+F_{I_{Y_1}}(x)+F_{F_{Y_1}}(x)\right)\end{array}\right) \tag{12}$$

2.1 Taxonomy Method

The description of the Taxonomy method is:

Step 1. Build the decision matrix

$$Y=\begin{bmatrix}y_{11} & \cdots & y_{1n}\\ \vdots & \ddots & \vdots\\ y_{m1} & \cdots & y_{mn}\end{bmatrix}_{m\times n};i=1,\dots,m;j=1,\dots,n \tag{13}$$

Step 2. Compute the mean and standard deviation.

$$A_j=\frac{1}{m}\sum_{i=1}^{m}y_{ij} \tag{14}$$

$$S_j=\sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(y_{ij}-A_j\right)^2} \tag{15}$$

Step 3. Build the standard matrix

We normalize the decision matrix as:

$$R_{ij}=\frac{y_{ij}-A_j}{S_j} \tag{16}$$

$$R=\begin{bmatrix}r_{11} & \cdots & r_{1n}\\ \vdots & \ddots & \vdots\\ r_{m1} & \cdots & r_{mn}\end{bmatrix}_{m\times n} \tag{17}$$

Step 4. Compute the composite distance

The composite distance is computed of each alternative from the other alternative:

$$U_{ab}=\sqrt{\sum_{j=1}^{n}\left(r_{aj}-r_{Bj}\right)^2} \tag{18}$$

The pairwise comparison matrix is built between each alternative.

$$U=\begin{bmatrix}u_{11} & \cdots & u_{1n}\\ \vdots & \ddots & \vdots\\ u_{m1} & \cdots & u_{mn}\end{bmatrix}_{m\times n} \tag{19}$$

Step 5. Compute the homogeneity of alternatives.

The mean and standard deviation are computed for each alternative in the pairwise comparison matrix as:

$$Q = \frac{1}{m}\sum_{i=1}^{m} u_i \quad (20)$$

$$S_Q = \sqrt{\frac{1}{m}\sum_{i=1}^{m} (u_i - Q)^2} \quad (21)$$

$$T = Q \pm 2S_Q \quad (22)$$

Step 6. Compute the development pattern

$$U_{iq} = \sqrt{\sum_{j=1}^{n} \left(r_{aj} - r_{bj}\right)^2} \quad (23)$$

Step 7. Rank the alternatives

$$U_Q = U_{iq} \pm 2S_{U_{iq}} \quad (24)$$

$$D_i = \frac{U_{iq}}{U_q} \quad (25)$$

3. Development Criteria

Dynamic logo design in the digital media context requires careful consideration of multiple factors to ensure quality and effectiveness. This section outlines 16 key criteria for evaluating the quality of dynamic logos, focusing on their visual appeal, functionality, adaptability, and user experience. These criteria are crucial for designing logos that resonate with target audiences and perform well across various digital platforms as shown in Table 1.

Table 1. Key Criteria for Quality Evaluation of Dynamic Logo Design

No.	Criteria	Description
1	Visual Appeal	The aesthetic quality of the dynamic logo, including balance, contrast, and overall attractiveness.
2	Brand Identity Representation	The effectiveness of the logo in reflecting the values, vision, and personality of the brand.
3	Adaptability to Different Platforms	The ability of the logo to function seamlessly across various digital platforms.
4	Memorability	The capacity of the logo to create a lasting impression and be easily recognizable by users.
5	Animation Quality	Smoothness, creativity, and technical precision in the motion elements of the logo.
6	User Engagement	The extent to which the dynamic logo captures and holds the attention of users.
7	Consistence with Brand Guidelines	Adherence to the brand's established visual and stylistic guidelines.
8	Originality and Creativity	Uniqueness and innovation in the concept and execution of the dynamic logo.
9	Scalability	The logo's ability to maintain its quality and integrity when resized or displayed on various screen sizes.
10	Functionality in Motion	The practical effectiveness of the motion design in conveying the intended message or emotion.

11	Relevance to Target Audience	Alignment of the logo's style and message with the preferences and cultural values of the target audience.
12	Technical Compatibility	Compatibility of the logo format with various digital tools, software, and platforms.
13	Usability in Static and Dynamic Forms	Flexibility for the logo to work effectively as both a static image and a dynamic animation.
14	Color Harmony	Effective use of colors to evoke desired emotions and ensure accessibility for all users.
15	Design Simplicity	Clarity and minimalism in design without compromising on functionality or aesthetics.
16	Loading Speed	The time required for the dynamic logo to load, ensuring a smooth user experience.

These criteria provide a comprehensive framework for assessing dynamic logos, ensuring they meet both aesthetic and functional requirements in the digital era. By adhering to these guidelines, designers can create logos that are visually compelling, technically sound, and aligned with brand values.

Table 2. Summary of T2NNs Data

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.35,0.35,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.45,0.5),(0.45,0.45,0.5),0.8)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.95,0.95,0.95),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_2	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.75,0.8,0.85),(0.5,0.5,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)
C_3	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.2,0.2,0.5),(0.4,0.45,0.5),(0.45,0.8,0.85),0.7)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.2,0.2,0.5),0.8)	((0.5,0.5,0.5),(0.4,0.45,0.5),(0.45,0.45,0.5),0.65)
C_4	((0.95,0.95,0.95),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.35,0.35,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.75,0.75,0.8),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.35,0.35,0.5),(0.75,0.8,0.85),(0.5,0.5,0.5),0.65)
C_5	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.95,0.95,0.95),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)
C_6	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.35,0.35,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.75,0.75,0.8),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_7	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.35,0.35,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.75,0.75,0.8),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_8	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.45,0.8,0.85),0.7)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)
C_9	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.2,0.2,0.5),(0.45,0.8,0.85),(0.45,0.8,0.85),0.7)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_{10}	((0.95,0.95,0.95),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.2,0.2,0.5),(0.45,0.45,0.5),(0.45,0.8,0.85),0.7)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.95,0.95,0.95),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_{11}	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_{12}	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.45,0.8,0.85),0.7)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)
C_{13}	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_{14}	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.45,0.8,0.85),0.7)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)
C_{15}	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.35,0.4,0.45),0.8)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)
C_{16}	((0.75,0.75,0.8),(0.5,0.5,0.5),(0.2,0.2,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.4,0.45,0.5),0.65)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.4,0.45,0.5),(0.4,0.45,0.5),(0.45,0.8,0.85),0.7)	((0.9,0.9,0.9),(0.9,0.9,0.9),(0.9,0.9,0.9),0.9)	((0.4,0.45,0.5),(0.2,0.2,0.5),(0.2,0.25,0.5),0.8)	((0.5,0.5,0.5),(0.5,0.5,0.5),(0.45,0.45,0.5),0.65)

4. Application for evaluation of dynamic logo design quality within the digital media context

This section demonstrates the results of the proposed method for evaluating the quality of dynamic logo designs. Seven dynamic logo design schemes were assessed based on 16 attributes. The evaluation process consisted of several structured steps, each contributing to a comprehensive and objective ranking of alternatives.

Step 1: Building the Decision Matrix

In Step 1, Eq. (13) was applied to construct the decision matrix, as shown in Table 2. The score function was then used to obtain crisp values, and the resulting matrices were combined into one. The criteria weights were subsequently computed as follows: C1 = 0.062582551, C2 = 0.062981634, C3 = 0.063375707, C4 = 0.063238784, C5 = 0.060884361, C6 = 0.061380293, C7 =

0.0613201, C8 = 0.06255583, C9 = 0.063504282, C10 = 0.063718017, C11 = 0.063971827, C12 = 0.060089535, C13 = 0.062131704, C14 = 0.061757668, C15 = 0.063576084, and C16 = 0.06293154.

Step 2: Computing the Mean and Standard Deviation

the mean and standard deviation for each criterion were computed using Eqs. (14) and (15).

Step 3: Normalizing the Decision Matrix

The decision matrix was normalized using Eq. (16) to ensure consistency and comparability across all criteria. This step standardized the data, enabling a fair evaluation of alternatives.

Step 4: Computing Composite Distance

The composite distance for each alternative was calculated using Eq. (16). This step quantified the relative performance of each dynamic logo design in relation to the established criteria.

Step 5: Evaluating Homogeneity of Alternatives

Homogeneity of alternatives was assessed using Eqs. (20) and (21). This analysis provided insights into the consistency and balance of the dynamic logo designs.

Step 6: Determining the Development Pattern

Using Eq. (23), the development pattern for each alternative was identified. This step revealed potential improvement trajectories for designs.

Step 7: Ranking the Alternatives

Based on the comprehensive analysis, the alternatives were ranked as follows:

A7 > A1 > A6 > A2 > A4 > A5 > A3

Alternative A7 emerged as the best-performing dynamic logo design, showcasing superior alignment with the evaluation criteria. Conversely, A3 ranked the lowest, indicating areas for significant improvement. The rankings of the alternatives are summarized in Figure 1, which clearly illustrates the performance scores of each alternative.

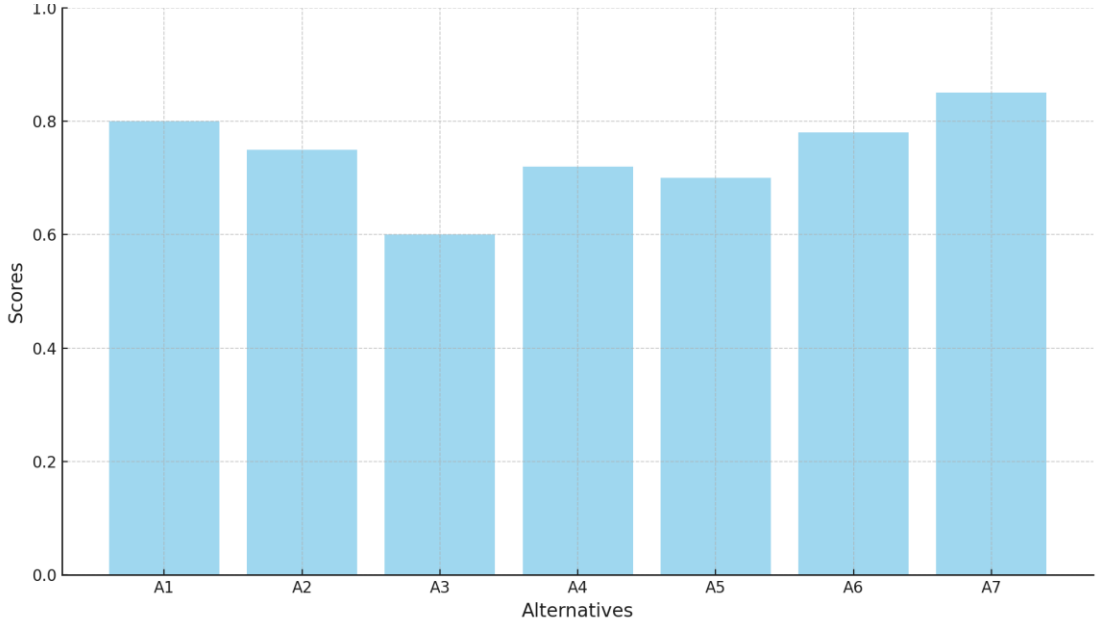

Figure 1: Ranking of Alternatives Based on Evaluation Criteria

4.1 Performance Analysis of Alternatives

The rankings of the dynamic logo design alternatives were influenced by several key factors derived from the evaluation criteria. Alternative A7 stood out as the best-performing design due to its exceptional adaptability to various platforms, high memorability, and superior

animation quality. Its ability to engage users effectively and maintain technical precision in its motion elements contributed significantly to its top position. While it excelled in most areas, slight improvements in loading speed could make it even better.

Alternative A1 demonstrated strong brand identity representation and consistent adherence to brand guidelines, making it a reliable choice. Its scalability across different screen sizes further strengthened its performance. However, it showed moderate creativity compared to A7, which limited its ranking.

Alternative A6 performed well in aligning with the preferences and cultural values of the target audience. It also showed effective use of color harmony, creating a visually appealing design. Despite these strengths, it struggled with the practical functionality of its motion design, which slightly impacted on its overall ranking.

In contrast, Alternative A3 ranked the lowest, primarily due to its lack of visual appeal and memorability. It failed to adapt effectively to various platforms, and its slower loading speed hindered its usability. While it had a simplistic and technically compatible design, these factors were insufficient to elevate its ranking among the alternatives.

The performance analysis highlights the strengths and weaknesses of each alternative, providing valuable insights into refining dynamic logo designs. By addressing the identified shortcomings, designers can enhance the quality and effectiveness of their work, ensuring alignment with both technical requirements and user expectations.

4.2 Comparative analysis

The T2NN- Taxonomy technique is compared with distance measures under T2NN. The comparative results are described in Table 3.

Table 3. Order for different techniques

Techniques	Order
T2NN- VIKOR technique	A7>A6>A1>A2>A5>A4>A3
T2NN- TOPSIS technique	A7>A2>A6>A1>A4>A5>A3
T2NN- EDAS technique	A7>A1>A4>A6>A2>A5>A3
T2NN- Taxonomy technique	A7>A1>A6>A2>A4>A5>A3

Based on comparative analysis, the ranking order of the existing techniques is the same as the order produced by the T2NN- Taxonomy technique. This suggests that the ranking order of the various techniques is consistent, which verifies the rationality and effectiveness of the T2NN- Taxonomy approach. The key advantages of T2NN- Taxonomy technique is summarized: (1) T2NN- Taxonomy technique not only addresses the uncertainty inherent in MCDA problems, but also incorporates psychological behavioral factors; (2) T2NN- Taxonomy approach effectively combines and describes the behavior of Taxonomy and average methods when applied to MCDA.

4.3 Sensitivity Analysis

In this part we change the parameter values in the T2NN- Taxonomy methodology to show the stability of the rank under different cases. As is Eqs. (24 and 25) we change the parameter

to plus and mins into four cases. Then we rank the alternatives. The results show the rank of alternatives is stable in different cases. The rank of alternatives is shown in Table 4.

Table 4. Order for different cases

Cases	Order
Case 1	A7>A6>A1>A2>A5>A4>A3
Case 2	A7>A2>A6>A1>A4>A5>A3
Case 3	A7>A1>A4>A2>A6>A5>A3
Case 4	A7>A1>A6>A6>A2>A5>A3

5. Conclusion

In the digital media landscape, evaluating the quality of dynamic logo design is crucial for several reasons. Firstly, it ensures that the logo effectively captures and retains audience attention, which is vital for brand recognition. A well-designed dynamic logo can convey a brand's identity and values clearly, enhancing its image and consistency across platforms. Additionally, quality evaluation helps optimize technical performance, ensuring smooth animations that are compatible with various devices and platforms. This is essential for maintaining seamless user experience. Furthermore, by assessing user engagement, brands can create logos that foster interaction and leave a lasting impression. Overall, this evaluation strengthens brand presence and effectiveness in the digital space. The evaluation of dynamic logo design quality within the digital media context is MCDA. Consequently, the T2NN- Taxonomy technique is described to grapple with MCDA for evaluation of dynamic logo design quality within the digital media context. The key contribution of this research is described: (1) the novel MCDA is described based on Taxonomy and average technique under T2NN; (2) The objective weights are described through average technique; (3) The new MCDA technique based on T2NN- Taxonomy technique is described for evaluation of dynamic logo design quality within the digital media context; (4) numerical example for evaluation of dynamic logo design quality within the digital media context and comparative analysis are described to validate the T2NN-Taxonomy.

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Pseudo Functions in N-Cylindrical Fuzzy Neutrosophic Topological Spaces

Basker P [1,*]

[1] Associate Professor, Department of Mathematics, Chandigarh University, Punjab – 140 413. India.
E-mail: mcpdbasker@gmail.com and basker.e11236@cumail.in

* Correspondence: mcpdbasker@gmail.com and basker.e11236@cumail.in

Abstract: An interpretations supplied the study investigators the current research investigation is presented coupled with an exploration subsequent to trends to whatever individuals have provided. Let's start through a beginning to defined a new set in a space $nCy\mathbb{F}$ called an $nCy\bar{\alpha}$ ≀-incorporates the ideas of closed and open sets and examine the map of a function Δ from $nCy\mathbb{F}^1$ to $nCy\mathbb{F}^2$ has been stated to be $nCy{\sim}Pse(\bar{\alpha})$-OP, $nCy{\sim}Pse(\bar{\alpha})$-CL and decisively $nCy\bar{\alpha}$-$OpFn$ and offer an understanding of the achieved outcomes by introducing the notion of the spaces in a $nCyFNTS's$ is referred to as ${}_{nCy\bar{\alpha}}\delta^0$-space, ${}_{nCy\bar{\alpha}}\delta^1$-space and ${}_{nCy\bar{\alpha}}\delta^2$-space. In addition to the examples, assumptions, and theorems, the distance between two n-$CyFNS$, in conjunction with their attributes and fundamental operations, was defined.

Keywords: nCy-pseudo alpha open functions, nCy-pseudo alpha closed functions, nCy-delta spaces, nCy- decisively alpha functions, nCy $\alpha\delta$0-space, nCy $\alpha\delta$1-space, n-CyFNS.

1. Introduction

Topological spaces are a key notion in mathematics, notably in topology, which investigates the features of space that remain constant during Continuous distortions such as twisting and bending but not ripping either glueing. In layman's words, a topological space is a set that contains a collection of subsets known as "open sets" that meet certain qualities, such as proximity and continuity. These open sets determine the topology of the space and allow us to define concepts such as convergence, continuity, compactness, and connectedness, which are important in many fields of mathematics, including analysis, geometry, and algebraic topology. One of the distinguishing characteristics of topological spaces is their generality. Topological spaces, as opposed to more rigid geometric structures such as Euclidean spaces, may be very flexible and abstract, allowing mathematicians to examine a diverse variety of forms and structures, from common lines and planes to more exotic and complicated ones. Overall, topological spaces provide a rich framework for exploring the shape and structure of mathematical objects, and they act as a link between different branches of mathematics, allowing ideas and techniques from one to be applied to another.

FTS build on the notion of classical TS by integrating fuzziness or uncertainty into the definitions of open sets and other topological features. This addition is especially beneficial when dealing with circumstances where specific limits or distinctions are not clearly specified. In a fuzzy topological space, rather of having crisp, well-defined open sets, we have "fuzzy" open sets that assign a level of being a member to each point. The level of being a member specifies how much point belongs to the set. The higher the degree of membership, the closer the point is to being completely contained within the set. The formal definition of a fuzzy topological space entails replacing the traditional concept of open sets with fuzzy sets, which are distinguished by membership functions. These membership

functions assign a value between 0 and 1 to each point in the space, reflecting its degree of belonging to the set. Fuzzy topological spaces are used in a variety of domains, including fuzzy logic, approximation reasoning, pattern recognition, and decision making, when ambiguity and imprecision exist in the data or issue formulation. They provide an effective framework for dealing with ambiguous or partial information, as well as a more flexible and nuanced approach to modelling complex systems.

Intuitionistic fuzzy topological spaces are an extension of classical topological spaces that account for the inherent ambiguity and vagueness seen in real-world data and decision-making processes. This expansion provides intuitionistic fuzzy sets, which provide a more expressive approach to manage uncertainty than classical fuzzy sets. Unlike classical FS assign a single level of being a member to each element reflecting its degree of membership, a set, IFS include the ideas of being a member and none being a member degrees. They also provide a third parameter, the hesitation degree, which expresses the level of ambiguity or indecision about an element's membership status. Intuitionistic Fuzzy Topology, like classical topology, relies heavily on the idea of open sets. However, in intuitionistic fuzzy topology, open sets are defined as IFS. These sets have distinguished being a member, non-membership, hesitation functions, allowing for a more sophisticated portrayal of the thought of openness in space. IFTS have applications in a variety of disciplines where uncertainty and ambiguity play important roles, including decision making, pattern recognition, image processing, and expert systems. By giving a more detailed representation of uncertainty, they provide a strong foundation for modelling and analysing complex systems in a more realistic and flexible way.

Neutrosophic fuzzy topological spaces combine two important theories in mathematical modelling: neutrosophic set theory and fuzzy topology. These spaces handle both uncertainty and indeterminacy, making them very adaptable to real-world settings that include ambiguity. Neutrosophic theory of sets is a refinement of traditional set theory that includes aspects of uncertainty. In a neutrosophic the set, every component has one of the following values: truth, indeterminacy, or false being a member. Neutrosophic fuzzy topological spaces are a combination of two significant mathematical modelling theories: neutrosophic set theory and fuzzy topology. These spaces can manage both uncertainty and indeterminacy, making them well-suited to ambiguous actual events situations. Neutrosophic theory of sets is an adaptation on classical theory of sets that incorporates uncertainty. In a neutrosophic established every element is assigned one of three attributes: truth, indeterminacy, or false membership.

Zadeh [43] created the groundwork for the topic of uncertainty known as fuzzy sets. Topology was the primary discipline of mathematics where fuzzy set notions and ideas formed parallels. Chang [14] used Zadeh's notion to breathe new life into the concept of fuzzy topological spaces. Since then, several concepts from classical topology have been applied to fuzzy topological spaces. In the latter part of 1970 and the early part of 1980, numerous authors made significant contributions to this emerging discipline. Later, Atanassov [2], [3] created a new set known as the Intuitionistic Fuzzy Set (IFS), in which the total of the acceptance and rejection severity grades does not exceed one. Coker [15] later got IFTS via IFSs, while Lee [23] established the features of continuous, open, and closed maps.

In 2013, Yager [42] introduced the Pythagorean Fuzzy Set (PyFS) as an extension of IFS that assures that the square sum of its individual degrees is less than or equal to one. Olgun et al. and Parimala M et al. established the notion of a PNTS [24, 26, 27, 28]. Cuong [16] originated the concept of the Picture FS (PFS). He used three indices in PFS: membership degree P (x), neutral being a member degree I (x), and non-membership degree N (x), with the requirement that $0 \leq P(x) + I(x) + N(x) \leq 1$. Obviously, PFSs are better suited than IFS and PyFS for dealing with fuzziness and ambiguity. Razaq et al. [30] introduced the concept of image FTS. Later, Kahraman and Gündogdu

introduced spherical fuzzy sets (SFS) [18, 19]. SFS should meet the requirement that the squared total of being a member degree, none being a member degree, and hesitancy degree be equal to or less than one. Princy and Mohana [29] proposed spherical FTS.

Smarandache [33, 34] introduced the neutrosophic set, which is a generalisation of the IFS. Salama and Alblowi [36] developed the idea of NTS. They defined NTS as a generalisation of IFTS, as well as a neutrosophic set, in addition to each element's degree of membership, indeterminacy, and nonmembership. Smarandache [37,41] originally established the dependency degree (also known as the independence degree) of the fuzzy and neutrosophic components in 2006. P. Basker and Broumi Said [4,5,6,7,8,9,10,11,12,13,17,25,32,35,38,39] studied neutrosophic sets in the Applications of Sets and Functions by using an open set in Fuzzy Neutrosophic Topological Spaces, Neutrosophic Homeomorphisms, (β_{pn}) - OS in Pythagorean Neutrosophic Topological Spaces, $N\psi_{\alpha}^{\#\,0}$-spaces and $N\psi_{\alpha}^{\#\,1}$-spaces. $\mathrm{I}^{(\mathrm{T})}$α-open and $\mathrm{I}^{(\mathrm{T})}$β–open sets. Neighbourhoods. $\mathcal{M}_{\mathrm{X}}\alpha\delta\langle\widetilde{\mathcal{H}}\rangle$ in $\mathcal{M}$-structures. Single-valued neutrosophic graphs, Correlation coefficient of interval neutrosophic set, Neutrosophic soft matrices.

Arockiarani and Jency [1] proposed the concept of a fuzzy neutrosophic set in which the total of all three membership characteristics does not exceed 3. Veereswari presented a FNTS with fundamental operations [40]. Sarannya Kumari et al. [30, 31, 20, 21, 22] recently proposed n-Cylindrical Fuzzy Neutrosophic Sets, with T and F regarded as dependent portions and I as independent components. Except for fuzzy neutrosophic sets, the n-CyFNS is the most extensive extension of fuzzy sets. In this situation, the level to which any of the neutral, positive, and negative members meet the criteria, $0 \leq \beta A(x) < 1$ and $0 \leq \alpha An(x) + \gamma An(x) \leq 1$, $n > 1$, is an integer. They also specified the distance between two n-CyFNS, as well as their attributes and fundamental operations. In this study, we define topological space in the n-CyFNS environment. This is a novel form of fuzzy neutrosophic set, with dependent components T and F and independent components I. We defined n-CyFN topological space and n-CyFN open sets. We also started the n-CyFN base, n-CyFN subbase, and various related experiments. In this paper the following abbreviation are used OS-Open Set, CS-Closed Set, *nCyFNTS*-nCylindrical Fuzzy Neutrosophic Topological Spaces, *n-CyFNS*-nCylindrical Fuzzy Neutrosophic Sets. The following are the preliminary definitions.

Definition 1.1. An n-CyFNS N on S is an entity of the type $N = \{< i, \alpha N(i), \beta N(i), \gamma N(i) > \mid i \in S\}$ where $\alpha N(i) \in [0,1]$, termed the degree of positive membership. of i in N, $\beta N(i) \in [0,1]$, Known as the degree of neutral membership of i in N and $\gamma N(i) \in [0,1]$, termed the degree of negative membership. of i in N, which satisfies the condition, $(\forall\, i \in S)$ $(0 \leq \beta N(i) \leq 1$ and $0 \leq \alpha Nn(i) + \gamma Nn(i) \leq 1, n > 1$, is an integer. Here T and F are dependent neutrosophic components and I is 100% independent.
For the convenience, $< i, \alpha N(i), \beta N(i), \gamma N(i) >$ is called as n-Cylindrical Fuzzy Neutrosophic Number (n-CyFNN) and is denoted as $A =< i, \alpha N(i), \beta N(i), \gamma N(i) >$.

Definition 1.2. Inclusion: $\forall$ two $N, O \in C_n(S), N \subseteq O \iff (\forall\, i \in S, \alpha N(i) \leq \alpha O(i)$ and $\beta N(i) \leq \beta O(i)$ and $\gamma N(i) \geq \gamma O(i))$ and $N = O \iff (N \subseteq O$ and $O \subseteq N)$.

Definition 1.3. Union: $\forall$ two $N, O \in C_n(S)$, the union of two n-CyFNSs N and O is $N \cup O(i) = \{\langle i, max\,(\alpha N(i), \alpha O(i)), max\,(\beta N(i), \beta O(i)), min\,(\gamma N(i), \gamma O(i))\rangle \mid i \in S\}$.

Definition 1.4. Intersection: $\forall$ two $N, O \in C_n(S)$,, the intersection of two n-CyFNSs N and O is $N \cap O(i) = \{\langle i, min\,(\alpha N(i), \alpha O(i)), min\,(\beta N(i), \beta O(i)), max\,(\gamma N(i), \gamma O(i))\rangle \mid i \in S\}$.

Definition 1.5. Complementation: For every $N \in C_n(S)$, the complement of an n-CyFNS N is $N^C = \{< i,\ \gamma A(i), \beta A(i), \alpha A(i) > \mid i \in S\}$

2. $nCy{\sim}\bar{\bar{\alpha}} < on$-Pseudo functions

Definition 2.1. An n-Cylindrical subset Y of a space $nCy\mathbb{F}$ is called an $nCy\bar{\bar{\alpha}}$ ≀-open set ($nCy\bar{\bar{\alpha}}$ ≀ OS) if Y is a subsets of $nCyI\left(nCyC\left(nCyI(Y)\right)\right)$ and a $nCy\bar{\bar{\alpha}}$ ≀-closed set if $nCyC\left(nCyI\left(nCyC(Y)\right)\right)$ is a subset of Y.

Example 2.2. Let $C_y = \{c1, c2, c3\}$ and $\rho_{Cy} = \{1_{CyN}, 0_{CyN}, Y_i\}, i = 1,2,3, \dots 8$, where, $Y_i = \{< n_1; \Re_j >, < n_2; \mathfrak{H}_k >, < n_2; \Im_l >\};\ j,k,l = 1,2,3, \dots 8$

Y_i	$\Re_j$	$\mathfrak{H}_j$	$\Im_l$
Y_1	$\Re_1 = \frac{49}{400}, \frac{58}{400}, \frac{175}{400}$	$\mathfrak{H}_1 = \frac{125}{400}, \frac{74}{400}, \frac{82}{400}$	$\Im_1 = \frac{285}{400}, \frac{122}{400}, \frac{115}{400}$
Y_2	$\Re_2 = \frac{38}{400}, \frac{49}{400}, \frac{49}{400}$	$\mathfrak{H}_2 = \frac{174}{400}, \frac{95}{400}, \frac{112}{400}$	$\Im_2 = \frac{177}{400}, \frac{98}{400}, \frac{79}{400}$
Y_3	$\Re_3 = \frac{82}{400}, \frac{192}{400}, \frac{74}{400}$	$\mathfrak{H}_3 = \frac{325}{400}, \frac{118}{400}, \frac{98}{400}$	$\Im_3 = \frac{192}{400}, \frac{57}{400}, \frac{48}{400}$
Y_4	$\Re_4 = \frac{58}{400}, \frac{125}{400}, \frac{82}{400}$	$\mathfrak{H}_4 = \frac{311}{400}, \frac{277}{400}, \frac{97}{400}$	$\Im_4 = \frac{115}{400}, \frac{78}{400}, \frac{74}{400}$
Y_5	$\Re_5 = \frac{192}{400}, \frac{134}{400}, \frac{115}{400}$	$\mathfrak{H}_5 = \frac{49}{400}, \frac{301}{400}, \frac{88}{400}$	$\Im_5 = \frac{49}{400}, \frac{48}{400}, \frac{58}{400}$
Y_6	$\Re_6 = \frac{74}{400}, \frac{57}{400}, \frac{285}{400}$	$\mathfrak{H}_6 = \frac{54}{400}, \frac{157}{400}, \frac{99}{400}$	$\Im_6 = \frac{54}{400}, \frac{59}{400}, \frac{95}{400}$
Y_7	$\Re_7 = \frac{82}{400}, \frac{95}{400}, \frac{74}{400}$	$\mathfrak{H}_7 = \frac{68}{400}, \frac{168}{400}, \frac{108}{400}$	$\Im_7 = \frac{134}{400}, \frac{221}{400}, \frac{95}{400}$
Y_8	$\Re_8 = \frac{125}{400}, \frac{124}{400}, \frac{49}{400}$	$\mathfrak{H}_8 = \frac{115}{400}, \frac{198}{400}, \frac{49}{400}$	$\Im_8 = \frac{82}{400}, \frac{117}{400}, \frac{74}{400}$

Here $\left\{< n_1; \frac{58}{400}, \frac{125}{400}, \frac{82}{400} >, < n_2; \frac{311}{400}, \frac{277}{400}, \frac{97}{400} >, < n_2; \frac{115}{400}, \frac{78}{400}, \frac{74}{400} >\right\}$ is a subsets of $nCyI\left(nCyC\left(nCyI\left(\left\{< n_1; \frac{58}{400}, \frac{125}{400}, \frac{82}{400} >, < n_2; \frac{311}{400}, \frac{277}{400}, \frac{97}{400} >, < n_2; \frac{115}{400}, \frac{78}{400}, \frac{74}{400} >\right\}\right)\right)\right)$. Thus $\left\{< n_1; \frac{58}{400}, \frac{125}{400}, \frac{82}{400} >, < n_2; \frac{311}{400}, \frac{277}{400}, \frac{97}{400} >, < n_2; \frac{115}{400}, \frac{78}{400}, \frac{74}{400} >\right\}$ is a $nCy\bar{\bar{\alpha}}$ ≀ OS.

Definition 2.3. A map $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ It has been stated to be $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP if every instance of $nCy\bar{\bar{\alpha}}$ ≀ OS in $nCy\mathbb{F}^1$ is OS in $nCy\mathbb{F}^2$. It is noticeable the concepts exist $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP and $nCy\bar{\bar{\alpha}}$- enduring correspond if the bijection function is used.

Example 2.4. Let $C_y = \{c1, c2\}$ and $\rho_{Cy} = \{1_{CyN}, 0_{CyN}, Y_i\}, i = 1,2,3, \dots 8$, where, $Y_i = \{< n_1; \Re_j >, < n_2; \mathfrak{H}_k >, < n_2; \Im_l >\};\ j,k,l = 1,2,3, \dots 8$

Y_i	$\Re_j$	$\mathfrak{H}_j$
Y_1	$\Re_1 = \frac{5}{8}, \frac{17}{40}, \frac{13}{40}$	$\mathfrak{H}_1 = \frac{17}{40}, \frac{9}{40}, \frac{1}{8}$
Y_2	$\Re_2 = \frac{20}{40}, \frac{27}{40}, \frac{15}{40}$	$\mathfrak{H}_2 = \frac{10}{40}, \frac{8}{40}, \frac{3}{8}$
Y_3	$\Re_3 = \frac{15}{40}, \frac{14}{40}, \frac{11}{40}$	$\mathfrak{H}_3 = \frac{22}{40}, \frac{11}{40}, \frac{5}{40}$

Y_4	$\Re_4 = \frac{13}{40}, \frac{17}{40}, \frac{13}{40}$	$\Im_4 = \frac{13}{40}, \frac{7}{40}, \frac{25}{40}$
Y_5	$\Re_5 = \frac{25}{40}, \frac{17}{40}, \frac{13}{40}$	$\Im_5 = \frac{19}{40}, \frac{5}{40}, \frac{5}{40}$
Y_6	$\Re_6 = \frac{22}{40}, \frac{17}{40}, \frac{13}{40}$	$\Im_6 = \frac{10}{40}, \frac{18}{40}, \frac{15}{40}$
Y_7	$\Re_7 = \frac{11}{40}, \frac{17}{40}, \frac{13}{40}$	$\Im_7 = \frac{12}{40}, \frac{19}{40}, \frac{5}{40}$
Y_8	$\Re_8 = \frac{21}{40}, \frac{17}{40}, \frac{13}{40}$	$\Im_8 = \frac{1}{40}, \frac{13}{40}, \frac{7}{40}$

Clearly $(nC_y, nCy\mathbb{F}^1)$ is an $nCy\bar{\bar{\alpha}}$ ≀ open set and $(nC_y, nCy\mathbb{F}^2)$ is an open set. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is defined by $\Delta(c1) = n_2, \Delta(c2) = n_1$, then Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP.

Theorem 2.5. A map $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$ -OP $\Leftrightarrow$ $\forall$ portion $L^{\#}$ of $nCy\mathbb{F}^1$, $\Delta\left(nCy\bar{\bar{\alpha}}_{In}(L^{\#})\right) \subset nCyIN\left(\Delta(L^{\#})\right)$.

Proof: Here Δ be a $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP. As of presently, we've $nCyIN(L^{\#}) \subset L^{\#}$ and $nCy\bar{\bar{\alpha}}_{In}(L^{\#})$ is a $nCy\bar{\bar{\alpha}}$ ≀ OS . Thus, we emerge at that $\Delta\left(nCy\bar{\bar{\alpha}}_{In}(L^{\#})\right) \subset \Delta(L^{\#})$. As $\Delta\left(nCy\bar{\bar{\alpha}}_{In}(L^{\#})\right)$ is OS, $\Delta\left(nCy\bar{\bar{\alpha}}_{In}(L^{\#})\right) \subset nCyIN\left(\Delta(L^{\#})\right)$. On the other hand, suppose that $L^{\#}$ is a $nCy\bar{\bar{\alpha}}$ ≀ OS in $nCy\mathbb{F}^1$. Then, $\Delta$of $L^{\#}$is equal to Δ of $nCy\bar{\bar{\alpha}}_{In}$ of $L^{\#} \subset nCyIN\left(\Delta(L^{\#})\right)$ but $nCyIN$ of $\Delta(L^{\#})$ is a subset of $\Delta(L^{\#})$ Thereby, Δ of $L^{\#}$ is equal to $nCyIN$ of $\Delta(L^{\#})$ and consequently Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP.

Theorem 2.6. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$- OP, then $nCy\bar{\bar{\alpha}}_{In}\left(\Delta^{-1}(L^{\#})\right) \subset \Delta^{-1}\left(nCyIN(L^{\#})\right)$ $\forall$ portion $L^{\#}$ of $nCy\mathbb{F}^2$.

Proof. Here $L^{\#}$ be any unpredictability subset within $nCy\mathbb{F}^2$. Then, $nCy\bar{\bar{\alpha}}_{In}\left(\Delta^{-1}(L^{\#})\right)$ is a $nCy\bar{\bar{\alpha}}$ ≀ OS in $nCy\mathbb{F}^1$ and Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$ -OP, then $\Delta\left(nCy\bar{\bar{\alpha}}_{In}\left(\Delta^{-1}(L^{\#})\right)\right) \subset nCyIN\left(\Delta\left(\Delta^{-1}(L^{\#})\right)\right) \subset nCyIN(L^{\#})$. Thereby, $nCy\bar{\bar{\alpha}}_{In}\left(\Delta^{-1}(L^{\#})\right) \subset \Delta^{-1}\left(nCyIN(L^{\#})\right)$. Remember that a subset $K_!$ is referred to as a point l of $nCy\mathbb{F}^1$ is $nCy\bar{\bar{\alpha}}$ ≀-neighborhood, $\exists$ a $nCy\bar{\bar{\alpha}}$ ≀ OS L_* the kind that $l \in L_* \subset K_!$.

Theorem 2.7. Concerning a map $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$, Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP $\Leftrightarrow$ $\forall$ subset $L_1^{\#}$ of $nCy\mathbb{F}^1$, $\Delta(nCy\bar{\bar{\alpha}}_{In}) \subset nCyIN\left(\Delta(L_1^{\#})\right)$ $\Leftrightarrow$ $\forall$ $l \in nCy\mathbb{F}^1$ and $\forall$ $nCy\bar{\bar{\alpha}}$ ≀-neighbourhood $L_1^{\#}$ of l in $nCy\mathbb{F}^1$, $\exists$ a neighbourhood $\Delta(L_1^{\#})$ of $\Delta(l)$ in $nCy\mathbb{F}^2$ such that $L^{\#}{}_2 \subset \Delta(L_1^{\#})$.

Proof: Let $l \in nCy\mathbb{F}^1$ and $L_1^{\#}$ be an arbitrary $nCy\bar{\bar{\alpha}}$ ≀-neighbourhood $L_1^{\#}$ of l in $nCy\mathbb{F}^1$. Then $\exists$ a $nCy\bar{\bar{\alpha}}$ ≀ OS, $L^{\#}{}_2$ in $nCy\mathbb{F}^1$ such that $l \in L^{\#}{}_2$ in $L_1^{\#}$, we've Δ of $L_2^{\#}$ is a $\Delta(nCy\bar{\bar{\alpha}}_{In}) \subset nCyIN\left(\Delta(L^{\#}{}_2)\right)$ and hence $\Delta(L^{\#}{}_2) = nCyIN\left(\Delta(L^{\#}{}_2)\right)$. Therefore, it follows that $\Delta(L^{\#}{}_2)$ is OS in $nCy\mathbb{F}^2$ such that $\Delta(l) \in \Delta(L^{\#}{}_2) \subset \Delta(L_1^{\#})$.

Let $L_1^{\#}$ be an arbitrary $nCy\bar{\bar{\alpha}}$ ≀ OS in $nCy\mathbb{F}^1$. Then $\forall$ $m \in \Delta(L_1^{\#})$, $\exists$ a neighbourhood $L^{\#}{}_{2_m}$ of m in $nCy\mathbb{F}^2$ such that $L^{\#}{}_{2_m} \subset \Delta(L_1^{\#})$. As $L^{\#}{}_{2_m}$ is a neighbourhood of m, $\exists$ an OS $L^{\#}{}_{3_m}$ in $nCy\mathbb{F}^2$ such that $m \in L^{\#}{}_{3_m}$ is in $L^{\#}{}_{2_m}$. Thus Δ of $L_1^{\#}$ is a $\cup$ of $\left\{L^{\#}{}_{3_m} : m \in \Delta \text{ of } L_1^{\#}\right\}$ which is an OS in $nCy\mathbb{F}^2$.This implies that Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP.

Theorem 2.8. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP $\Leftrightarrow$ $\forall$ subset $D_{\#}$ of $nCy\mathbb{F}^2$ and $\forall$ $nCy\bar{\bar{\alpha}}$ ≀ CS, $P^{(1}$ of $nCy\mathbb{F}^1$ containing $\Delta^{-1}(D_{\#})$, $\exists$ a CS $P^{(2}$ of $nCy\mathbb{F}^2$ containing $D_{\#}$ ∣ $\Delta^{-1}(P^{(2}) \subset P^{(1}$.

Proof: In case Δ is $nCy\bar{\bar{\alpha}} \wr OS$. Let $D_{\#} \subset nCy\mathbb{F}^2$ and $P^{(1}$ be a $nCy\bar{\bar{\alpha}} \wr CS$ of $nCy\mathbb{F}^1$ containing $\Delta^{-1}(D_{\#})$. Now place $P^{(2} = nCy\mathbb{F}^2 - \Delta(nCy\mathbb{F}^1 - P^{(1})$. It is obvious that $\Delta^{-1}\ of\ P^{(2}\ is\ in\ P^{(1} \Longrightarrow D_{\#}$ is in $P^{(2}$. Since Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP, We acquire $P^{(2}$ as a CS of $nCy\mathbb{F}^2$. Furthermore, we've $\Delta^{-1}\left(P^{(2}\right) \subset P^{(1}$.

In contrast, let $L^{\#}$ be a $nCy\bar{\bar{\alpha}} \wr OS$ of $nCy\mathbb{F}^1$ and place $D_{\#} = nCy\mathbb{F}^2 \backslash \Delta(L^{\#})$. Then $nCy\mathbb{F}^1 \backslash L^{\#}$ is a $nCy\bar{\bar{\alpha}} \wr CS$ in set in $nCy\mathbb{F}^1$ containing $\Delta^{-1}(D_{\#})$. By supposition, $\exists$ a CS, $P^{(1}$ of $nCy\mathbb{F}^2 \mid D_{\#} \subset P^{(1}$ and $\Delta^{-1}\ of\ P^{(1} \subset nCy\mathbb{F}^1 \backslash L^{\#}$. Thus, we acquire $\Delta(L^{\#}) \subset nCy\mathbb{F}^2 \backslash P^{(1}$. Conversely, nevertheless, it implies that $D_{\#} \subset P^{(1}$, $nCy\mathbb{F}^2 \backslash P^{(1} \subset nCy\mathbb{F}^2 \backslash D_{\#} = \Delta(L^{\#})$. Hence, we get $\Delta(L^{\#}) = nCy\mathbb{F}^2 \backslash P^{(1}$ which is OS and hence Δ is a $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP function.

Theorem 2.9. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP $\Leftrightarrow$ $\Delta^{-1}\left(nCyCL(D_{\#})\right) \subset nCy\bar{\bar{\alpha}}_{Cl}\left(\Delta^{-1}(D_{\#})\right)$ $\forall$ subset $D_{\#}$ of $nCy\mathbb{F}^2$.

Proof: Suppose that Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$ -OP. For any subset $D_{\#}$ of $nCy\mathbb{F}^2$, $\Delta^{-1}(D_{\#}) \subset nCy\bar{\bar{\alpha}}_{Cl}\left(\Delta^{-1}(D_{\#})\right)$. Therefore, $\exists$ a CS $P^{(1}$ in $nCy\mathbb{F}^2 \mid D_{\#} \subset P^{(1}$ and $\Delta^{-1}\ of\ P^{(1} \subset nCy\bar{\bar{\alpha}}_{Cl}(\Delta^{-1}\ of\ D_{\#})$. Therefore, we obtain $\Delta^{-1}\left(nCyCL(D_{\#})\right) \subset \Delta^{-1}\left(P^{(1}\right) \subset nCy\bar{\bar{\alpha}}_{Cl}\left(\Delta^{-1}(D_{\#})\right)$.

In contrast, let $D_{\#} \subset nCy\mathbb{F}^2$ and Δ be a $nCy\bar{\bar{\alpha}} \wr CS$ of $nCy\mathbb{F}^1$ containing $\Delta^{-1}(D_{\#})$. Put $L^{\#}{}_3 = nCyCL_Y(D_{\#})$, then we have $D_{\#} \subset L^{\#}{}_3$ and $L^{\#}{}_3$ is CS and $\Delta^{-1}(L^{\#}{}_3) \subset nCy\bar{\bar{\alpha}}_{Cl}\left(\Delta^{-1}(D_{\#})\right) \subset P^{(1}$.Then, Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP.

Lemma 2.10. The two maps $\Delta_{\mathrm{I}}: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ and $\Delta_{\mathrm{II}}: nCy\mathbb{F}^2 \longrightarrow nCy\mathbb{F}^3$ and $\Delta_{\mathrm{II}} \circ \Delta_{\mathrm{I}}: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^3$ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-$OP$. If Δ_{II} is nCy-Cont. injective, then Δ_{I} is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-$OP$.

Proof: Let $L^{\#}$ be a $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$. Then $(\Delta_{\mathrm{II}} \circ \Delta_{\mathrm{I}})(L^{\#})$ is nCy-OS in $nCy\mathbb{F}^3$ since $\Delta_{\mathrm{II}} \circ \Delta_{\mathrm{I}}$ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP. Further Δ_{II} is an injective nCy-Cont. function, $\Delta_{\mathrm{I}}(L^{\#}) = \Delta_{\mathrm{I}}{}^{-1}(\Delta_{\mathrm{II}} \diamond \Delta_{\mathrm{I}})(L^{\#})$ is open in $nCy\mathbb{F}^2$. This demonstrates the fact Δ_{I} is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-$OP$.

Definition 2.11. A map $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is said to be $nCy{\sim}Pse(\bar{\bar{\alpha}})$-CL if the image of each $nCy\bar{\bar{\alpha}} \wr CS$ in $nCy\mathbb{F}^1$ is closed in $nCy\mathbb{F}^2$. Clearly, every $nCy{\sim}Pse(\bar{\bar{\alpha}})$-CL map is nCy-CS as well as $nCy\bar{\bar{\alpha}} \wr CS$.

Example 2.12. Let $C_y = \{c1, c2, c3\}$ and $\rho_{Cy} = \left\{1_{CyN}, 0_{CyN}, Y_i\right\}, i = 1,2,3, \ldots 8$, where,
$Y_i = \left\{< n_1; \Re_j >, < n_2; \Im_k >, < n_2; \Im_l >\right\};\ j, k, l = 1,2,3, \ldots 8$

Y_i	$\Re_j$	$\Im_j$	$\Im_l$
Y_1	$\Re_1 = \frac{103}{200}, \frac{95}{200}, \frac{110}{200}$	$\Im_1 = \frac{44}{200}, \frac{155}{200}, \frac{82}{200}$	$\Im_1 = \frac{112}{200}, \frac{93}{200}, \frac{55}{200}$
Y_2	$\Re_2 = \frac{112}{200}, \frac{87}{200}, \frac{111}{200}$	$\Im_2 = \frac{66}{200}, \frac{114}{200}, \frac{110}{200}$	$\Im_2 = \frac{101}{200}, \frac{98}{200}, \frac{110}{200}$
Y_3	$\Re_3 = \frac{42}{200}, \frac{63}{200}, \frac{121}{200}$	$\Im_3 = \frac{97}{200}, \frac{101}{200}, \frac{87}{200}$	$\Im_3 = \frac{136}{200}, \frac{101}{200}, \frac{122}{200}$
Y_4	$\Re_4 = \frac{13}{200}, \frac{93}{200}, \frac{39}{200}$	$\Im_4 = \frac{112}{200}, \frac{91}{200}, \frac{92}{200}$	$\Im_4 = \frac{112}{200}, \frac{97}{200}, \frac{111}{200}$
Y_5	$\Re_5 = \frac{31}{200}, \frac{67}{200}, \frac{54}{200}$	$\Im_5 = \frac{40}{200}, \frac{81}{200}, \frac{93}{200}$	$\Im_5 = \frac{121}{200}, \frac{79}{200}, \frac{121}{200}$

Y_6	$\Re_6 = \frac{53}{200}, \frac{73}{200}, \frac{67}{200}$	$\Im_6 = \frac{68}{200}, \frac{79}{200}, \frac{67}{200}$	$\aleph_6 = \frac{18}{200}, \frac{88}{200}, \frac{65}{200}$
Y_7	$\Re_7 = \frac{81}{200}, \frac{68}{200}, \frac{93}{200}$	$\Im_7 = \frac{69}{200}, \frac{78}{200}, \frac{79}{200}$	$\aleph_7 = \frac{89}{200}, \frac{97}{200}, \frac{79}{200}$
Y_8	$\Re_8 = \frac{47}{200}, \frac{55}{200}, \frac{73}{200}$	$\Im_8 = \frac{87}{200}, \frac{73}{200}, \frac{96}{200}$	$\aleph_8 = \frac{97}{200}, \frac{101}{200}, \frac{89}{200}$

Here $(nC_y, nCy\mathbb{F}^1)$ is $nCy\bar{\bar{\alpha}}\wr\ CS$ and $(nC_y, nCy\mathbb{F}^2)$ is CS. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is defined by $\Delta(c1) = n_2, \Delta(c2) = n_1$ and $\Delta(c3) = n_3$ then Δ is $nCy{\sim}Pse(\bar{\bar{\alpha}})$-CL.

Remark 2.13. Every $nCy\text{-}CS$ map need not be $nCy{\sim}Pse(\bar{\bar{\alpha}})$-CL as demonstrated by the subsequent example.

Example 2.14. Let $C_y = \{c1, c2, c3\}$ and $\rho_{Cy} = \{1_{CyN}, 0_{CyN}, Y_i\}, i = 1\ to\ 6$, where,
$Y_i = \{< n_1; \Re_j >, < n_2; \Im_k >, < n_2; \aleph_l >\};\ j, k, l = 1\ to\ 6$

Y_i	$\Re_j$	$\Im_j$	$\aleph_l$
Y_1	$\Re_1 = \frac{247}{2000}, \frac{447}{2000}, \frac{647}{2000}$	$\Im_1 = \frac{347}{2000}, \frac{147}{2000}, \frac{447}{2000}$	$\aleph_1 = \frac{47}{2000}, \frac{147}{2000}, \frac{347}{2000}$
Y_2	$\Re_2 = \frac{47}{2000}, \frac{47}{2000}, \frac{147}{2000}$	$\Im_2 = \frac{247}{2000}, \frac{147}{2000}, \frac{347}{2000}$	$\aleph_2 = \frac{17}{2000}, \frac{17}{2000}, \frac{57}{2000}$
Y_3	$\Re_3 = \frac{57}{2000}, \frac{47}{2000}, \frac{47}{2000}$	$\Im_3 = \frac{347}{2000}, \frac{447}{2000}, \frac{147}{2000}$	$\aleph_3 = \frac{147}{2000}, \frac{147}{2000}, \frac{247}{2000}$
Y_4	$\Re_4 = \frac{247}{2000}, \frac{57}{2000}, \frac{47}{2000}$	$\Im_4 = \frac{37}{2000}, \frac{147}{2000}, \frac{447}{2000}$	$\aleph_4 = \frac{247}{2000}, \frac{547}{2000}, \frac{447}{2000}$
Y_5	$\Re_5 = \frac{111}{2000}, \frac{212}{2000}, \frac{412}{2000}$	$\Im_5 = \frac{212}{2000}, \frac{111}{2000}, \frac{401}{2000}$	$\aleph_5 = \frac{97}{2000}, \frac{157}{2000}, \frac{347}{2000}$
Y_6	$\Re_6 = \frac{177}{2000}, \frac{421}{2000}, \frac{347}{2000}$	$\Im_6 = \frac{111}{2000}, \frac{212}{2000}, \frac{247}{2000}$	$\aleph_6 = \frac{347}{2000}, \frac{347}{2000}, \frac{347}{2000}$

Here $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is defined by $\Delta(c1) = n_2, \Delta(c2) = n_1$ and $\Delta(c3) = n_3$ then Δ is $nCy\text{-}CS$ map need not be $nCy{\sim}Pse(\bar{\bar{\alpha}})$-CL.

Theorem 2.15. The $nCy\mathbb{F}^1$ and $nCy\mathbb{F}^2$ be $n\text{-}CyFNTS's$ and the map $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is a $nCy{\sim}Pse(\bar{\bar{\alpha}})\text{-}CL \iff \Delta(nCy\mathbb{F}^1)$ is $nCy\text{-}CS$ in $nCy\mathbb{F}^2$ and $\Delta(L^\#)\backslash\Delta(nCy\mathbb{F}^1\backslash L^\#)$ is $nCy\text{-}OS$ in $\Delta(nCy\mathbb{F}^1)$ in situations where $L^\#$ is $nCy\bar{\bar{\alpha}}\wr OS$ in $nCy\mathbb{F}^1$.

Proof: In case $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is a $nCy{\sim}Pse(\bar{\bar{\alpha}})\text{-}CL$ map. Since $nCy\mathbb{F}^1$ is $nCy\bar{\bar{\alpha}}\wr CS$, $\Delta(nCy\mathbb{F}^1)$ is $nCy\text{-}CS$ in $nCy\mathbb{F}^2$ and $\Delta\ of\ L^\#\backslash\Delta\ of\ nCy\mathbb{F}^1\backslash L^\# = \Delta\ of\ L^\# \cap \Delta\ of\ nCy\mathbb{F}^1\backslash\Delta(nCy\mathbb{F}^1\backslash L^\#)$ is $nCy\text{-}OS$ in $\Delta(nCy\mathbb{F}^1)$ when $L^\#$ is $nCy\bar{\bar{\alpha}}\wr OS$ in $nCy\mathbb{F}^1$.

In case $\Delta(nCy\mathbb{F}^1)$ is $nCy\text{-}CS$ in $nCy\mathbb{F}^2$, $\Delta(L^\#)\backslash\Delta(nCy\mathbb{F}^1\backslash L^\#)$ is $nCy\text{-}OS$ in $\Delta(nCy\mathbb{F}^1)$ when $L^\#$ is $nCy\bar{\bar{\alpha}}\wr OS$ in $nCy\mathbb{F}^1$, and let $E_\#$ be $nCy\text{-}CS$ in $nCy\mathbb{F}^1$. Then $\Delta(E_\#) = \Delta(nCy\mathbb{F}^1)\backslash(\Delta(nCy\mathbb{F}^1\backslash E_\#)\backslash \Delta(E_\#))$ is $nCy\text{-}CS$ in $\Delta(nCy\mathbb{F}^1)$ and hence, $nCy\text{-}CS$ in $nCy\mathbb{F}^2$.

Corollary 2.16. The $nCy\mathbb{F}^1$ and $nCy\mathbb{F}^2$ be $n\text{-}CyFNTS's$ and let $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ be a $nCy\bar{\bar{\alpha}}$-ENDURING $nCy{\sim}Pse(\bar{\bar{\alpha}})\text{-}CL$ surjective map. Then n-CyFNT on $nCy\mathbb{F}^2$ is $\{\Delta(L^{\#})\backslash\Delta(nCy\mathbb{F}^1\backslash L^{\#}): L^{\#}$ is $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1\}$.

Proof: Let $Q_{\cdot}^{\#}$ be $nCy\text{-}OS$ in $nCy\mathbb{F}^2$. Then $\Delta^{-1}(Q_{\cdot}^{\#})$ is $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$, and $\Delta\left(\Delta^{-1}(Q_{\cdot}^{\#})\right)\backslash \Delta\left(nCy\mathbb{F}^1\backslash\Delta^{-1}(Q_{\cdot}^{\#})\right) = Q_{\cdot}^{\#}$ As a result, all $nCy\text{-}OS$ in $nCy\mathbb{F}^2$ are in the form of $\Delta(L^{\#})\backslash\Delta(nCy\mathbb{F}^1\backslash L^{\#})$, $L^{\#}$ is $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$. In contrast, all sets of the kind $\Delta(L^{\#})\backslash\Delta(nCy\mathbb{F}^1\backslash L^{\#})$, $L^{\#}$ is $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$, are $nCy\text{-}OS$ in $nCy\mathbb{F}^2$.

Definition 2.17. A $nCyFNTS's$ $nCy\mathbb{F}^1$ is referred to as ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space If each set of two unique points f_1 and f_2 of $nCy\mathbb{F}^1$, $\exists$ a $nCy\bar{\bar{\alpha}} \wr OS$ having one point but excluding the other.

Theorem 2.18. A $nCyFNTS's$ $nCy\mathbb{F}^1$ is a ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space $\Leftrightarrow$ $nCy\bar{\bar{\alpha}}$-closures of unique points are unique.

Proof: Let f_1 and f_2 be unique points of $nCy\mathbb{F}^1$. Since $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space, $\exists$ a $nCy\bar{\bar{\alpha}} \wr OS$ $M_{\#\#}$ such that $f_1 \in M_{\#\#}$ and $f_2 \notin M_{\#\#}$. thereby, $nCy\mathbb{F}^1 - M_{\#\#}$ is a $nCy\bar{\bar{\alpha}} \wr CS$ containing f_2 but not f_1. But $nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$ is the $\cap$ of all $nCy\bar{\bar{\alpha}} \wr CS's$ containing f_2. Thus $f_2 \in nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$ But $f_1 \notin nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$ as $f_1 \notin nCy\mathbb{F}^1 - M_{\#\#}$. Therefore, $nCy\bar{\bar{\alpha}}_{Cl}\{f_1\} \neq nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$.

In contrast, let $nCy\bar{\bar{\alpha}}_{Cl}\{f_1\} \neq nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$ for $f_1 \neq f_2 \implies \exists$ a minimum of a single point $f_3 \in nCy\mathbb{F}^1$ such that $f_3 \in nCy\bar{\bar{\alpha}}_{Cl}\{f_1\}$ but $f_3 \notin nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$. We assert $f_1 \notin nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$, because if $f_1 \in nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$ then $\{f_1\} \subset nCy\bar{\bar{\alpha}}_{Cl}\{f_2\} \implies nCy\bar{\bar{\alpha}}_{Cl}\{f_1\} \subset nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$. So $f_3 \in nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$, which contradicts itself. Hence $f_1 \notin nCy\bar{\bar{\alpha}}_{Cl}\{f_2\} \Longrightarrow f_1 \in nCy\mathbb{F}^1 - nCy\bar{\bar{\alpha}}_{Cl}\{f_2\}$ which is a $nCy\bar{\bar{\alpha}} \wr OS$ having just f_1 and not f_2. Hence $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space.

Definition 2.19. A map $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is referred to as decisively $nCy\bar{\bar{\alpha}}\text{-}OpFn.$ if the image of every $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$ is OS in $nCy\mathbb{F}^2$

Example 2.20. Let $C_y = \{c1, c2, c3\}$ and $\rho_{Cy} = \{1_{CyN}, 0_{CyN}, Y_i\}, i = 1,2,3, \ldots 8$, where,
$Y_i = \{< n_1; \Re_j >, < n_2; \Im_k >, < n_2; \Im_l >\};\ j,k,l = 1,2,3, \ldots 8$

Y_i	$\Re_j$	$\Im_j$	$\Im_l$
Y_1	$\Re_1 = \frac{49}{400}, \frac{58}{400}, \frac{175}{400}$	$\Im_1 = \frac{125}{400}, \frac{74}{400}, \frac{82}{400}$	$\Im_1 = \frac{285}{400}, \frac{122}{400}, \frac{115}{400}$
Y_2	$\Re_2 = \frac{38}{400}, \frac{49}{400}, \frac{49}{400}$	$\Im_2 = \frac{174}{400}, \frac{95}{400}, \frac{112}{400}$	$\Im_2 = \frac{177}{400}, \frac{98}{400}, \frac{79}{400}$
Y_3	$\Re_3 = \frac{82}{400}, \frac{192}{400}, \frac{74}{400}$	$\Im_3 = \frac{325}{400}, \frac{118}{400}, \frac{98}{400}$	$\Im_3 = \frac{192}{400}, \frac{57}{400}, \frac{48}{400}$
Y_4	$\Re_4 = \frac{58}{400}, \frac{125}{400}, \frac{82}{400}$	$\Im_4 = \frac{311}{400}, \frac{277}{400}, \frac{97}{400}$	$\Im_4 = \frac{115}{400}, \frac{78}{400}, \frac{74}{400}$
Y_5	$\Re_5 = \frac{192}{400}, \frac{134}{400}, \frac{115}{400}$	$\Im_5 = \frac{49}{400}, \frac{301}{400}, \frac{88}{400}$	$\Im_5 = \frac{49}{400}, \frac{48}{400}, \frac{58}{400}$
Y_6	$\Re_6 = \frac{74}{400}, \frac{57}{400}, \frac{285}{400}$	$\Im_6 = \frac{54}{400}, \frac{157}{400}, \frac{99}{400}$	$\Im_6 = \frac{54}{400}, \frac{59}{400}, \frac{95}{400}$
Y_7	$\Re_7 = \frac{82}{400}, \frac{95}{400}, \frac{74}{400}$	$\Im_7 = \frac{68}{400}, \frac{168}{400}, \frac{108}{400}$	$\Im_7 = \frac{134}{400}, \frac{221}{400}, \frac{95}{400}$

Y_8	$\Re_8 = \frac{125}{400}, \frac{124}{400}, \frac{49}{400}$	$\Im_8 = \frac{115}{400}, \frac{198}{400}, \frac{49}{400}$	$\Im_8 = \frac{82}{400}, \frac{117}{400}, \frac{74}{400}$

Clearly $(nC_y, nCy\mathbb{F}^1)$ is an $nCy\bar{\bar{\alpha}}\wr OS$ and $(nC_y, nCy\mathbb{F}^2)$ is a OS. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is defined by $\Delta(c1) = n_1, \Delta(c2) = n_2$ and $\Delta(c3) = n_3$ then Δ is $nCy\bar{\bar{\alpha}}$-$OpFn$..

Theorem 2.21. In a bijection $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is a decisively $nCy\bar{\bar{\alpha}}$-$OpFn$. and $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space, then $nCy\mathbb{F}^2$ is also ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space.

Proof: Let g_1 and g_2 be two separate points of $nCy\mathbb{F}^2$. Since Δ is bijective $\exists$ distinct points f_1 and f_2 of $nCy\mathbb{F}^1 \mid \Delta(f_1) = g_1$ and $\Delta(f_2) = g_2$. Since $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space $\exists$ a $nCy\bar{\bar{\alpha}}\wr OS$, $G \mid x_1 \in G$ and $x_2 \notin G$. Therefore $g_1 = \Delta\, of\, f_1 \in \Delta\, of\, G$ and $g_2 = \Delta\, of\, f_2 \notin \Delta\, of\, G$. Since Δ being decisively $nCy\bar{\bar{\alpha}}$-$OpFn$., $\Delta(G)$ is $nCy\bar{\bar{\alpha}}\wr OS$ in $nCy\mathbb{F}^2$. Thus, $\exists$ a $nCy\bar{\bar{\alpha}}\wr OS$ $\Delta(G)$ in $nCy\mathbb{F}^2 \mid g_1 \in \Delta(G)$ and $g_2 \notin \Delta(G)$. Therefore $nCy\mathbb{F}^2$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space.

Definition 2.22. An $nCyFNTS's$ $nCy\mathbb{F}^1$ is referred to as ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space In the event that any two distinct points f and g, $\exists$ a $nCy\bar{\bar{\alpha}}\wr OS$ $G_\#$ and $H_\# \mid f\ is\ G_\#$, $g\ is\ not\ G_\#$ and $f\ is\ not\ H_\#$, $g\ is\ H_\#$.

Theorem 2.23. An $nCyFNTS's$ $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space $\Leftrightarrow$ singletons are $nCy\bar{\bar{\alpha}}\wr CS$.

Proof: Let $nCy\mathbb{F}^1$ be a ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space and $f \in nCy\mathbb{F}^1$. Let $g \in nCy\mathbb{F}^1 - \{f\}$. Then for $f \neq g$, $\exists$ $nCy\bar{\bar{\alpha}}\wr OS$ ${L^\#}_g$ that $g \in {L^\#}_g$ and $f\ is\ not\ {L^\#}_g$. Furthermore, $g\ is\ {L^\#}_g \subset nCy\mathbb{F}^1 - \{f\}$. That is $nCy\mathbb{F}^1 - \{f\} = \cup \left\{{L^\#}_g : g \in nCy\mathbb{F}^1 - \{f\}\right\}$, which is $nCy\bar{\bar{\alpha}}\wr OS$. Hence $\{f\}$ is $nCy\bar{\bar{\alpha}}\wr CS$.

In contrast, suppose $\{f\}$ is $nCy\bar{\bar{\alpha}}\wr CS\ \forall\ f \in nCy\mathbb{F}^1$. Let f and $g \in nCy\mathbb{F}^1$ with $f\ is\ not\ equal\ to\ g$. Now $f\ is\ not\ equal\ to\ g \Rightarrow g \in nCy\mathbb{F}^1 - \{f\}$. Thereby $nCy\mathbb{F}^1 - \{f\}$ is $nCy\bar{\bar{\alpha}}\wr CS$ containing g but not f. Similarly, $nCy\mathbb{F}^1 - \{g\}$ is $nCy\bar{\bar{\alpha}}\wr CS$ containing f but not g. Thus $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space.

Theorem 2.24. The object being ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space remains intact under bijection and decisively $nCy\bar{\bar{\alpha}}$-$OpFn$.

Proof: Let $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ possess bijection and decisively $nCy\bar{\bar{\alpha}}$-$OpFn$. Let $nCy\mathbb{F}^1$ be a ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space and l_1, l_2 be any 2-unique points of $nCy\mathbb{F}^2$. Since Δ is bijective $\exists$ unique points k_1 k_2 of $nCy\mathbb{F}^1 \mid l_1 = \Delta(k_1)$ and $l_2 = \Delta(k_2)$. Now $nCy\mathbb{F}^1$ being a ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space, $\exists$ $nCy\bar{\bar{\alpha}}\wr OS$ P_{t1} and $P_{t2} \mid k_1 \in P_{t1}$, $k_2 \notin P_{t1}$ and $k_1 \notin P_{t2}$, $k_2 \in P_{t2}$. Thereby $l_1 = \Delta(k_1) \in \Delta(P_{t1})$ but $l_2 = \Delta(k_2) \notin \Delta(P_{t1})$ and $l_2 = \Delta(k_2) \in \Delta(P_{t2})$ and $l_1 = \Delta(k_1) \notin \Delta(P_{t2})$. Here Δ possess decisively $nCy\bar{\bar{\alpha}}$-$OpFn$., $\Delta(P_{t1})$ and $\Delta(P_{t2})$ are $nCy\bar{\bar{\alpha}}\wr OS$ of $nCy\mathbb{F}^2 \mid l_1 \in \Delta(P_{t1})$ but $l_2 \notin \Delta(P_{t1})$ and $l_2 \in \Delta(P_{t2})$ and $l_1 \notin \Delta(P_{t2})$. Thus $nCy\mathbb{F}^2$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space.

Theorem 2.25. Let $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ be bijective and decisively $nCy\bar{\bar{\alpha}}$-$OpFn$. In case $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space $\Longrightarrow nCy\mathbb{F}^2$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space.

Proof: Let g_1, g_2 be any 2-unique points of $nCy\mathbb{F}^2$. Since Δ is bijective $\exists$ unique points f_1, f_2 of $nCy\mathbb{F}^1 \mid g_1 = \Delta(f_1)$ and $g_2 = \Delta(f_2)$. Now $nCy\mathbb{F}^1$ being a ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space, $\exists$ $nCy\bar{\bar{\alpha}}\wr OS$ $C_\#$ and $D_\#$ that $f_1\ in\ C_\#$, $f_2\ not\ in\ C_\#$ and $f_1\ not\ in\ D_\#$, $f_2\ in\ D_\#$. Thereby $g_1 = \Delta(f_1) \in \Delta$ of $C_\#$but $g_2 = \Delta$ of $f_2 \notin \Delta\, of\, C_\#$ and $y_2 = \Delta$ of $(f_2) \in \Delta$ of $(D_\#)$ and $g_1 = \Delta$ of $(f_1)\ not\ in\ \Delta(D_\#)$. Now $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space $\Longrightarrow C_\#$ and $D_\#$ are OS in $nCy\mathbb{F}^1$ and Δ is decisively $nCy\bar{\bar{\alpha}}$-$OpFn$., $\Delta(C_\#)$ and $\Delta(D_\#)$ are $nCy\bar{\bar{\alpha}}\wr OS$ of $nCy\mathbb{F}^2$. Thus $\exists$ $nCy\bar{\bar{\alpha}}\wr OS \mid g_1 \in \Delta(C_\#)$ but $g_2 \notin \Delta(C_\#)$ and $g_2 \in \Delta(D_\#)$ and $g_1 \notin \Delta(D_\#)$. Thus $nCy\mathbb{F}^2$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space.

Example 2.26. Let $C_y = \{c1, c2, c3\}$ and $\rho_{Cy} = \{1_{CyN}, 0_{CyN}, Y_i\}, i = 1,2,3, \ldots 8$, where,

$Y_i = \{< n_1; \Re_j >, < n_2; \Im_k >\};\ j,k = 1,2,3,\dots 8$

Y_i	$\Re_j$	$\Im_j$
Y_1	$\Re_1 = \frac{15}{40}, \frac{11}{40}, \frac{9}{40}$	$\Im_1 = \frac{15}{40}, \frac{17}{40}, \frac{13}{40}$
Y_2	$\Re_2 = \frac{22}{40}, \frac{27}{40}, \frac{5}{40}$	$\Im_2 = \frac{11}{40}, \frac{17}{40}, \frac{13}{40}$
Y_3	$\Re_3 = \frac{27}{40}, \frac{17}{40}, \frac{21}{40}$	$\Im_3 = \frac{23}{40}, \frac{13}{40}, \frac{15}{40}$
Y_4	$\Re_4 = \frac{18}{40}, \frac{13}{40}, \frac{13}{40}$	$\Im_4 = \frac{7}{40}, \frac{17}{40}, \frac{21}{40}$
Y_5	$\Re_5 = \frac{19}{40}, \frac{16}{40}, \frac{23}{40}$	$\Im_5 = \frac{15}{40}, \frac{15}{40}, \frac{17}{40}$
Y_6	$\Re_6 = \frac{15}{40}, \frac{11}{40}, \frac{13}{40}$	$\Im_6 = \frac{11}{40}, \frac{17}{40}, \frac{17}{40}$
Y_7	$\Re_7 = \frac{9}{40}, \frac{12}{40}, \frac{8}{40}$	$\Im_7 = \frac{13}{40}, \frac{13}{40}, \frac{18}{40}$
Y_8	$\Re_8 = \frac{11}{40}, \frac{13}{40}, \frac{11}{40}$	$\Im_8 = \frac{18}{40}, \frac{11}{40}, \frac{15}{40}$

Clearly $(nC_y, nCy\mathbb{F}^1)$ is an ${}_{nCy\bar{\alpha}}\delta^1$-space and $(nC_y, nCy\mathbb{F}^2)$ is an ${}_{nCy\bar{\alpha}}\delta^1$-space. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is defined by $\Delta(c1) = n_2, \Delta(c2) = n_1$ and $\Delta(c3) = n_3$, then Δ is decisively $nCy\bar{\alpha}$-$OpFn$.

Theorem 2.27. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy\bar{\alpha}$- ENDURING injection and $nCy\mathbb{F}^2$ is $nCy\delta_{-1}$ then $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\alpha}}\delta^1$-space.

Proof: Let $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ be $nCy\bar{\alpha}$-ENDURING injection and $nCy\mathbb{F}^2$ is $nCy\delta_{-1}$. The 2-unique points n_1, n_2 of $nCy\mathbb{F}^1 \exists$ unique points o_1, o_2 of $nCy\mathbb{F}^2 \mid o_1 = \Delta(n_1)$ and $o_2 = \Delta$ of (n_2). Since $nCy\mathbb{F}^2$ is $nCy\delta_{-1}$-space, $\exists$ OS $L^{\#(1}$ and $L^{\#(2}$ in $nCy\mathbb{F}^2 \mid o_1 \in L^{\#(1}$, $o_2 \notin L^{\#(1}$ and $o_1 \notin L^{\#(2}$, $o_2 \in L^{\#(2}$. i.e., $n_1 \in \Delta^{-1}$of $\left(L^{\#(1}\right)$, $n_1 \notin \Delta^{-1}$ of $\left(L^{\#(2}\right)$ and $n_2 \in \Delta^{-1}$ of $\left(L^{\#(2}\right)$, $n_2 \notin \Delta^{-1}$ of $\left(L^{\#(1}\right)$. Since Δ is $nCy\bar{\alpha}$-ENDURING Δ^{-1} $of\left(L^{\#(1}\right)$, $\Delta^{-1}of\left(L^{\#(2}\right)$ are $nCy\bar{\alpha} \wr OS$ in $nCy\mathbb{F}^1$. Thus, the 2-unique points n_1, n_2 of $nCy\mathbb{F}^1 \ \exists\ nCy\bar{\alpha} \wr OS, \Delta^{-1}of\left(L^{\#(1}\right)$ and $\Delta^{-1}of\left(L^{\#(2}\right) \mid n_1 \in \Delta^{-1}of\left(L^{\#(1}\right)$, $n_1 \notin \Delta^{-1}of\left(L^{\#(2}\right)$ and $n_2 \in \Delta^{-1}of\left(L^{\#(2}\right)$, $n_2 \notin \Delta^{-1}\left(L^{\#(1}\right)$. Thereby $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\alpha}}\delta^1$-space.

Theorem 2.28. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy\bar{\alpha}$-IR injective function and $nCy\mathbb{F}^2$ is ${}_{nCy\bar{\alpha}}\delta^1$-space then $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\alpha}}\delta^1$-space.

Proof: Let n_1, n_2 be pair of unique points in $nCy\mathbb{F}^1$. Since Δ is injective, $\exists$ unique points o_1, o_2 of $nCy\mathbb{F}^2 \mid o_1 = \Delta(n_1)$ and $o_2 = \Delta(n_2)$. Since $nCy\mathbb{F}^2$ is ${}_{nCy\bar{\alpha}}\delta^1$-space, $\exists\ nCy\bar{\alpha} \wr OS$, $Qs^{\#1}$ and $Qs^{\#2}$ in $nCy\mathbb{F}^2 \mid o_1 \in Qs^{\#1}$, $o_2 \notin Qs^{\#1}$ and $o_1 \notin Qs^{\#2}$, $o_2 \in Qs^{\#2}$. i.e., $n_1 \in \Delta^{-1}(Qs^{\#1})$, $n_1 \notin \Delta^{-1}(Qs^{\#2})$ and $n_2 \in \Delta^{-1}(Qs^{\#2})$, $n_2 \notin \Delta^{-1}(Qs^{\#1})$. Since Δ is $nCy\bar{\alpha}$-IR, $\Delta^{-1}(Qs^{\#1})$, $\Delta^{-1}(Qs^{\#2})$ are $nCy\bar{\alpha} \wr OS$ in $nCy\mathbb{F}^1$. Thus, the 2-unique points n_1, n_2 of $nCy\mathbb{F}^1 \ \exists\ nCy\bar{\alpha} \wr OS$, $\Delta^{-1}(Qs^{\#1})$ and $\Delta^{-1}(Qs^{\#2}) \mid n_1 \in \Delta^{-1}(Qs^{\#1})$, $n_1 \notin \Delta^{-1}(Qs^{\#2})$ and $n_2 \in \Delta^{-1}(Qs^{\#2})$, $n_2 \notin \Delta^{-1}(Qs^{\#1})$. Thereby $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\alpha}}\delta^1$-space.

Definition 2.29. An $nCyPNTS$ is referred to as ${}_{nCy\bar{\alpha}}\delta^2$-space if for any pair of unique points c^1 & c^2, $\exists$ disjoint $nCy\bar{\alpha} \wr OS\ N^{l1}$ & $N^{l2} \mid c^1 \in N^{l1}$ & $c^2 \in N^{l2}$.

Theorem 2.30. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy\bar{\bar{\alpha}}$-ENDURING injection and $nCy\mathbb{F}^2$ is $nCy\delta_{-2}$ then $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space.

Proof: Let $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ be $nCy\bar{\bar{\alpha}}$-ENDURING injection and $nCy\mathbb{F}^2$ is $nCy\delta_{-2}$. The 2- unique points $e_{1,}\ e_2$ of $nCy\mathbb{F}^1\ \exists$ unique points $f_{1,}\ f_2$ of $nCy\mathbb{F}^2 \mid f_1 = \Delta(e_1)$ and $f_2 = \Delta(e_2)$. Since $nCy\mathbb{F}^2$ is $nCy\delta_{-2}$-space, $\exists$ distinct $nCyOS\ \ De_{\#1}$ and $De_{\#2}$ in $nCy\mathbb{F}^2 \mid f_1 \in De_{\#1},\ \Delta_2 \in De_{\#2}$. i.e., $e_{1,} \in \Delta^{-1}\ of(De_{\#1})$ and $e_2 \in \Delta^{-1}\ of(De_{\#2})$. Since Δ is $nCy\bar{\bar{\alpha}}$-ENDURING $\Delta^{-1}\ of(De_{\#1})$, $\Delta^{-1}\ of(De_{\#2})$ are $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$. Further Δ is injective, $\Delta^{-1}\ of(De_{\#1}) \cap \Delta^{-1}\ of(De_{\#2}) = \Delta^{-1}\ of(De_{\#1} \cap De_{\#2}) = \Delta^{-1}\ of(\phi) = \phi$. Thus, for 2-unique points $e_{1,}\ e_2$ of $nCy\mathbb{F}^1\ \exists$ disjoint $nCy\bar{\bar{\alpha}} \wr OS$, $\Delta^{-1}\ of(De_{\#1})$ and $\Delta^{-1}\ of(De_{\#2})$ such that $e_1 \in \Delta^{-1}\ of(De_{\#1})$ and $e_2 \in \Delta^{-1}\ of(De_{\#2})$. Therefore $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space.

Theorem 2.31. If $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is $nCy\bar{\bar{\alpha}}$-IR injective function and $nCy\mathbb{F}^2$ is and ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space then $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space.

Proof: Let $e_{1,}\ e_2$ be pair of unique points in $nCy\mathbb{F}^1$. Since Δ is injective $\exists$ distinct points $f_{1,}\ f_2$ of $nCy\mathbb{F}^2 \mid f_1 = \Delta(e_1)$ and $f_2 = \Delta(e_2)$. Since $nCy\mathbb{F}^2$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space $\exists\ nCy\bar{\bar{\alpha}} \wr OS$, $De_{\#1}$ and $De_{\#2}$ in $nCy\mathbb{F}^2 \mid f_1 \in De_{\#1},\ f_2 \in De_{\#2}$. i.e., $e_{1,} \in \Delta^{-1}of(De_{\#1})$ and $e_2 \in \Delta^{-1}of(De_{\#2})$. Since Δ is $nCy\bar{\bar{\alpha}}$-IR injective $\Delta^{-1}of(De_{\#1})$, $\Delta^{-1}of(De_{\#2})$ are $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$.Thus, for 2-unique points $e_{1,}\ e_2$ of $nCy\mathbb{F}^1\ \ \exists\ nCy\bar{\bar{\alpha}} \wr OS$, $\Delta^{-1}of(De_{\#1})$ and $\Delta^{-1}of(De_{\#2}) \mid e_1 \in \Delta^{-1}of(De_{\#1})$ and $e_2 \in \Delta^{-1}of(De_{\#2})$. Thereby $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space.

Example 2.32. Let $C_y = \{c1, c2, c3\}$ and $\rho_{Cy} = \{1_{CyN}, 0_{CyN}, Y_i\}, i = 1,2,3, \ldots 8$, where,
$Y_i = \{< n_1; \Re_j >, < n_2; \Im_k >, < n_2; \Im_l >\};\ j, k, l = 1,2,3, \ldots 8$

Y_i	$\Re_j$	$\Im_j$	$\Im_l$
Y_1	$\Re_1 = \frac{117}{2000}, \frac{223}{2000}, \frac{417}{2000}$	$\Im_1 = \frac{417}{2000}, \frac{113}{2000}, \frac{112}{2000}$	$\Im_1 = \frac{223}{2000}, \frac{89}{2000}, \frac{418}{2000}$
Y_2	$\Re_2 = \frac{407}{2000}, \frac{421}{2000}, \frac{417}{2000}$	$\Im_2 = \frac{147}{2000}, \frac{101}{2000}, \frac{101}{2000}$	$\Im_2 = \frac{223}{2000}, \frac{117}{2000}, \frac{155}{2000}$
Y_3	$\Re_3 = \frac{152}{2000}, \frac{147}{2000}, \frac{153}{2000}$	$\Im_3 = \frac{147}{2000}, \frac{112}{2000}, \frac{152}{2000}$	$\Im_3 = \frac{112}{2000}, \frac{223}{2000}, \frac{155}{2000}$
Y_4	$\Re_4 = \frac{223}{2000}, \frac{217}{2000}, \frac{89}{2000}$	$\Im_4 = \frac{89}{2000}, \frac{89}{2000}, \frac{417}{2000}$	$\Im_4 = \frac{117}{2000}, \frac{101}{2000}, \frac{203}{2000}$
Y_5	$\Re_5 = \frac{101}{2000}, \frac{217}{2000}, \frac{153}{2000}$	$\Im_5 = \frac{101}{2000}, \frac{117}{2000}, \frac{417}{2000}$	$\Im_5 = \frac{147}{2000}, \frac{117}{2000}, \frac{112}{2000}$
Y_6	$\Re_6 = \frac{112}{2000}, \frac{223}{2000}, \frac{418}{2000}$	$\Im_6 = \frac{147}{2000}, \frac{417}{2000}, \frac{113}{2000}$	$\Im_6 = \frac{112}{2000}, \frac{223}{2000}, \frac{417}{2000}$
Y_7	$\Re_7 = \frac{101}{2000}, \frac{203}{2000}, \frac{153}{2000}$	$\Im_7 = \frac{117}{2000}, \frac{417}{2000}, \frac{203}{2000}$	$\Im_7 = \frac{117}{2000}, \frac{223}{2000}, \frac{101}{2000}$
Y_8	$\Re_8 = \frac{407}{2000}, \frac{203}{2000}, \frac{155}{2000}$	$\Im_8 = \frac{421}{2000}, \frac{112}{2000}, \frac{407}{2000}$	$\Im_8 = \frac{417}{2000}, \frac{112}{2000}, \frac{203}{2000}$

Clearly $(nC_y, nCy\mathbb{F}^1)$ is an ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space and $(nC_y, nCy\mathbb{F}^2)$ is an ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space. Here $\Delta: nCy\mathbb{F}^1 \longrightarrow nCy\mathbb{F}^2$ is defined by $\Delta(c1) = n_1, \Delta(c2) = n_2$ and $\Delta(c3) = n_3$ then Δ is $nCy\bar{\bar{\alpha}}$-IR injective function.

Theorem 2.33. An $nCyPNTS$, $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space $\Leftrightarrow$ For each $e \neq f$, $\exists$ a $nCy\bar{\bar{\alpha}} \wr OS$, $De_{\#1} \mid e \in De_{\#1}$ and $f \notin nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1}) \Leftrightarrow$ For each $e \in nCy\mathbb{F}^1$, $\{e\} = \cap \{nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1}) : De_{\#1}$ is a $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$ and $e \in De_{\#1}\}$.

Proof: Let $e \in nCy\mathbb{F}^1$ and $e \neq f$, $\exists$ disjoint $nCy\bar{\bar{\alpha}} \wr OS$, $De_{\#1}$ & $De_{\#2} \mid e \in De_{\#1}$ & $f \in De_{\#2}$. Clearly, $nCy\mathbb{F}^1 - De_{\#2}$ is $nCy\bar{\bar{\alpha}} \wr CS$. Since $De_{\#1} \cap De_{\#2} = \phi$, $De_{\#1} \subset nCy\mathbb{F}^1 - De_{\#2}$. Thereby $nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1}) \subset nCy\bar{\bar{\alpha}}_{Cl}(nCy\mathbb{F}^1 - De_{\#2}) = nCy\mathbb{F}^1 - De_{\#2}$. Now $\notin nCy\mathbb{F}^1 - De_{\#2} \Rightarrow f \notin nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1})$.

For each $e \neq f$, $\exists$ a $nCy\bar{\bar{\alpha}} \wr OS, De_{\#1} \mid e \in De_{\#1}$ & $f \notin nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1})$. So $f \notin \cap \{nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1}) : De_{\#1}$ is a $nCy\bar{\bar{\alpha}} \wr OS$ in $nCy\mathbb{F}^1$ and $e \in De_{\#1}\} = \{e\}$.

Let $e, f \in nCy\mathbb{F}^1$ and $e \neq f$. By supposition $\exists$ a $nCy\bar{\bar{\alpha}} \wr OS$, $De_{\#1} \mid e \in De_{\#1}$ and $nCy\bar{\bar{\alpha}}_{Cl}(De_{\#1})$. This $\Rightarrow \exists$ a $nCy\bar{\bar{\alpha}} \wr CS$, $D_{\#2} \mid f \notin De_{\#2}$. Thereby $f \in nCy\mathbb{F}^1 - De_{\#2}$ & $nCy\mathbb{F}^1 - De_{\#2}$ is $nCy\bar{\bar{\alpha}} \wr OS$. Thus, $\exists$ 2-disjoint $nCy\bar{\bar{\alpha}} \wr OS$, $De_{\#1}$ and $nCy\mathbb{F}^1 - De_{\#2} \mid e \in De_{\#1}$ and $f \in nCy\mathbb{F}^1 - De_{\#2}$. Thereby $nCy\mathbb{F}^1$ is ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space.

Conclusion: The analysis's interpretations of the present investigation are presented, together with an evaluation of the latest developments to which they have influenced. Start from the beginning to defined a new set in a space $nCy\mathbb{F}$ called an $nCy\bar{\bar{\alpha}} \wr$- integrated The ideas of CS and OS and considered the map of a function Δ from $nCy\mathbb{F}^1$ to $nCy\mathbb{F}^2$ demonstrates been declared to be $nCy{\sim}Pse(\bar{\bar{\alpha}})$-OP, $nCy{\sim}Pse(\bar{\bar{\alpha}})$-CL and decisively $nCy\bar{\bar{\alpha}}$-$OpFn$ and offered a comprehension of the accomplished achievements by presented the concept of spaces in $nCyFNTS's$ is considered to as ${}_{nCy\bar{\bar{\alpha}}}\delta^0$-space, ${}_{nCy\bar{\bar{\alpha}}}\delta^1$-space and ${}_{nCy\bar{\bar{\alpha}}}\delta^2$-space. Furthermore to the examples, assumptions, and theorems, the space separating two n-$CyFNS$, furthermore, their unique features and core functions were outlined.

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Neutrosophic Approach to Solving First-Order Differential Equations: Applications to Heat Convection with Uncertain Initial Conditions

Said Broumi[1], M. Shanmugapriya[2], R. Sundareswaran[3]

[1]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco

[1, 2]Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Chennai - 603 110, India

Email: s.broumi@flbenmsik.ma, shanmugapriyam@ssn.edu.in, sundareswaranr@ssn.edu.in

*Corresponding author detail: s.broumi@flbenmsik.ma

Abstract: This study explores the solution of first-order differential equations (DE) using trapezoidal neutrosophic numbers ($TrapN_{number}$) as initial conditions. It examines various forms of $TrapN_{number}$ based on the dependencies of truth (T), indeterminacy (I), and falsity (F). The application first order DE is illustrated through heat conduction problems in fluids. The temperature distribution $T_1(x,\alpha)$, $T_2(x,\alpha)$, $T_1'(x,\beta)$, $T_1''(x,\gamma)$, $T_2'(x,\beta)$ and $T_2''(x,\gamma)$ are analyzed through tables and graphs. A solution procedure for the system of first-order ODEs is developed and demonstrated with numerical examples.

Keywords: difference equation; trapezoidal neutrosophic number, trapezoidal single valued neutrosophic number

1. Introduction

F_{set} theory, significantly developed by Zadeh [1], is defined by the membership function $\mu_{\tilde{A}}(x)$. Atanassov [2, 3] extended this concept by introducing a non-membership function $\vartheta_{\tilde{A}}(x)$, leading to IF_{set}. Liu and Yuan [4] then combined triangular F_{sets} with IF_{set}. to create triangular IF_{sets}. Later, Ye [5] replaced triangular $F_{numbers}$ with trapezoidal $F_{numbers}$ in both truth and falsity membership functions, resulting in trapezoidal IF_{sets}. However, traditional fuzzy and intuitionistic fuzzy logic lack the concept of indeterminacy. Smarandache [6-7] introduced indeterminacy term, enhancing the framework beyond fuzzy logic. Thus, neutrosophic logic consists of three components: truth value ($T_{\tilde{A}_{N_{set}}}$), indeterminacy value ($I_{\tilde{A}_{N_{set}}}$), and false value($F_{\tilde{A}_{N_{set}}}$), with these membership values defined within the non-standard interval $]-0,1+[$. Smarandache [8] introduced the concept of N_{sets} as a generalization of IF_{sets} expanding the framework of fuzzy logic by incorporating indeterminacy. This structure allows for the representation of uncertainty, imprecision, and inconsistency.

First-order systems of ordinary differential equations have a wide range of applications, including population dynamics and disease spread, electrical circuits and fluid dynamics, motion and heat transfer, economic growth and market analysis, pollutant dispersion and ecosystem dynamics, and drug dosage and physiological processes. There are several approaches to solve first-order ODEs. Several researchers have utilized neutrosophic logic, including $SVN_{numbers}$, $TriN_{numbers}$, and $TrapN_{numbers}$, to solve DEs and explore their applications. Sumathi and Mohana Priya [9] presented a novel perspective on neutrosophic DEs, exploring how neutrosophic concepts can be applied to

solve and interpret DEs. Sumathi and Crispin Sweety [10] proposed a new method for solving DEs using $TrapN_{numbers}$, showcasing a specific application of neutrosophic logic in mathematical modeling. Alamin et al. [11] explored solutions and interpretations of neutrosophic homogeneous DEs, enhancing the understanding of discrete systems in the context of neutrosophic logic. Mondal et al. [12] address the solution of a system of differential equations within an intuitionistic fuzzy environment. Chakraborty et al. [13] have explored various linear and non-linear forms of $TrapN_{numbers}$ and discuss de-neutrosophication techniques. Shanmugapriya et al. [14] explored the solution of system of first-order simultaneous DEs within a neutrosophic environment, which allows handling indeterminacy, truth, and falsity values in the context of uncertainty. Their research emphasizes applications in time-cost optimization and sequencing problems.

1.1 Research gap

The research gap in this study emphasizes the limited exploration of $TrapN_{numbers}$ as initial conditions in solving first-order DEs. It highlights the need for systematic analysis of their relationships among truth, indeterminacy, and falsity, as well as their application to heat conduction problems in fluids. Addressing these gaps could improve the use of neutrosophic logic in modeling uncertainty.

1.2 Objective

The aim of this study is to investigate solutions for first-order DEs using $TrapN_{numbers}$ as initial conditions. It seeks to explore various forms of $TrapN_{numbers}$ based on the relationships among truth, indeterminacy, and falsity. Additionally, the study aims to apply these concepts to heat conduction problems in fluids, analyzing temperature distributions through specific functions and presenting findings through tables and graphs.

1.3 Novelty

The novelty of this present study lies in the use of $TrapN_{numbers}$ as initial conditions for solving first-order differential equations (DEs). This innovative approach explores various forms of $TrapN_{numbers}$, capturing the relationships between truth, indeterminacy, and falsity values, which extends traditional methods of solving DEs by incorporating uncertainty. Additionally, applying these neutrosophic concepts to heat conduction problems in fluids, particularly in analyzing temperature distributions, offers a fresh perspective on addressing complex physical systems with uncertain parameters.

1.4 Applications

Neutrosophic logic significantly enhances modeling in various fields by addressing uncertainties inherent in complex systems. In population dynamics, it enables more accurate modeling of species interactions and growth rates. For electrical circuits, this approach facilitates the analysis of circuit behaviors under uncertain conditions. In fluid dynamics, neutrosophic logic aids in understanding fluid behaviors across different states of uncertainty. It also plays a crucial role in economic growth by evaluating models that may involve incomplete or uncertain data. Furthermore, it assists in pollutant dispersion by predicting the spread of pollutants with varying degrees of certainty. Lastly, in drug dosage optimization, neutrosophic logic allows for the tailoring of drug delivery methods

based on uncertain patient responses, ultimately improving decision-making across these diverse disciplines.

2. Preliminaries

In this section, the fundamental concepts of neutrosophic logic utilized in this article are presented as follows:

Definition 2.1. N_{set}: [1] Let X be a universe set. An N_{set} $\tilde{A}_{N_{set}}$ on X is defined as:
$\tilde{A}_{N_{set}} = \left\{\langle x, T_{\tilde{A}_{N_{set}}}(x), I_{\tilde{A}_{N_{set}}}(x), F_{\tilde{A}_{N_{set}}}(x)\rangle : x \in X\right\}$,
where

i. $T_{\tilde{A}_{N_{set}}}(x): X \to {}^{-}]0,1[^{+}$ is called the truth membership (T_{mem}), representing the degree of confidence in the membership of $x \in X$ in $\tilde{A}_{N_{set}}$

ii. $I_{\tilde{A}_{N_{set}}}(x): X \to {}^{-}]0,1[^{+}$ is called the indeterminacy membership (I_{mem}), representing the degree of uncertainty in the membership of $x \in X$ in $\tilde{A}_{N_{set}}$

iii. $F_{\tilde{A}_{N_{set}}}(x): X \to {}^{-}]0,1[^{+}$ is called the falsity membership (F_{mem})representing the degree of skepticism in the membership of $x \in X$ in $\tilde{A}_{N_{set}}$.

These membership functions satisfy the condition: that ${}^{-}0 \leq T_{\tilde{A}_{N_{set}}}(x) + I_{\tilde{A}_{N_{set}}}(x) + F_{\tilde{A}_{N_{set}}}(x) \leq 3^{+}$.

Definition 2.2. SVN_{set}: [2] An N_{set} $\tilde{A}_{N_{set}}$ on X is said to be SVN_{set} $\left(\widetilde{SA}_{N_{set}}\right)$ if x is a single-valued independent variable. The set $\widetilde{SA}_{N_{set}}$is defined as:
$\widetilde{SA}_{N_{set}} = \left\{\langle x, T_{\widetilde{SA}_{N_{set}}}(x), I_{\widetilde{SA}_{N_{set}}}(x), F_{\widetilde{SA}_{N_{set}}}(x)\rangle : x \in X\right\}$,
where the functions $T_{\widetilde{SA}_{N_{set}}}(x), I_{\widetilde{SA}_{N_{set}}}(x), F_{\widetilde{SA}_{N_{set}}}(x): X \to {}^{-}]0,1[^{+}$ represents the $T_{mem}, I_{mem,}$ and F_{mem} functions, respectively, for the element $x \in X$ in $\widetilde{SA}_{N_{set}}$, such that:
${}^{-}0 \leq T_{\widetilde{SA}_{N_{set}}}(x) + I_{\widetilde{SA}_{N_{set}}}(x) + F_{\widetilde{SA}_{N_{set}}}(x) \leq 3^{+}$.

Definition 2.3. (α, β, γ)-cut: [3] The (α, β, γ)-cut N_{set} is $\tilde{A}_{N_{set}}$ defined as:
$\tilde{A}^{(\alpha,\beta,\gamma)}_{N_{set}} = \left\{\langle T_{\tilde{A}_{N_{set}}}(x), I_{\tilde{A}_{N_{set}}}(x), F_{\tilde{A}_{N_{set}}}(x)\rangle : x \in X, T_{\tilde{A}_{N_{set}}}(x) \geq \alpha, I_{\tilde{A}_{N_{set}}}(x) \leq \beta, F_{\tilde{A}_{N_{set}}}(x) \leq \gamma\right\}$,
where $\alpha, \beta, \gamma \in [0,1]$ are fixed values, and they satisfy the condition:
$\alpha + \beta + \gamma \leq 3$.

Definition 2.4. N_{number}: [3] A N_{set} $\tilde{A}_{N_{set}}$ over the set of real numbers $\mathbb{R}$ is said to be N_{number} if it satisfies the following properties:

i. $\tilde{A}_{N_{set}}$ is normal: if there exist $x_0 \in \mathbb{R}$ such that $T_{\tilde{A}_{N_{set}}}(x_0) = 1$. $\left(I_{\tilde{A}_{N_{set}}}(x_0) = F_{\tilde{A}_{N_{set}}}(x_0) = 0\right)$.

ii. $\tilde{A}_{N_{set}}$ is convex for the $T_{\tilde{A}_{N_{set}}}(x)$ function.
$(ie)\ T_{\tilde{A}_{N_{set}}}(\mu x_1 + (1-\mu)x_2) \geq \min\left(T_{\tilde{A}_{N_{set}}}(x_1), T_{\tilde{A}_{N_{set}}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ & $\mu \in [0,1]$.

iii. $\tilde{A}_{N_{set}}$ is concave set for the $I_{\tilde{A}_{N_{set}}}(x)$ and $F_{\tilde{A}_{N_{set}}}(x)$ functions.
$(ie)\ \ I_{\tilde{A}_{N_{set}}}(\mu x_1 + (1-\mu)x_2) \geq \max\left(I_{\tilde{A}_{N_{set}}}(x_1), I_{\tilde{A}_{N_{set}}}(x_2)\right)$ and $F_{\tilde{A}_{N_{set}}}(\mu x_1 + (1-\mu)x_2) \geq \max\left(F_{\tilde{A}_{N_{set}}}(x_1), F_{\tilde{A}_{N_{set}}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ & $\mu \in [0,1]$.

Definition 2.5. $TrapN_{number}$: [3] A $TrapN_{number}$, $\tilde{A}_{N_{set}}$ is a subset of N_{number}, $\tilde{A}_{N_{set}}$in $\mathbb{R}$ with the following $T, I, and\ F$ functions which is given by the following:

$$T_{\tilde{A}_{N_{set}}}(x)=\begin{cases}\left(\frac{x-a_1}{a_2-a_1}\right)u_{\tilde{A}_{N_{set}}}\ for\ a_1\le x\le a_2\\ u_{\tilde{A}_{N_{set}}}\qquad for\ a_2\le x\le a_3\\ \left(\frac{a_4-x}{a_4-a_3}\right)u_{\tilde{A}_{N_{set}}}\ for a_3\le x\le a_4\\ 0\qquad otherwise\end{cases};\ I_{\tilde{A}_{N_{set}}}(x)=\begin{cases}\left(\frac{a_2-x}{a_2-a_1}\right)v_{\tilde{A}_{N_{set}}}\ for\ a_1\le x\le a_2\\ v_{\tilde{A}_{N_{set}}}\qquad for\ a_2\le x\le a_3\\ \left(\frac{a_4-x}{a_4-a_3}\right)u_{v_{N_{set}}}\ for\ a_3\le x\le a_4\\ 1\qquad otherwise\end{cases}$$

$$F_{\tilde{A}_{N_{set}}}(x)=\begin{cases}\left(\frac{a_2-x}{a_2-a_1}\right)w_{\tilde{A}_{N_{set}}}\ for\ a_1\le x\le a_2\\ w_{\tilde{A}_{N_{set}}}\qquad for\ a_2\le x\le a_3\\ \left(\frac{a_4-x}{a_4-a_3}\right)w_{\tilde{A}_{N_{set}}}\ for a_3\le x\le a_4\\ 1\qquad otherwise\end{cases}$$

where $^{-}0\le T_{\tilde{A}_{N_{set}}}(x)+I_{\tilde{A}_{N_{set}}}(x)+F_{\tilde{A}_{N_{set}}}(x)\le 3^{+}, x\in\tilde{A}_{N_{set}}$.

Definition 2.6. (α,β,γ) - cut of a $TrapN_{number}$, $\tilde{A}_{N_{set}}$: The (α,β,γ)-cut of a $TrapN_{number}$ $\tilde{A}_{N_{set}}$, where: $\tilde{A}_{N_{set}}=\langle(a_1,a_2,a_3,a_4);u_{\tilde{A}_{N_{set}}},v_{\tilde{A}_{N_{set}}},w_{\tilde{A}_{N_{set}}}\rangle$ is defined as:

$$\left(\tilde{A}_{N_{set}}\right)_{\alpha,\beta,\gamma}=\left[T_{\tilde{A}_{N1_{set}}}(\alpha),T_{\tilde{A}_{N2_{set}}}(\alpha);I_{\tilde{A}_{N1_{set}}}(\beta),I_{\tilde{A}_{N2_{set}}}(\beta);F_{\tilde{A}_{N1_{set}}}(\gamma),F_{\tilde{A}_{N2_{set}}}(\gamma)\right],$$

where

i. T_{mem}: $T_{\tilde{A}_{N1_{set}}}(\alpha)=[a_1+\alpha(a_2-a_1)]u_{\tilde{A}_{N_{set}}}, T_{\tilde{A}_{N2_{set}}}(\alpha)=[a_4-\alpha(a_4-a_3)]u_{\tilde{A}_{N_{set}}}$.

ii. I_{mem}: $I_{\tilde{A}_{N1_{set}}}(\beta)=[a_2-\beta(a_2-a_1)]v_{\tilde{A}_{N_{set}}}, I_{\tilde{A}_{N2_{set}}}(\beta)=[a_3+\beta(a_4-a_3)]v_{\tilde{A}_{N_{set}}}$.

iii. F_{mem}: $F_{\tilde{A}_{N1_{set}}}(\gamma)=[a_2-\gamma(a_2-a_1)]w_{\tilde{A}_{N_{set}}}, F_{\tilde{A}_{N2_{set}}}(\gamma)=[a_3+\gamma(a_4-a_3)]w_{\tilde{A}_{N_{set}}}$.

here $\alpha,\beta,\gamma\in(0,1]\ and\quad ^{-}0<\alpha+\beta+\gamma\le 3^{+}$.

Linear Generalized N_{number}:

Definition: 2.7. The quantity of the $T,I,and\ F$ are not dependent then the $TrapSVN_{number}$ $\tilde{A}_{N_{set}}=\langle(a_1,a_2,a_3,a_4);(b_1,b_2,b_3,b_4);(c_1,c_2,c_3,c_4)\rangle$ whose $T_{mem},I_{mem,}$ and F_{mem} functions are defined as follows:

$$T_{\tilde{A}_{N_{set}}}(x)=\begin{cases}\left(\frac{x-a_1}{a_2-a_1}\right)\ for\ a_1\le x<a_2\\ 1\qquad for\ a_2<x\le a_3\\ \left(\frac{a_4-x}{a_4-a_3}\right)\ for a_3<x\le a_4\\ 0\qquad otherwise\end{cases};I_{\tilde{A}_{N_{set}}}(x)=\begin{cases}\left(\frac{b_2-x}{b_2-b_1}\right)\ for\ b_1\le x<b_2\\ 0\qquad for\ b_2<x\le b_3\\ \left(\frac{x-b_3}{b_4-b_3}\right)\ for\ b_3<x\le b_4\\ 1\qquad otherwise\end{cases}$$

$$F_{\tilde{A}_{N_{set}}}(x)=\begin{cases}\left(\frac{c_2-x}{c_2-c_1}\right)\ for\ c_1\le x<c_2\\ 0\qquad for\ c_2<x\le c_3\\ \left(\frac{x-\ c_3}{c_4-c_3}\right)\ for\ c_3<x\le c_4\\ 1\qquad otherwise\end{cases}$$

where $^{-}0\le T_{\tilde{A}_{N_{set}}}(x)+I_{\tilde{A}_{N_{set}}}(x)+F_{\tilde{A}_{N_{set}}}(x)\le 3^{+}, x\in\tilde{A}_{N_{set}}$.

Definition: 2.8 The (α,β,γ)-cut of a $TrapSVN_{number}$ $\tilde{A}_{N_{set}}=\langle(a_1,a_2,a_3,a_4);(b_1,b_2,b_3,b_4);(c_1,c_2,c_3,c_4)\rangle$ of definition 2.7 is defined as follows:

$$\left(\tilde{A}_{N_{set}}\right)_{\alpha,\beta,\gamma} = \left[T_{\tilde{A}_{N1_{set}}}(\alpha), T_{\tilde{A}_{N2_{set}}}(\alpha); I_{\tilde{A}_{N1_{set}}}(\beta), I_{\tilde{A}_{N2_{set}}}(\beta); F_{\tilde{A}_{N1_{set}}}(\gamma), F_{\tilde{A}_{N2_{set}}}(\gamma)\right],$$

where $T_{\tilde{A}_{N1_{set}}}(\alpha) = [a_1 + \alpha(a_2 - a_1)], T_{\tilde{A}_{N2_{set}}}(\alpha) = [a_4 - \alpha(a_4 - a_3)]; I_{\tilde{A}_{N1_{set}}}(\alpha) = [b_2 - \beta(b_2 - b_1)], I_{\tilde{A}_{N2_{set}}}(\alpha) = [b_3 + \beta(b_4 - b_3)]; F_{\tilde{A}_{N1_{set}}}(\alpha) = [c_2 - \gamma(c_2 - c_1)], F_{\tilde{A}_{N2_{set}}}(\alpha) = [c_3 + \gamma(c_4 - c_3)]$,

here $\alpha, \beta, \gamma \in (0,1]\ and\quad {}^{-}0 < \alpha + \beta + \gamma \leq 3^{+}$.

Definition: 2.9. The quantity of $I, and\ F$ are dependent then the $\tilde{A}_{N_{set}} = \langle (a_1, a_2, a_3, a_4); (b_1, b_2, b_3, b_4);\ v_{\tilde{A}_{N_{set}}}, w_{\tilde{A}_{N_{set}}} \rangle$ whose $T_{mem}, I_{mem,}$ and F_{mem} functions are defined as follows: $TrapSVN_{number}$

$$T_{\tilde{A}_{N_{set}}}(x) = \begin{cases} \left(\dfrac{x - a_1}{a_2 - a_1}\right) & for\ a_1 \leq x < a_2 \\ 1 & for\ a_2 < x \leq a_3 \\ \left(\dfrac{a_4 - x}{a_4 - a_3}\right) & for a_3 < x \leq a_4 \\ 0 & otherwise \end{cases}; I_{\tilde{A}_{N_{set}}}(x) = \begin{cases} \left(\dfrac{b_2 - x + v_{\tilde{A}_{N_{set}}}(x - b_1)}{b_2 - b_1}\right) & for\ b_1 \leq x < b_2 \\ v_{\tilde{A}_{N_{set}}} & for\ b_2 < x \leq b_3 \\ \left(\dfrac{x - b_3 + v_{\tilde{A}_{N_{set}}}(b_4 - x)}{b_4 - b_3}\right) & for\ b_3 < x \leq b_4 \\ 1 & otherwise \end{cases}$$

$$F_{\tilde{A}_{N_{set}}}(x) = \begin{cases} \left(\dfrac{b_2 - x + w_{\tilde{A}_{N_{set}}}(x - b_1)}{b_2 - b_1}\right) & for\ b_1 \leq x < b_2 \\ w_{\tilde{A}_{N_{set}}} & for\ b_2 < x \leq b_3 \\ \left(\dfrac{x - b_2 + w_{\tilde{A}_{N_{set}}}(b_3 - x)}{b_3 - b_2}\right) & for\ b_3 < x \leq b_3 \\ 1 & otherwise \end{cases}$$

where ${}^{-}0 \leq T_{\tilde{A}_{N_{set}}}(x) + I_{\tilde{A}_{N_{set}}}(x) + F_{\tilde{A}_{N_{set}}}(x) \leq 2^{+}, x \in \tilde{A}_{N_{set}}$.

Definition: 2.10 The (α, β, γ)-cut of a $TrapSVN_{number}$ $A_{TNe} = \langle (a_1, a_2, a_3, a_4); (b_1, b_2, b_3, b_4);\ v_{\tilde{A}_{N_{set}}}, w_{\tilde{A}_{N_{set}}} \rangle$ of definition 2.9 is defined as follows:

$(A_{TNe})_{\alpha,\beta,\gamma} = \left[T_{A_{TNe1}}(\alpha), T_{A_{TNe2}}(\alpha); I_{A_{TNe1}}(\beta), I_{A_{TNe2}}(\beta); F_{A_{TNe1}}(\gamma), f_{A_{TNe2}}(\gamma)\right]$, where $T_{A_{TNe1}}(\alpha) = [a_1 + \alpha(a_2 - a_1)], T_{A_{TNe2}}(\alpha) = [a_4 - \alpha(a_4 - a_3)]; I_{A_{TNe1}}(\beta) = \left[\frac{b_2 - v_{\tilde{A}_{N_{set}}} b_1 - \beta(b_2 - b_1)}{1 - v_{\tilde{A}_{N_{set}}}}\right], I_{A_{TNe2}}(\beta) = \left[\frac{b_3 - v_{\tilde{A}_{N_{set}}} b_4 + \beta(b_4 - b_3)}{1 - v_{\tilde{A}_{N_{set}}}}\right]; F_{A_{TNe1}}(\gamma) = \left[\frac{b_2 - w_{\tilde{A}_{N_{set}}} b_1 - \gamma(b_2 - b_1)}{1 - w_{\tilde{A}_{N_{set}}}}\right], F_{A_{TNe2}}(\gamma) = \left[\frac{b_3 - w_{\tilde{A}_{N_{set}}} b_4 + \gamma(b_4 - b_3)}{1 - w_{\tilde{A}_{N_{set}}}}\right]$,

here $0 < \alpha \leq 1, v_{\tilde{A}_{N_{set}}} < \beta \leq 1, w_{\tilde{A}_{N_{set}}} < \gamma \leq 1\ and - 0 < \beta + \gamma \leq 1 + and\ - 0 < \alpha + \beta + \gamma \leq 2 +$.

Definition: 2.11 The quantity of the $T, I, and\ F$ are dependent then the $TrapSVN_{number}\ of\ T_{mem}, I_{mem,}$ and F_{mem} function defined as

$$T_{A_{TNe}}(x) = \begin{cases} \left(\dfrac{x - a_1}{a_2 - a_1}\right) u_{\tilde{A}_{N_{set}}} & for\ a_1 \leq x < a_2 \\ 1 & for\ a_2 < x \leq a_3 \\ \left(\dfrac{a_4 - x}{a_4 - a_3}\right) u_{\tilde{A}_{N_{set}}} & for a_3 < x \leq a_4 \\ 0 & otherwise \end{cases};$$

$$I_{A_{TNe}}(x) = \begin{cases} \left(\dfrac{a_2 - x + u_{\tilde{A}_{N_{set}}}(x - a_1)}{a_2 - a_1}\right) & for\ a_1 \le x < a_2 \\ u_{\tilde{A}_{N_{set}}} & for\ a_2 < x \le a_3 \\ \left(\dfrac{x - a_3 + u_{\tilde{A}_{N_{set}}}(a_4 - x)}{a_4 - a_3}\right) & for\ a_3 < x \le a_4 \\ 1 & otherwise \end{cases};$$

$$F_{A_{TNe}}(x) = \begin{cases} \left(\dfrac{a_2 - x + w_{\tilde{A}_{N_{set}}} x - a_1)}{a_2 - a_1}\right) & for\ a_1 \le x < a_2 \\ w_{\tilde{A}_{N_{set}}} & for\ a_2 < x \le a_3 \\ \left(\dfrac{x - a_2 + w_{\tilde{A}_{N_{set}}}(a_3 - x)}{a_3 - a_2}\right) & for\ a_3 < x \le a_3 \\ 1 & otherwise \end{cases}$$

where $-0 \le T_{A_{TNe}}(x) + I_{A_{TNe}}(x) + F_{A_{TNe}}(x) \le 1+, x \in A_{TNe}$.

Definition: 2.12 (α, β, γ)-cut of a $TrapSVN_{number}$ $A_{TNe} = \langle (a_1, a_2, a_3, a_4); u_{\tilde{A}_{N_{set}}}, v_{\tilde{A}_{N_{set}}}, w_{\tilde{A}_{N_{set}}} \rangle$ of definition 2.11 is defined as follows:

$(A_{TNe})_{\alpha,\beta,\gamma} = \left[T_{A_{TNe1}}(\alpha), T_{A_{TNe2}}(\alpha); I_{A_{TNe1}}(\beta), I_{A_{TNe2}}(\beta); F_{A_{TNe1}}(\gamma), f_{A_{TNe2}}(\gamma)\right]$, where, $T_{A_{TNe1}}(\alpha) = \left[a_1 + \dfrac{\alpha}{u_{\tilde{A}_{N_{set}}}}(a_2 - a_1)\right], T_{A_{TNe2}}(\alpha) = \left[a_4 - \dfrac{\alpha}{u_{\tilde{A}_{N_{set}}}}(a_4 - a_3)\right]; I_{A_{TNe1}}(\beta) = \left[\dfrac{a_2 - v_{\tilde{A}_{N_{set}}} a_1 - \beta(a_2 - a_1)}{1 - v_{\tilde{A}_{N_{set}}}}\right], I_{A_{TNe2}}(\beta) = \left[\dfrac{a_3 - v_{\tilde{A}_{N_{set}}} a_4 + \beta(a_4 - a_3)}{1 - v_{\tilde{A}_{N_{set}}}}\right]; F_{A_{TNe1}}(\gamma) = \left[\dfrac{a_2 - w_{\tilde{A}_{N_{set}}} a_1 - \gamma(a_2 - a_1)}{1 - w_{\tilde{A}_{N_{set}}}}\right], F_{A_{TNe2}}(\gamma) = \left[\dfrac{a_3 - w_{\tilde{A}_{N_{set}}} a_4 + \gamma(a_4 - a_3)}{1 - w_{\tilde{A}_{N_{set}}}}\right]$,

here, $0 < \alpha \le u_{\tilde{A}_{N_{set}}}, v_{\tilde{A}_{N_{set}}} < \beta \le 1, w_{\tilde{A}_{N_{set}}} < \gamma \le 1$ and $-0 < \alpha + \beta + \gamma \le 1+$.

3. Solution of Neutrosophic boundary value problem

In this section, we discuss the solution of first-order ODE with a $TrapN_{numbers}$ and a different form of linear $TrapN_{numbers}$ as initial conditions.

3.1 Solution of First-Order ODE using $TrapN_{number}$

Let us consider the first order differential equation $\frac{dy}{dx} = ky$ (1)

with boundary condition $y(x_0) = \tilde{a}$, where $\tilde{a} = \langle (a_1, a_2, a_3, a_4); u_{\tilde{A}_{N_{set}}}, v_{\tilde{A}_{N_{set}}}, w_{\tilde{A}_{N_{set}}} \rangle$ is a $TrapN_{numbers}$.

Case (i) when k is a positive constant $(ie) k > 0$

The (α, β, γ)-cut of Eq. (1) is

$$\frac{d}{dx}[y_1(x,\alpha), y_2(x,\alpha); y_1'(x,\beta), y_2'(x,\beta); y_1''(x,\gamma), y_2''(x,\gamma)]$$
$$= k[y_1(x,\alpha), y_2(x,\alpha); y_1'(x,\beta), y_2'(x,\beta); y_1''(x,\gamma), y_2''(x,\gamma)] \quad (2)$$

with the initial condition

$$y(x_0;\alpha,\beta,\gamma) = \langle [a_1+\alpha(a_2-a_1)]u_{\tilde{A}_{N_{set}}}, [a_4-\alpha(a_4-a_3)]u_{\tilde{A}_{N_{set}}};\ [a_2-\beta(a_2-a_1)]v_{\tilde{A}_{N_{set}}}, [a_3 + \beta(a_4-a_3)]v_{\tilde{A}_{N_{set}}}; [a_2-\gamma(a_2-a_1)]w_{\tilde{A}_{N_{set}}}, [a_3 + \gamma(a_4-a_3)]w_{\tilde{A}_{N_{set}}}\rangle \quad (3)$$

Solution: The general solution of Eq.(1) is $y_1(x,\alpha) = e^{kx+c_1}$ (4)

Applying boundary condition in Eq.(4) we get

$$c_1 = \log[a_1+\alpha(a_2-a_1)]u_{\tilde{A}_{N_{set}}}\, e^{-kx_0} \quad (5)$$

substituting Eq.(5) in Eq.(4), we get

$$y_1(x,\alpha) = [a_1+\alpha(a_2-a_1)]u_{\tilde{A}_{N_{set}}} e^{k(x-x_0)},\ y_2(x,\alpha) = [a_4-\alpha(a_4-a_3)]u_{\tilde{A}_{N_{set}}} e^{k(x-x_0)}$$

similarly,

$$y_1'(x,\beta) = [a_2-\beta(a_2-a_1)]v_{\tilde{A}_{N_{set}}} e^{k(x-x_0)}, y_2'(x,\beta) = [a_3+\beta(a_4-a_3)]v_{\tilde{A}_{N_{set}}} e^{k(x-x_0)}$$

$$y_1''(x,\gamma) = [a_2-\gamma(a_2-a_1)]w_{\tilde{A}_{N_{set}}} e^{k(x-x_0)}, y_2''(x,\gamma) = [a_3+\gamma(a_4-a_3)]w_{\tilde{A}_{N_{set}}} e^{k(x-x_0)}.$$

Case (ii) when k is a negative constant $(ie) k<0$, let us take $k=-p$ and $p>0$

The (α,β,γ)-cut of Eq. (1) is

$$\frac{d}{dx}[y_1(x,\alpha), y_2(x,\alpha); y_1'(x,\beta), y_2'(x,\beta); y_1''(x,\gamma), y_2''(x,\gamma)]$$

$$= -p[y_1(x,\alpha), y_2(x,\alpha); y_1'(x,\beta), y_2'(x,\beta); y_1''(x,\gamma), y_2''(x,\gamma)] \quad (6)$$

with the initial condition

$$y(x_0;\alpha,\beta,\gamma) = \langle [a_1+\alpha(a_2-a_1)]u_{\tilde{A}_{N_{set}}}, [a_4-\alpha(a_4-a_3)]u_{\tilde{A}_{N_{set}}};\ [a_2-\beta(a_2-a_1)]v_{\tilde{A}_{N_{set}}}, [a_3 + \beta(a_4-a_3)]v_{\tilde{A}_{N_{set}}}; [a_2-\gamma(a_2-a_1)]w_{\tilde{A}_{N_{set}}}, [a_3 + \gamma(a_4-a_3)]w_{\tilde{A}_{N_{set}}}\rangle \quad (7)$$

then the general solution of the above Eqs.(6) and (7) are as follows:

$$y_1(x,\alpha) = [a_1+\alpha(a_2-a_1)]u_{\tilde{A}_{N_{set}}} e^{-k(x-x_0)}, y_2(x,\alpha) = [a_4-\alpha(a_4-a_3)]u_{\tilde{A}_{N_{set}}} e^{-k(x-x_0)}$$

$$y_1'(x,\beta) = [a_2-\beta(a_2-a_1)]v_{\tilde{A}_{N_{set}}} e^{-k(x-x_0)}, y_2'(x,\beta) = [a_3+\beta(a_4-a_3)]v_{\tilde{A}_{N_{set}}} e^{-k(x-x_0)}$$

$$y_1''(x,\gamma) = [a_2-\gamma(a_2-a_1)]w_{\tilde{A}_{N_{set}}} e^{-k(x-x_0)}, y_2''(x,\gamma) = [a_3+\gamma(a_4-a_3)]w_{\tilde{A}_{N_{set}}} e^{-k(x-x_0)}$$

3.2 Solutions of First-Order ODE using Linear Generalized $\boldsymbol{N_{number}}$

Category (i): ***when the quantity of the truth, indeterminacy and falsity are not dependent***

The general solution of the DE $\frac{dy}{dx} = ky$, with boundary condition $y(x_0) = \tilde{a}$, where $\tilde{a} = \langle (a_1,a_2,a_3,a_4); (b_1,b_2,b_3,b_4);\ (c_1,c_2,c_3,c_4)\rangle$ is a $TrapSVN_{number}$ is as follows:

Case(i) when $k>0$

$$y_1(x,\alpha) = [a_1+\alpha(a_2-a_1)]e^{k(x-x_0)}, y_2(x,\alpha) = [a_4-\alpha(a_4-a_3)]e^{k(x-x_0)}$$

$$y_1'(x,\beta) = [b_2-\beta(b_2-b_1)]e^{k(x-x_0)}, y_2'(x,\beta) = [b_3+\beta(b_4-b_3)]e^{k(x-x_0)}$$

$$y_1''(x,\gamma) = [c_2-\gamma(c_2-c_1)]e^{k(x-x_0)}, y_2''(x,\gamma) = [c_3+\gamma(c_4-c_3)]e^{k(x-x_0)}$$

Case(ii) when $k<0$

$$y_1(x,\alpha) = [a_1+\alpha(a_2-a_1)]e^{-k(x-x_0)}, \quad y_2(x,\alpha) = [a_4-\alpha(a_4-a_3)]e^{-k(x-x_0)}$$

$$y_1'(x,\beta) = [b_2-\beta(b_2-b_1)]e^{-k(x-x_0)}, \quad y_2'(x,\beta) = [b_3+\beta(b_4-b_3)]e^{-k(x-x_0)}$$

$$y_1''(x,\gamma) = [c_2-\gamma(c_2-c_1)]e^{-k(x-x_0)}, \quad y_2''(x,\gamma) = [c_3+\gamma(c_4-c_3)]e^{-k(x-x_0)}$$

Category* (ii): *when the quantity of indeterminacy and falsity are dependent

The general solution of the DE $\frac{dy}{dx} = ky, (k > 0)$ with boundary condition $y(x_0) = \tilde{a}$, where $\tilde{a} = \langle (a_1, a_2, a_3, a_4); (b_1, b_2, b_3, b_4);\ v_{\tilde{A}_{N_{set}}}, w_{\tilde{A}_{N_{set}}} \rangle$ is a $TrapSVN_{number}$ is as follows:

Case(i) when $k > 0$

$$y_1(x, \alpha) = [a_1 + \alpha(a_2 - a_1)]e^{k(x-x_0)}, y_2(x, \alpha) = [a_4 - \alpha(a_4 - a_3)]e^{k(x-x_0)}$$

$$y_1'(x, \beta) = \left[\frac{b_2 - v_{\tilde{A}_{N_{set}}} b_1 - \beta(b_2 - b_1)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}, y_2'(x, \beta) = \left[\frac{b_3 - v_{\tilde{A}_{N_{set}}} b_4 + \beta(b_4 - b_3)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}$$

$$y_1''(x, \gamma) = \left[\frac{b_2 - w_{\tilde{A}_{N_{set}}} b_1 - \gamma(b_2 - b_1)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}, y_2''(x, \gamma) = \left[\frac{b_3 - w_{\tilde{A}_{N_{set}}} b_4 + \gamma(b_4 - b_3)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}$$

Case(ii) when $k < 0$

$$y_1(x, \alpha) = [a_1 + \alpha(a_2 - a_1)]e^{-k(x-x_0)}, y_2(x, \alpha) = [a_4 - \alpha(a_4 - a_3)]e^{-k(x-x_0)}$$

$$y_1'(x, \beta) = \left[\frac{b_2 - v_{\tilde{A}_{N_{set}}} b_1 - \beta(b_2 - b_1)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}, y_2'(x, \beta) = \left[\frac{b_3 - v_{\tilde{A}_{N_{set}}} b_4 + \beta(b_4 - b_3)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}$$

$$y_1''(x, \gamma) = \left[\frac{b_2 - w_{\tilde{A}_{N_{set}}} b_1 - \gamma(b_2 - b_1)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}, y_2''(x, \gamma) = \left[\frac{b_3 - w_{\tilde{A}_{N_{set}}} b_4 + \gamma(b_4 - b_3)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}.$$

Category* (iii): *when the quantity of the truth, indeterminacy and falsity are dependent

The general solution of the DE $\frac{dy}{dx} = ky, (k > 0)$ with boundary condition $y(x_0) = \tilde{a}$, where $\tilde{a} = \langle (a_1, a_2, a_3, a_4); u_{\tilde{A}_{N_{set}}}, v_{\tilde{A}_{N_{set}}}, w_{\tilde{A}_{N_{set}}} \rangle$ is a $TrapSVN_{number}$ is as follows:

Case(i) when $k > 0$

$$y_1(x, \alpha) = \left[a_1 + \frac{\alpha}{u_{\tilde{A}_{N_{set}}}}(a_2 - a_1)\right] e^{k(x-x_0)}, y_2(x, \alpha) = \left[a_4 - \frac{\alpha}{u_{\tilde{A}_{N_{set}}}}(a_4 - a_3)\right] e^{k(x-x_0)}$$

$$y_1'(x, \beta) = \left[\frac{a_2 - v_{\tilde{A}_{N_{set}}} a_1 - \beta(a_2 - a_1)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}, y_2'(x, \beta) = \left[\frac{a_3 - v_{\tilde{A}_{N_{set}}} a_4 + \beta(a_4 - a_3)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}$$

$$y_1''(x, \gamma) = \left[\frac{a_2 - w_{\tilde{A}_{N_{set}}} a_1 - \gamma(a_2 - a_1)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}, y_2''(x, \gamma) = \left[\frac{a_3 - w_{\tilde{A}_{N_{set}}} a_4 + \gamma(a_4 - a_3)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{k(x-x_0)}$$

Case(ii) when $k < 0$

$$y_1(x, \alpha) = \left[a_1 + \frac{\alpha}{u_{\tilde{A}_{N_{set}}}}(a_2 - a_1)\right] e^{-k(x-x_0)}, y_2(x, \alpha) = \left[a_4 - \frac{\alpha}{u_{\tilde{A}_{N_{set}}}}(a_4 - a_3)\right] e^{-k(x-x_0)}$$

$$y_1'(x, \beta) = \left[\frac{a_2 - v_{\tilde{A}_{N_{set}}} a_1 - \beta(a_2 - a_1)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}, y_2'(x, \beta) = \left[\frac{a_3 - v_{\tilde{A}_{N_{set}}} a_4 + \beta(a_4 - a_3)}{1 - v_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}$$

$$y_1''(x, \gamma) = \left[\frac{a_2 - w_{\tilde{A}_{N_{set}}} a_1 - \gamma(a_2 - a_1)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}, y_2''(x, \gamma) = \left[\frac{a_3 - w_{\tilde{A}_{N_{set}}} a_4 + \gamma(a_4 - a_3)}{1 - w_{\tilde{A}_{N_{set}}}}\right] e^{-k(x-x_0)}.$$

4. Application of the First-Order DE to the Heat Convection in Fluid

4.1 Newtons Law of Cooling

Newtons Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large.
If we let $T(t)$ be the temperature of the object at time t, T_s be the bulk environmental temperature, $\alpha = \frac{h}{\rho c V'}$ and A_s the constant surface area then we can formulate Newton's law of Cooling as differential equation:

$$\frac{dT(t)}{dt} = -\alpha A_s[T(t) - T_s] \tag{8}$$

with initial condition $T(t)|_{t=0} = T(0) = T_0$.

4.2 Example: Solution of Heat Convection in Fluid

The initial temperature of an object is 80℃ we place it in a refrigerator and maintained it in 5℃. If $\alpha = 0.002m^2s$ and $A_s = 0.2m^2$. What will be the temperature after 8 minutes?

Crisp solution:

We have the initial condition $T(0) = 80℃$, the temperature of the surrounding $T_s = 30℃$, $\alpha = 0.002m^2s$ and $A_s = 0.2m^2$. Using this condition in Eq. (8) then the analytical solutions become $T(t) = 5 + 75e^{-0.0004t}$. At $t = 8$, we get $T(8) = 79.7604$.

Solution:

The standard heat convection equation and the corresponding boundary conditions are:

$$\frac{dT(t)}{dt} = -\alpha A_s[T(t) - T_s]$$

with initial condition $T(t)|_{t=0} = T(0) = T_0$.

After applying neutrosophic parameters the above equation is reformulated to the neutrosophic differential equation.

$$(ie)\frac{d}{dt}[T_1(t,\alpha), T_2(t,\alpha); T_1'(t,\beta), T_2'(t,\beta); T_1''(t,\gamma), T_2''(t,\gamma)]$$
$$= -\alpha A_s\big[[T_1(t,\alpha), T_2(t,\alpha); T_1'(t,\beta), T_2'(t,\beta); T_1''(t,\gamma), T_2''(t,\gamma)] - T_s\big]$$

with the initial condition

$$T(t_0;\alpha,\beta,\gamma) = \langle [a_1 + \alpha(a_2 - a_1)]u_{\tilde{A}_{N_{set}}}, [a_4 - \alpha(a_4 - a_3)]u_{\tilde{A}_{N_{set}}};\ [a_2 - \beta(a_2 - a_1)]v_{\tilde{A}_{N_{set}}}, [a_3 + \beta(a_4 - a_3)]v_{\tilde{A}_{N_{set}}}; [a_2 - \gamma(a_2 - a_1)]w_{\tilde{A}_{N_{set}}}, [a_3 + \gamma(a_4 - a_3)]w_{\tilde{A}_{N_{set}}}\rangle$$

(i) Using $TrapN_{number}$

If the initial temperature is a $TrapN_{number}$ $\tilde{a} = \langle (3,4,5,6); 0.7, 0.6, 0.4\rangle$ then the solution of the above example is:

$[T(t)]_{\alpha,\beta,\gamma} = [T_1(t,\alpha), T_2(t,\alpha); T_1'(t,\beta), T_2'(t,\beta); T_1''(t,\gamma), T_2''(t,\gamma)]$

where

$T_1(t,\alpha) = [3 + \alpha]0.7[5 + 75e^{-0.0004t}];\ T_2(t,\alpha) = [6 - \alpha]0.7[5 + 75e^{-0.0004t}]$

$T_1'(t,\beta) = [4 - \beta]0.6[5 + 75e^{-0.0004t}];\ T_2'(t,\beta) = [5 + \beta]0.6[5 + 75e^{-0.0004t}]$

$T_1''(t,\gamma) = [4 - \gamma]0.4[5 + 75e^{-0.0004t}];\ T_2''(t,\gamma) = [5 + \gamma]0.4[5 + 75e^{-0.0004t}]$

here, $0 < \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1\ and - 0 < \alpha + \beta + \gamma \leq 3 +$.

(ii) *Using Linear Generalized N_{number}*

***Category* (i)**:

If the initial temperature is a $TrapSVN_{number}$ $\tilde{a} = \langle(1,2,3,4); (5,6,7,8); (9,10,11,12)\rangle$ then the solution of the above example is:

$[T(t)]_{\alpha,\beta,\gamma} = [T_1(t,\alpha), T_2(t,\alpha); T_1'(t,\beta), T_2'(t,\beta); T_1''(t,\gamma), T_2''(t,\gamma)]$

where

$T_1(t,\alpha) = [1+\alpha][5+75e^{-0.0004t}];\ T_2(t,\alpha) = [4-\alpha][5+75e^{-0.0004t}]$

$T_1'(t,\beta) = [5-\beta][5+75e^{-0.0004t}];\ T_2'(t,\beta) = [7+\beta][5+75e^{-0.0004t}]$

$T_1''(t,\gamma) = [10-\gamma][5+75e^{-0.0004t}];\ T_2''(t,\gamma) = [11+\gamma][5+75e^{-0.0004t}]$

here, $0 < \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1\ and - 0 < \alpha+\beta+\gamma \leq 3+.$

Category (ii):

If the initial temperature is a $TrapSVN_{number}$ $\tilde{a} = \langle(1,2,3,4); (5,6,7,8);\ 0.1,0.2\rangle$ then the solution of the above example is:

$[T(t)]_{\alpha,\beta,\gamma} = [T_1(t,\alpha), T_2(t,\alpha); T_1'(t,\beta), T_2'(t,\beta); T_1''(t,\gamma), T_2''(t,\gamma)]$

where

$T_1(t,\alpha) = [1+\alpha][5+75e^{-0.0004t}];\ T_2(t,\alpha) = [4-\alpha][5+75e^{-0.0004t}]$

$T_1'(t,\beta) = \left[\frac{3.5-\beta}{0.9}\right][5+75e^{-0.0004t}];\ T_2'(t,\beta) = \left[\frac{6.2+\beta}{0.9}\right][5+75e^{-0.0004t}]$

$T_1''(t,\gamma) = \left[\frac{5-\gamma}{0.8}\right][5+75e^{-0.0004t}];\ T_2''(t,\gamma) = \left[\frac{5.4+\gamma}{0.8}\right][5+75e^{-0.0004t}]$

here, $0 < \alpha \leq 1, v_A < \beta \leq 1, w_A < \gamma \leq 1\ and - 0 < \beta+\gamma \leq 1+\ and\ - 0 < \alpha+\beta+\gamma \leq 2+.$

Category (iii):

If the initial temperature is a $TrapSVN_{number}$ $\tilde{a} = \langle(1,2,3,4); 0.1,0.2,0.3\rangle$ then the solution of the above example is:

$[T(t)]_{\alpha,\beta,\gamma} = [T_1(t,\alpha), T_2(t,\alpha); T_1'(t,\beta), T_2'(t,\beta); T_1''(t,\gamma), T_2''(t,\gamma)]$

where

$T_1(t,\alpha) = \left[1+\frac{\alpha}{0.1}\right][5+75e^{-0.0004t}];\ T_2(t,\alpha) = \left[4-\frac{\alpha}{0.1}\right][5+75e^{-0.0004t}]$

$T_1'(t,\beta) = \left[\frac{1.8-\beta}{0.8}\right][5+75e^{-0.0004t}];\ T_2'(t,\beta) = \left[\frac{2.2+\beta}{0.8}\right][5+75e^{-0.0004t}]$

$T_1''(t,\gamma) = \left[\frac{1.7-\gamma}{0.7}\right][5+75e^{-0.0004t}];\ T_2''(t,\gamma) = \left[\frac{1.8+\gamma}{0.7}\right][5+75e^{-0.0004t}]$

here, $0 < \alpha \leq u_A, v_A < \beta \leq 1, w_A < \gamma \leq 1$ and $- 0 < \alpha+\beta+\gamma \leq 1+.$

The solution of the temperature distribution of $T(t) = 5+75e^{-0.0004t}$ for various t and $0 < \alpha, \beta, \gamma \leq 1$ $TrapN_{number}$ and linear generalized N_{number} with different categories are depicted in Tables 1-4. These tables summarize the behavior of truth, indeterminacy, and falsity under various (α, β, γ) - cut values at $t = 8$. It is observed that the results of $T(t)$ satisfies the condition of strong solution.

Table 1. $TrapN_{number}$ solution for the different values of α, β and γ at $t = 8$

α	$T_1(t,\alpha)$	$T_2(t,\alpha)$	β	$T_1'(t,\beta)$	$T_2'(t,\beta)$	γ	$T_1''(t,\gamma)$	$T_2''(t,\gamma)$
0.1	173.0800	329.4103	0.1	186.6392	244.0667	0.1	124.4261	162.7111
0.2	178.6632	323.8271	0.2	181.8536	248.8523	0.2	121.2357	165.9015
0.3	184.2464	318.2439	0.3	177.0680	253.6380	0.3	118.0453	169.0920
0.4	189.8297	312.6607	0.4	172.2824	258.4236	0.4	114.8549	172.2824
0.5	195.4129	307.0774	0.5	167.4968	263.2092	0.5	111.6645	175.4728
0.6	200.9961	301.4942	0.6	162.7111	267.9948	0.6	108.4741	178.6632
0.7	206.5793	295.9110	0.7	157.9255	272.7805	0.7	105.2837	181.8536
0.8	212.1626	290.3277	0.8	153.1399	277.5661	0.8	102.0932	185.0440
0.9	217.7458	284.7445	0.9	148.3543	282.3517	0.9	98.9028	188.2345
1	223.3290	279.1613	1	143.5686	287.1373	1	95.7124	191.4249

Table 2. $TrapSVN_{number}$ solution for the different values of α, β and γ at $t = 8$ when the quantity of the truth, indeterminacy and falsity are not dependent

α	$T_1(t,\alpha)$	$T_2(t,\alpha)$	β	$T_1'(t,\beta)$	$T_2'(t,\beta)$	γ	$T_1''(t,\gamma)$	$T_2''(t,\gamma)$
0.1	87.7364	311.0654	0.1	470.5862	566.2987	0.1	789.6277	885.3402
0.2	95.7124	303.0894	0.2	462.6102	574.2747	0.2	781.6517	893.3162
0.3	103.6884	295.1134	0.3	454.6341	582.2508	0.3	773.6757	901.2923
0.4	111.6645	287.1373	0.4	446.6581	590.2268	0.4	765.6996	909.2683
0.5	119.6405	279.1613	0.5	438.6821	598.2028	0.5	757.7236	917.2444
0.6	127.6166	271.1853	0.6	430.7060	606.1789	0.6	749.7476	925.2204
0.7	135.5926	263.2092	0.7	422.7300	614.1549	0.7	741.7715	933.1964
0.8	143.5686	255.2332	0.8	414.7539	622.1309	0.8	733.7955	941.1725
0.9	151.5447	247.2571	0.9	406.7779	630.1070	0.9	725.8194	949.1485
1	159.5207	239.2811	1	398.8019	638.0830	1	717.8434	957.1246

Table 3. $TrapSVN_{number}$ solution for the different values of α, β and γ at $t = 8$ when the quantity of indeterminacy and falsity are dependent

α	$T_1(t,\alpha)$	$T_2(t,\alpha)$	β	$T_1'(t,\beta)$	$T_2'(t,\beta)$	γ	$T_1''(t,\gamma)$	$T_2''(t,\gamma)$
0.10	87.7364	311.0654	0.20	469.7000	567.1849	0.300	468.5922	568.292
0.15	91.7244	307.0774	0.25	465.2689	571.6160	0.325	466.0997	570.7852
0.20	95.7124	303.0894	0.30	460.8377	576.0472	0.350	463.6072	573.2777
0.25	99.7004	299.1014	0.35	456.4066	580.4783	0.375	461.1147	575.7702
0.30	103.6884	295.1134	0.40	451.9755	584.9094	0.400	458.6222	578.2627
0.35	107.6765	291.1254	0.45	447.5443	589.3406	0.425	456.1296	580.7552
0.40	111.6645	287.1373	0.50	443.1132	593.7717	0.450	453.6371	583.2478
0.45	115.6525	283.1493	0.55	438.6821	598.2028	0.475	451.1446	585.7403
0.50	119.6405	279.1613	0.60	434.2509	602.6340	0.500	448.6521	588.2328
0.55	123.6285	275.1733	0.65	429.8198	607.0651	0.525	446.1596	590.7253

Table 4. $TrapSVN_{number}$ solution for the different values of α, β and γ at $t = 8$ when the quantity of the truth, indeterminacy and falsity are dependent

α	$T_1(t,\alpha)$	$T_2(t,\alpha)$	β	$T_1'(t,\beta)$	$T_2'(t,\beta)$	γ	$T_1''(t,\gamma)$	$T_2''(t,\gamma)$
0.01	87.7364	311.0654	0.250	154.5357	244.2661	0.325	156.6721	242.1297
0.02	95.7124	303.0894	0.275	152.0432	246.7586	0.350	153.8235	244.9783
0.03	103.6884	295.1134	0.300	149.5507	249.2511	0.375	150.9750	247.8269
0.04	111.6645	287.1373	0.325	147.0582	251.7437	0.400	148.1264	250.6754

0.05	119.6405	279.1613	0.350	144.5656	254.2362	0.425	145.2778	253.5240
0.06	127.6166	271.1853	0.375	142.0731	256.7287	0.450	142.4292	256.3726
0.07	135.5926	263.2092	0.400	139.5806	259.2212	0.475	139.5806	259.2212
0.08	143.5686	255.2332	0.425	137.0881	261.7137	0.500	136.7320	262.0698
0.09	151.5447	247.2571	0.450	134.5956	264.2062	0.525	133.8835	264.9184
0.1	159.5207	239.2811	0.475	132.1031	266.6987	0.550	131.0349	267.7670

5. Results and Discussion

5.1 Analysis of 3D contour plots

The temperature distribution of $T_1(t,\alpha) = [3+\alpha]0.7[5+75e^{-0.0004t}]$; $T_2(t,\alpha) = [6-\alpha]0.7[5+75e^{-0.0004t}]$ for $0 < \alpha \leq 1$ and $100 \leq t \leq 1000$, which is shown if Fig. 1 and 2. The temperature distribution of $T_1(t,\alpha) = [1+\alpha][5+75e^{-0.0004t}], T_2(t,\alpha) = [4-\alpha][5+75e^{-0.0004t}]$ for $0 < \alpha \leq 1$ and $100 \leq t \leq 1000$, which is shown if Fig. 3 and 4. The temperature distribution of $T_1(t,\alpha) = [1+\alpha][5+75e^{-0.0004t}], T_2(t,\alpha) = [4-\alpha][5+75e^{-0.0004t}]$ for $0 < \alpha \leq 0.55$ and $100 \leq t \leq 1000$, which is shown if Fig. 5 and 6. The temperature distribution of $T_1(t,\alpha) = \left[1+\frac{\alpha}{0.1}\right][5+75e^{-0.0004t}], T_2(t,\alpha) = \left[4-\frac{\alpha}{0.1}\right][5+75e^{-0.0004t}]$ for $0 < \alpha \leq 0.1$ and $100 \leq t \leq 1000$, which is shown if Fig. 7 and 8. It is noted that, the upsurging values of α, β, γ increase $T_1(t,\alpha)$ and decrease $T_2(t,\alpha)$ where as $T_1'(t,\beta)$ and $T_1''(t,\gamma)$ are diminishing functions and $T_2'(t,\beta)$ and $T_2''(t,\gamma)$ are elevation functions. Therefore, the mentioned heat convection problem satisfies the conditions of the strong solutions of a neutrosophic difference equation [12] $(ie)\, T_1(t,\alpha) \leq T_2(t,\alpha), T_1'(t,\beta) \leq T_2'(t,\beta), T_1''(t,\gamma) \leq T_2''(t,\gamma)$ and $\frac{\partial}{\partial\alpha}[T_1(t,\alpha)] > 0$, $\frac{\partial}{\partial\alpha}[T_2(t,\alpha)] < 0$; $\frac{\partial}{\partial\beta}[T_1'(t,\beta)] < 0, \frac{\partial}{\partial\beta}[T_2'(t,\beta)] > 0$; $\frac{\partial}{\partial\gamma}[T_1''(t,\gamma)] < 0, \frac{\partial}{\partial\gamma}[T_2''(t,\gamma)] > 0$. Hence the obtained solution is a strong solution.

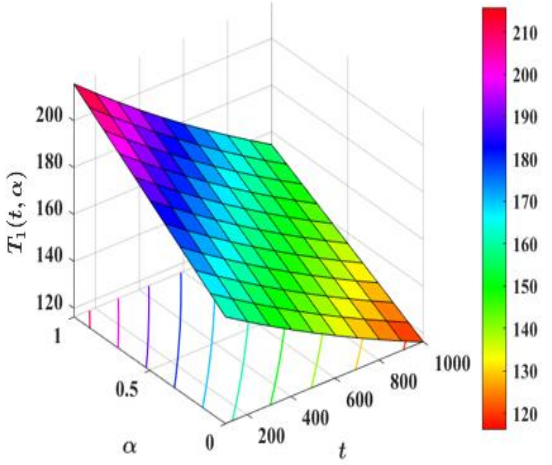


Fig. 1. Impact of t in $T_1(t,\alpha)$

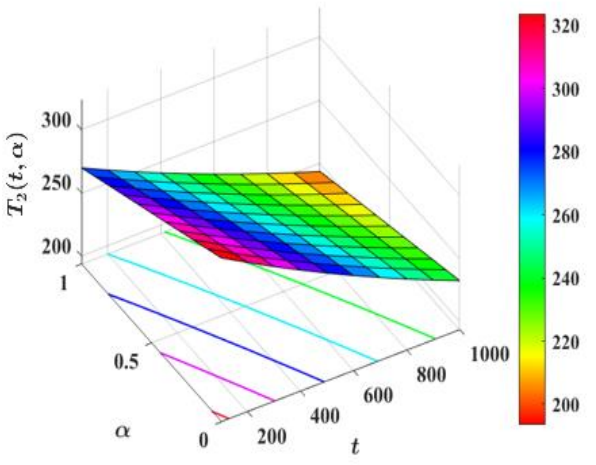


Fig. 2. Impact of t in $T_2(t,\alpha)$

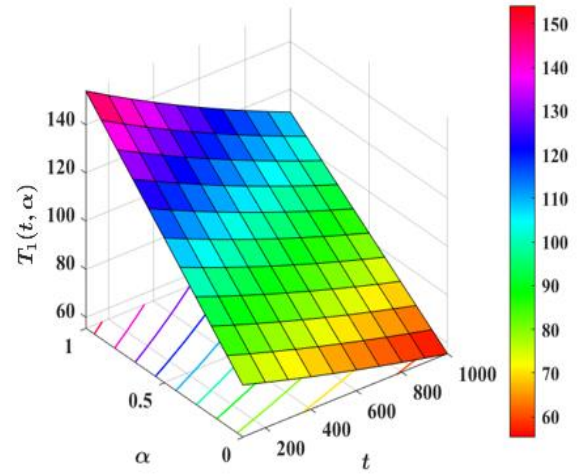


Fig. 3. Impact of t in $T_1(t,\alpha)$

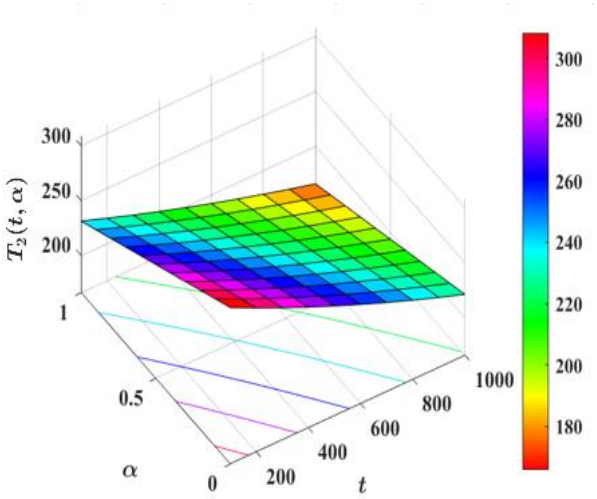


Fig. 4. Impact of t in $T_2(t,\alpha)$

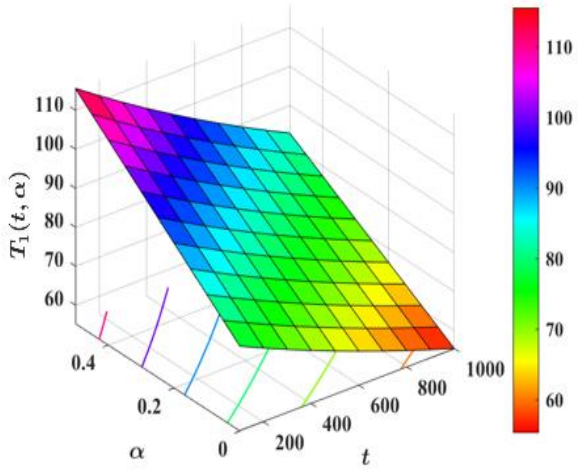


Fig. 5. Impact of t in $T_1(t,\alpha)$

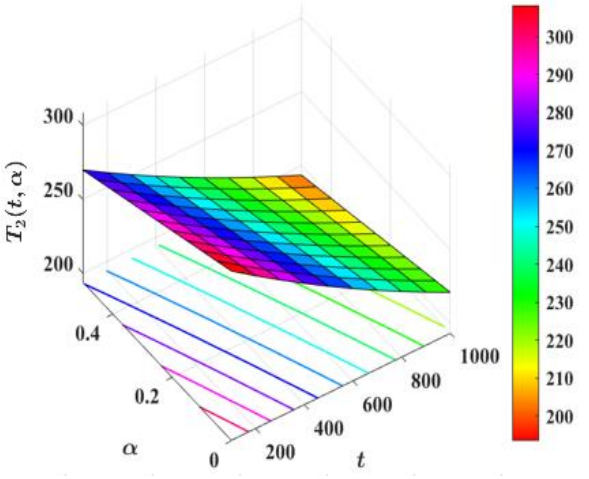


Fig. 6. Impact of t in $T_2(t,\alpha)$

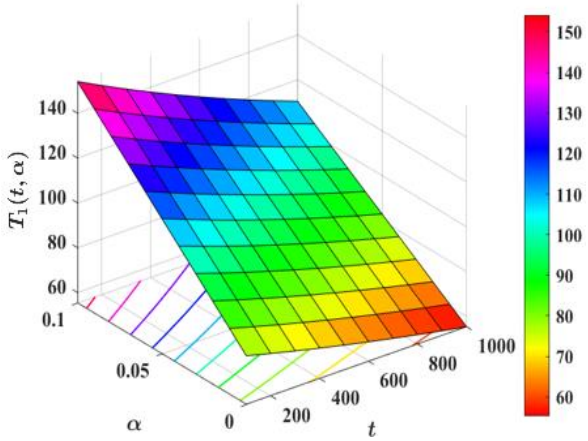


Fig. 7. Impact of t in $T_1(t,\alpha)$

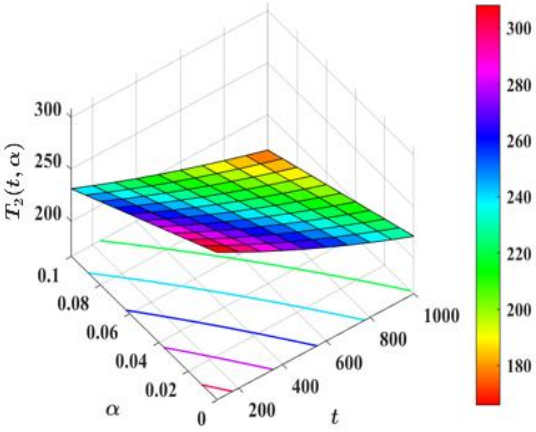


Fig. 8. Impact of t in $T_2(t,\alpha)$

5.2 Analysis of $TrapN_{number}$ and $TrapSVN_{number}$ plots

Figure 9 demonstrates that, as $\alpha - cut$ values increase, the lower bound of truth membership, $T_1(t,\alpha)$ rises, while the upper bound, $T_2(t,\alpha)$ decreases. This trend shows a diminishing uncertainty as the values approach the exact solution. Similarly, Figure 10 shows that for indeterminacy membership, both the lower bound $T_1'(t,\beta)$ and $T_2'(t,\beta)$ upper bound decrease with increasing $\beta - cut$ values, confirming that indeterminacy is minimized as the $\beta -$ cut values approach. In figures 11 $(a),(b)$ *and* (c), when the dependency of falsity on $\gamma - cut$ is introduced, the lower bound $T_1''(t,\gamma)$ shows a decreasing trend, while the upper bound $T_2''(t,\gamma)$ rises. This reveals that falsity becomes more important at higher $\gamma - cut$ values. Finally, figures 12 $(a),(b)$ *and* (c) presents the truth, indeterminacy, and falsity are considered dependent. The data suggests that as $\alpha - cuts$ values increase and $\beta - cut$ and higher $\gamma - cut$ values decrease, the lower bound of truth membership rises, and indeterminacy and falsity gradually diminish. This points to a convergence toward the exact solution when the dependencies are balanced.

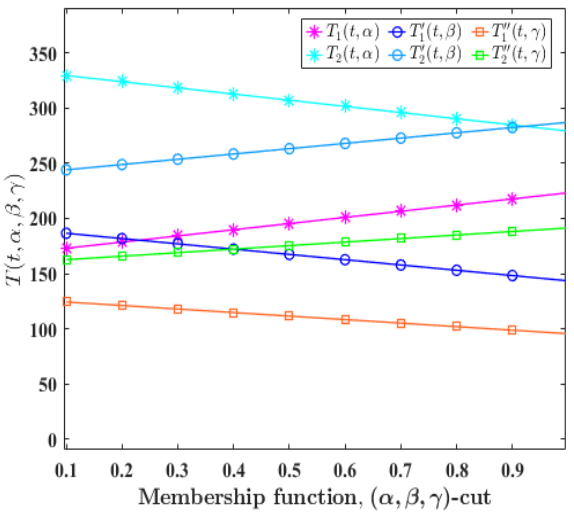


Fig. 9. Impact of t in $T(t,\alpha,\beta,\gamma)$ using $TrapN_{number}$

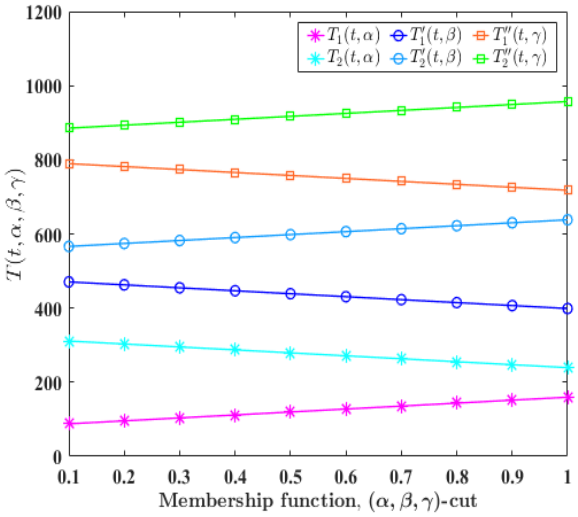
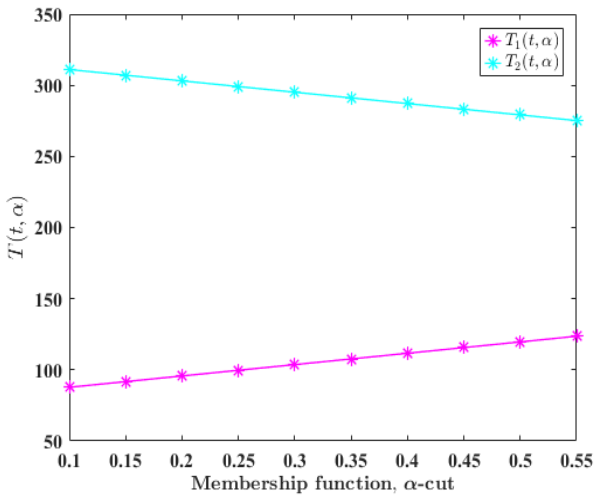
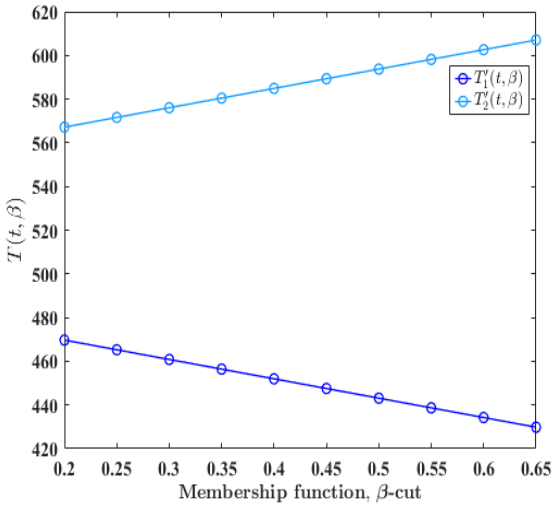


Fig. 10. Impact of t in $T(t,\alpha,\beta,\gamma)$ using $TrapSVN_{number}$

(a)



(b)



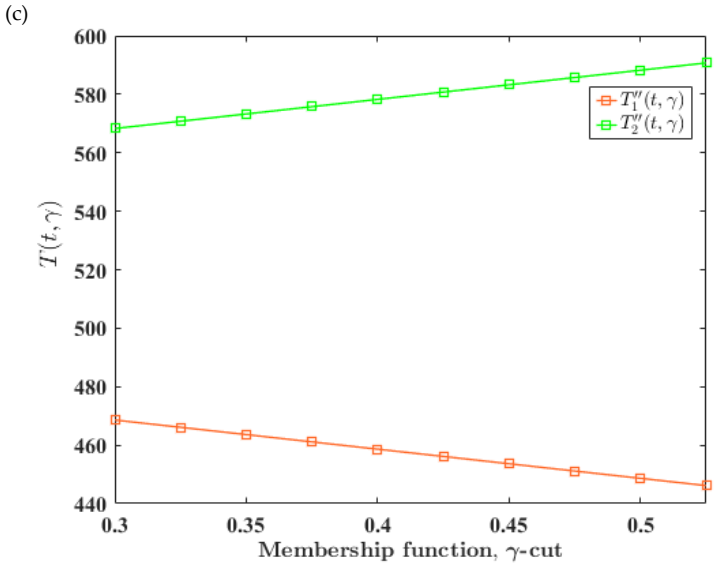
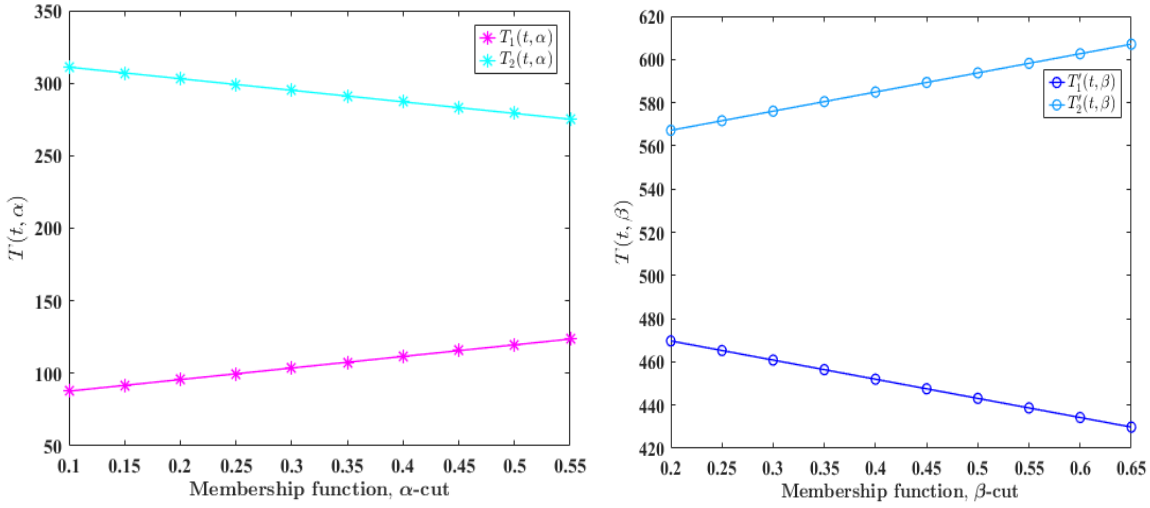


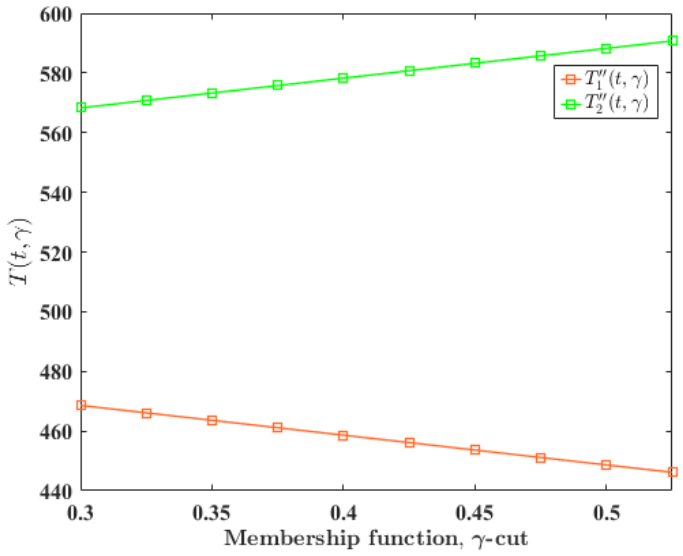
Fig. 11. $(a),(b),(c)$ Impact of t in $T(t,\alpha)$, $T(t,\beta)$, *and* $T(t,\gamma)$ using $TrapSVN_{number}$

6. Conclusion

In recent years, mathematical modeling that incorporates uncertainty or vagueness has found significant applications across various industries and engineering fields. In this research paper, we analyze the solution of first-order DE in neutrosophic environment. The $(\alpha,\beta,\gamma)-cut$ neutrosophic set and the strong and weak solutions of neutrosophic DE concepts are also applied for the DE with a $TrapN_{number}$ and a linear generalized $TrapN_{number}$as initial conditions. Furthermore, the application of this first-order DE to the heat convection problem in fluid dynamics provides a more comprehensive interpretation of differential equations in uncertain conditions.



(a) (b)



(c)

Fig. 12. $(a),(b),(c)$ Impact of t in $T(t,\alpha)$, $T(t,\beta)$, *and* $T(t,\gamma)$ using $TrapSVN_{number}$

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Conflict of Interest

The authors of this research study declare that they have no competing interests.

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Intelligent Elderly Care: Practicing Ambiguity Neutrosophic Theory for Optimization Contemporary Machine Learning Techniques in Elderly Care

Mahmoud M. Ismail [1], Ahmed A. Metwaly[2] , Osama ElKomy3,Alaa Al-Ghamry[2] ,and Mona Mohamed[4*]

1 Decision Support Department, Faculty of Computers and Informatics, Zagazig University, Zagazig, 44519, Egypt, mmsba@zu.edu.eg;
2 Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Zagazig 44519, Egypt, a.metwaly23@fci.zu.edu.eg.
3 Department of Information Technology, Faculty of Computers and Informatics, Zagazig University, Zagazig 44519, Egypt, omelkomy@fci.zu.edu.eg.
4 Higher Technological Institute, 10th of Ramadan City 44629, Egypt, Mona.fouad@hti.edu.eg;

* Correspondence: Mona.fouad@hti.edu.eg;

Abstract: One of the biggest challenges facing healthcare systems throughout the world is the aging population. The growing number of elderly citizens in need of specialized care is severely straining the available resources and care methods. It is sometimes difficult for traditional care techniques to address the complex and varied requirements of this expanding population. To fix these challenges, inclusion of contemporary technology is imperative and pragmatic solutions such as Internet of Thing (IoT), cloud computing (CC), and artificial intelligence (AI) techniques such as machine learning (ML), deep learning (DL). Such AI has potential role to make elderly care services (ECSs) to be intelligent ECSs (IECSs) through providing proactivity and earlier detection based on smart IoT sensors. Therefore, this study seeks to achieve two objectives. Firstly, leveraging the capabilities of ML techniques to revolutionize care delivery to be intelligent, optimize resource allocation, and proactive. Secondly, evaluating the robustness of utilized ML techniques in serving study's objectives. Accordingly, utilized ML Techniques consider alternatives (MLTs) that evaluate based on CRiteria Importance Through Inter-criteria Correlation (CRITIC) to obtain weights for criteria which alternatives evaluated based on. These weights are leveraging in Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) to rank MLTs alternatives. For bolstering the evaluation process, we are integrating uncertainty theory of Probabilistic Simplified Neutrosophic Set (PSNS) which effectively captures the inherent uncertainty and imprecision

Keywords: intelligent elderly care services; machine learning; contemporary technology; Probabilistic Simplified Neutrosophic Set; uncertainty.

1. 1. Introduction

1.1 Context and Motivations

There is a growing need for effective and efficient elderly care solutions as the world's population ages. The United Nations estimated that the number of people 60 and over will double from 1 billion in 2020 to almost 2.1 billion by 2050, making up 26% of the world's population. For families, social services, and healthcare systems, this sharp rise poses serious difficulties [1].Economically[2], [3] the aging population has an impact on budgetary policy. Financial hardship may result from the growing load on pension funds and social security systems if they are not properly handled. Socially [4], a lack of personalized support is characteristic of traditional care models that are finding it more and more difficult to address the needs of the aging population. New methods of providing care that emphasize proactive treatments, individualized assistance, and effective resource use are required due to the growing incidence of chronic illnesses and cognitive decline.

Consequently, elder care services have been becoming an essential aspect of the strategy as they provide a more accessible way to serve the elderly, with a focus on those who are handicapped or semi-impaired. Hence, it is imperative to necessitate innovative approaches to support older adults in maintaining their health, independence, and overall quality of life.

There has been a surge in several methodologies that can be applied to improve the efficiency of elderly care practices by addressing their mental health needs. First, both research and the industrial community should exert much effort to provide clear guidance to disabled elderly individuals.

regarding the usage of intelligent systems and services in their daily lives[5].This can help alleviate the misconceptions or cognitive biases that elderly people can have regarding elderly care services.

Second, care systems require continuous development of intelligent functions that can learn the common patterns in the care requirements of a particular population, thereby becoming able to meet the real, rudimentary requirements of disabled elderly individuals [6].Third, while providing rudimentary services is imperative, enhancing the reliability of these care services and making them easy to personalize to the needs of corresponding individuals is highly recommended. This can be offered through intelligent systems that can be tailored to the explicit needs and preferences of elderly individuals [7]. For instance, intelligent services can be provided ranging from elementary detection to personalized health sustenance, which allows taking care of elderly individuals based on their exclusive conditions.

Thereby[8] provided an innovative solution to the problems of caring for the elderly. This solution is formed in Information and communication technology (ICT) which provides a wealth of chances to improve people's quality of life and services. Utilizing ICT[9] for elderly people includes a variety of uses, such as online support communities, mobile health apps, telehealth services, and systems for distant surveillance. Figure 1 showcases samples of technologies ICT and its positive roles in converting traditional elder care services into intelligent elder care services (IECSs).

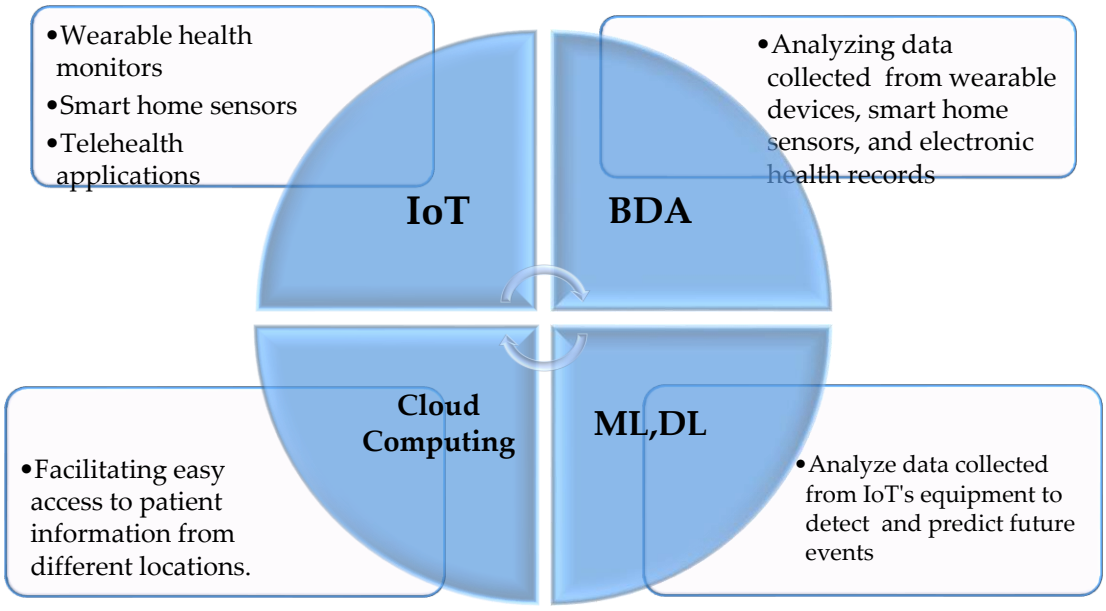
•Wearable health monitors
•Smart home sensors
•Telehealth applications
IoT
•Analyzing data collected from wearable devices, smart home sensors, and electronic health records
BDA
Cloud Computing
•Facilitating easy access to patient information from different locations.
ML,DL
•Analyze data collected from IoT's equipment to detect and predict future events

Figure 1. Role of ICT Toward Digitizing Elder Care Services

1.2 Contributions: Bolstering IECSs

We formed the provided contributions of the study into a set of hypotheses.

H1: What are deployed methodologies for bolstering ECSs to be IECSs?

Herein, we are exploiting the ability of ML [10] to proactive and individualized care for elders by analyzing large, composite data, discovering, learning patterns, and performing predictions with no or negligible human intervention. As well as [11] stated that ML has the potential to personalize the solutions for activity monitoring, fall detection, and early diagnosis of age-related settings. Scholars of [12]are settling ML as a cornerstone of modern elder care technologies. Due to its ability to process real-time data from wearable sensors, ML algorithms became able to forecast possible anomalies and deliver timely alerts, which can ensure proactive safety and quality of life for elder people.

H2: Evaluating the ability of ML techniques for Predicting future occurrences based on data collected from smart equipment.

Neutrosophic theory, entrenched in handling uncertainty, indeterminacy, and imprecision, represents a vigorous framework for decision-making in a broad range of complex applications with uncertain environments [13].It represents an extension to fuzzy logic, in which neutrosophic sets introduce nuanced data representation, which accommodates contradictory, incomplete, or vague information [14]. With this representational ability, neutrosophic sets come to be valuable to elder care, in which decision-making always includes many criteria and uncertain settings. whereinc the integration of neutrosophic theory into healthcare makes it easy to accurately assess risk by considering the inherent uncertainties daily activities of elder people. Leveraged neutrosophic theory can improve decision-making abilities by offering reliable and adaptive solutions that can effectively meet the complex requirements of elderly individuals[15].

H3: What are utilized methodologies for evaluating and prioritizing ML techniques?

Multi-attribute decision-making (MADM) refers to the practice of decision-making that prioritizes solutions in alignment with existing information through specific methodologies [16]–[18] . Its theories and models are extensively utilized across various domains such as venture-capital decision-making, project evaluation, and industrial sector development evaluation [19], [20] . In recent decades, MADM has seen broad applications in different fields[21], [22] . The evaluation of community home-based elderly care services quality for disabled elderly individuals constitutes a MADM. Recently, the TOPSIS model [23] has been employed to rank alternatives and obtaining the optimal. As an extension of the traditional neutrosophic set, the Probabilistic Simplified Neutrosophic Set (PSNS) was presented to handle uncertainty, imprecision, and indeterminacy in a more organized and probabilistic way[24] . Like neutrosophic sets, PSNS represents an element with three components: truth membership (T), indeterminacy membership (I), and falsity membership (F), however, it assigns each of these components with probabilistic value, which allows a more flexible representation of information by integrating randomness as well as indeterminacy concurrently. This makes PSNS ease its practical implementation while preserving the competence to model uncertainty in real-world problems., making them an active tool for improving MADM.

Generally, we are leveraging ML techniques in this study to early detect of falls of elderly people in indoor environments. The performance of various ML algorithms is evaluated using PSNNs to capture uncertainty, imprecision, and indeterminacy in the evaluation process. wherein the evaluation is conducted based on a set of criteria. Hence, MCDM techniques of CRITIC and TOPSIS

are used to obtain weights for utilized criteria and utilize the obtained weights from CRITIC in TOPSIS to rank alternatives of ML algorithms. The findings and discussions of proof-of-concept experiments have demonstrated our framework's capacity to guarantee a thorough and reliable ranking, hence enhancing the decision-making process in elder care systems.

The structure of study is organized as follows. First, we discuss definitions and essential concepts Sect. 2. The, methodology procedures are described in Sect. 3. Next, we discuss Valdity of elderly care and related setups as given in Sect 4. Sect. 5 reports the comparative analysis. Sect. 5 concludes the work.

2. Preliminaries

The objective of this section is to exhibit the basic concepts of utilized techniques.

Definition 1: Wang et al. described Single Value Neutrosophic Sets (SVNSs) in [25] and formed as:

$$CC = \{(\theta, CT(\theta), CI(\theta), CF(\theta)) | \theta \in \Theta\} \quad (1)$$

where $CT(\theta), CI(\theta), CF(\theta)$ is membership, indeterminacy-membership and falsity-membership, $CT(\theta), CI(\theta), CF(\theta) \in [0,1]$, $0 \le CT(\theta) + CI(\theta) + CF(\theta) \le 3$.

Definition 2: PSNSs have been clarified in c as:

$$PSNS = \left\{\left(\theta, CT(\theta)(PCT(\theta)), CI(\theta)(PCI(\theta)), CF(\theta)(PCF(\theta))\right) | \theta \in \Theta\right\} \quad (2)$$

where $PCT(\theta), PCI(\theta), PCF(\theta)$ is possibility values of $CT(\theta), CI(\theta), CF(\theta)$. The probabilistic simplified neutrosophic number (PSNN) is listed as $PC = (CT(PCT), CI(PCI), CF(PCF))$.

Definition 3: Let two $PSNNs\ sets$, [24] clarified these sets as $PSNN_1 = (CT_1(PCT_1), CI_1(PCI_1), CF_1(PCF_1))$, $PSNN_2 = (CT_2(PCT_2), CI_2(PCI_2), CF_2(PCF_2))$, the basic operations are put forward:

(1) $$PSNN_1 \oplus PSNN_2 = \begin{pmatrix} CT_1 + CT_2 - CT_1 \cdot CT_2\left(2!\frac{PCT_1 \cdot PCT_2}{PCT_1 + PCT_2}\right), \\ CI_1 \cdot CI_2\left(2!\frac{PCI_1 \cdot PCI_2}{PCI_1 + PCI_2}\right), CF_1 \cdot CF_2\left(2!\frac{PCF_1 \cdot PCF_2}{PCF_1 + PCF_2}\right) \end{pmatrix};$$

(2) $$PSNN_1 \otimes PSNN_2 = \begin{pmatrix} CT_1 \cdot CT_2\left(2!\frac{PCT_1 \cdot PCT_2}{PCT_1 + PCT_2}\right), \\ CI_1 + CI_2 - CI_1 \cdot CI_2\left(2!\frac{PCI_1 \cdot PCI_2}{PCI_1 + PCI_2}\right), \\ CF_1 + CF_2 - CF_1 \cdot CF_2\left(2!\frac{PCF_1 \cdot PCF_2}{PCF_1 + PCF_2}\right) \end{pmatrix};$$

(3) $\lambda\, PSNN = \left(1 - (1 - CT)^{\lambda}(PCT), (CI)^{\lambda}(PCI), (CF)^{\lambda}(PCF)\right), \lambda > 0;$

(4) $(PSNN)^{\lambda} = \left((CT)^{\lambda}(PCT), 1 - (1 - CI)^{\lambda}(PCI), 1 - (1 - CF)^{\lambda}(PCF)\right), \lambda > 0$

Definition 4 [24] **:** Given $PSNN_1 = \left(CT_1(PCT_1), CI_1(PCI_1), CF_1(PCF_1)\right)$ and $PSNN_2 = \left(CT_2(PCT_2), CI_2(PCI_2), CF_2(PCF_2)\right)$, the PSNN Logarithmic distance (LD_{PSNN}) between PC_1 and PC_2 is constructed:

$$\boldsymbol{LD_{PSNN}(PSNN_1, PSNN_2) = \frac{1}{3}\left(\begin{array}{l} (\boldsymbol{CT_1 \times PCT_1})\, \boldsymbol{log}\, \dfrac{(\boldsymbol{CT_1 \times PCT_1})}{\dfrac{(\boldsymbol{CT_1 \times PCT_1}) + (\boldsymbol{CT_2 \times PCT_2})}{\boldsymbol{2}}} \\ +(\boldsymbol{CT_2 \times PCT_2})\, \boldsymbol{log}\, \dfrac{(\boldsymbol{CT_2 \times PCT_2})}{\dfrac{(\boldsymbol{CT_1 \times PCT_1}) + (\boldsymbol{CT_2 \times PCT_2})}{\boldsymbol{2}}} \\ +(\boldsymbol{CI_1 \times PCI_1})\, \boldsymbol{log}\, \dfrac{(\boldsymbol{CI_1 \times PCI_1})}{\dfrac{(\boldsymbol{CI_1 \times PCI_1}) + (\boldsymbol{CI_2 \times PCI_2})}{\boldsymbol{2}}} \\ +(\boldsymbol{CI_2 \times PCI_2})\, \boldsymbol{log}\, \dfrac{(\boldsymbol{CI_2 \times PCI_2})}{\dfrac{(\boldsymbol{CI_1 \times PCI_1}) + (\boldsymbol{CI_2 \times PCI_2})}{\boldsymbol{2}}} \\ +(\boldsymbol{CF_1 \times PCF_1})\, \boldsymbol{log}\, \dfrac{(\boldsymbol{CF_1 \times PCF_1})}{\dfrac{(\boldsymbol{CF_1 \times PCF_1}) + (\boldsymbol{CF_2 \times PCF_2})}{\boldsymbol{2}}} \\ +(\boldsymbol{CF_2 \times PCF_2})\, \boldsymbol{log}\, \dfrac{(\boldsymbol{CF_2 \times PCF_2})}{\dfrac{(\boldsymbol{CF_1 \times PCF_1}) + (\boldsymbol{CF_2 \times PCF_2})}{\boldsymbol{2}}} \end{array}\right)} \quad (\boldsymbol{3})$$

3. Methodology: Deployment and Evaluation Procedures

Herein, we are clarifying the procedures for implementing ML techniques to serve the study objectives. As well AS modeling the complex patterns from sensory information about elder people. After that, the role of PSNSs begins to promote harnessed MADM to prioritize alternatives of utilized ML techniques implemented in an uncertain environment**.**

Generally speaking, the utilized ML techniques in our study are illustrated in Table 3. The procedures for deploying these techniques are implemented as follows:

Step 1: we normalize all sensor measurements into a similar scale using min-max normalization, which prevents features with large ranges from dominating the model.

Step 2: we input the missing samples with mean values to maintain dataset integrity.

Step 3: we apply z-score to remove outliers

Step 4: we choose appropriate ML models to classify different older people's activities.

Step 5: we train these models under *k*-fold cross-validation strategy to ensure robust model evaluation.

Step 6: Evaluation Process

Evaluating the harnessed ML techniques using MADM techniques that work under the authority of PSNSs. Hence, the basic aspects of the evaluation process are determined as:

- Let $A = \{MLT_1, MLT_2, \cdots, MLT_m\}$ be alternatives, and $C = \{C_1, C_2, \cdots, C_n\}$Be attributes with weight w_c, where $w_c^j \in [0,1], \sum_{j=1}^{n} w_c^j = 1$.

- Suppose that the evaluated data is represented with PSNN-based decision matrix Ð. Then, the PSNN-TOPSIS technique is applied in conjunction with the CRITIC technique's weights to select the best alternative
- Construct PSNN-matrix $Ð = \left(Ð_{ij}\right)_{m\times n} = \left(CT_{ij}\left(PCT_{ij}\right), CI_{ij}\left(PCI_{ij}\right), CF_{ij}\left(PCF_{ij}\right)\right)_{m\times n}$
- Put forward the normalized matrix $Ð^{normalized} = \left[Ð_{ij}^{normalized}\right]_{m\times n}$ based on $Ð = \left(Ð_{ij}\right)_{m\times n}$.

$$Ð_{ij}^{normalized} = \left(CT_{ij}^{N}\left(PCT_{ij}^{N}\right), CI_{ij}^{N}\left(PCI_{ij}^{N}\right), CF_{ij}^{N}\left(PCF_{ij}^{N}\right)\right) = \begin{cases} \left(CT_{ij}\left(PCT_{ij}\right), CI_{ij}\left(PCI_{ij}\right), CF_{ij}\left(PCF_{ij}\right)\right) \\ \left(CF_{ij}\left(PCF_{ij}\right), 1 - CI_{ij}\left(1 - PCI_{ij}\right), CT_{ij}\left(PCT_{ij}\right)\right) \end{cases} \quad (4)$$

6.1. Generating criteria weight based on PSNN-CRITIC

The CRITIC technique[26] is utilized to put forward the weight. The decision steps of CRITIC technique are constructed

1. From the PSNN-matrix $Ð^{normalized} = \left[Ð_{ij}^{normalized}\right]_{m\times n}$, the PSNN correlation coefficient (r^{PSNN}) for attributes is put forward.

$$r_{jt}^{PSNN} = \frac{\sum_{i=1}^{m}\left(SV\left(Ð_{ij}^{normalized}\right) - SV\left(Ð_{j}^{normalized}\right)\right)\left(SV\left(Ð_{it}^{normalized}\right) - SV\left(Ð_{t}^{normalized}\right)\right)}{\sqrt{\sum_{i=1}^{m}\left(SV\left(Ð_{ij}^{normalized}\right) - SV\left(Ð_{j}^{normalized}\right)\right)^{2}}\sqrt{\sum_{i=1}^{m}\left(SV\left(Ð_{it}^{normalized}\right) - SV\left(Ð_{t}^{normalized}\right)\right)^{2}}},$$

for all $j, t = 1,2,\dots,n,$ (5)

$$SV\left(Ð_{j}^{normalized}\right) = \frac{1}{m}\sum_{i=1}^{m} SV\left(Ð_{ij}^{normalized}\right) = \frac{CT_{ij}^{N}\cdot PCT_{ij}^{N} + CI_{ij}^{N}\cdot PCI_{ij}^{N} + CF_{ij}^{N}\cdot PCF_{ij}^{N}}{3m}$$

and

$$SV\left(Ð_{it}^{normalized}\right) = \frac{1}{m}\sum_{i=1}^{m} SV\left(Ð_{it}^{normalized}\right) = \frac{CT_{it}^{N}\cdot PCT_{it}^{N} + CI_{it}^{N}\cdot PCI_{it}^{N} + CF_{it}^{N}\cdot PCF_{it}^{N}}{3m}$$

2. Computing standard deviation (PSNNSD) based on Eq.(6).

$$SD_{j}^{PSNN} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(SV\left(Ð_{ij}^{normalized}\right) - SV\left(Ð_{j}^{normalized}\right)\right)^{2}} \quad (6)$$

where $SV\left(Ð_{j}^{normalized}\right) = \frac{1}{m}\sum_{i=1}^{m} SV\left(Ð_{ij}^{normalized}\right)$.

3. Put forward the weight information.

$$w_{c}^{j} = \frac{SD_{j}^{PSNN}\sum_{t=1}^{n}\left(1 - r_{jt}^{PSNN}\right)}{\sum_{j=1}^{n}\left(SD_{j}^{PSNN}\sum_{t=1}^{n}\left(1 - r_{jt}^{PSNN}\right)\right)} \quad (7)$$

where $w_{c}^{j} \in [0,1]$ and $\sum_{j=1}^{n} w_{c}^{j} = 1$.

6.2: Ranking ML alternatives based on PSNN-TOPSIS

Put forward the PSNN positive ideal decision alternative ($PIDA_{PSNN}$) and PSNN negative ideal decision alternative ($NIDA_{PSNN}$) :

$$PIDA_{PSNN} = \{PIDA_{PSNN}^{j}\}, for\ all\ j = 1,2, \dots, n. \quad (8)$$

$$NIDA_{PSNN} = NIDA_{PSNN}^{j}, for\ all\ j = 1,2, \dots, n. \quad (9)$$

$$PIDA_{PSNN}^{j} = \begin{pmatrix} CT_j^{+N}(PCT_j^{+N}), \\ CI_j^{+N}(PCI_j^{+N}), CF_j^{+N}(PCF_j^{+N}) \end{pmatrix} \quad (10)$$

$$NIDA_{PSNN}^{j} = \begin{pmatrix} CT_j^{-N}(PCT_j^{-N}), \\ CI_j^{-N}(PCI_j^{-N}), VC_j^{-N}(PCF_j^{-N}) \end{pmatrix} \quad (11)$$

$$SV\left(CT_j^{+N}(PCT_j^{+N}), CI_j^{+N}(PCI_j^{+N}), CF_j^{+N}(PCF_j^{+N})\right) = \max_i SV\left(CT_{ij}^{N}(PCT_{ij}^{N}), CI_{ij}^{N}(PCI_{ij}^{N}), CF_{ij}^{N}(PCF_{ij}^{N})\right) \quad (12)$$

$$SV\left(CT_j^{-N}(PCT_j^{-N}), CI_j^{-N}(PCI_j^{-N}), VC_j^{-N}(PCF_j^{-N})\right) = \min_i SV\left(CT_{ij}^{N}(PCT_{ij}^{N}), CI_{ij}^{N}(PCI_{ij}^{N}), CF_{ij}^{N}(PCF_{ij}^{N})\right) \quad (13)$$

where

$$SV\left(CT_j^{+N}(PCT_j^{+N}), CI_j^{+N}(PCI_j^{+N}), CF_j^{+N}(PCF_j^{+N})\right) = \frac{CT_j^{+N} \cdot PCT_j^{+N} + CI_j^{+N} \cdot PCI_j^{+N} + CF_j^{+N} \cdot PCF_j^{+N}}{3} \quad (14)$$

$$SV\left(CT_j^{-N}(PCT_j^{-N}), CI_j^{-N}(PCI_j^{-N}), VC_j^{-N}(PCF_j^{-N})\right) = \frac{CT_j^{-N} \cdot PCT_j^{-N} + CI_j^{-N} \cdot PCI_j^{-N} + CF_j^{-N} \cdot PCF_j^{-N}}{3} \quad (15)$$

$$SV\left(CT_{ij}^{N}(PCT_{ij}^{N}), CI_{ij}^{N}(PCI_{ij}^{N}), CF_{ij}^{N}(PCF_{ij}^{N})\right) = \frac{CT_{ij}^{N} \cdot PCT_{ij}^{N} + CI_{ij}^{N} \cdot PCI_{ij}^{N} + CF_{ij}^{N} \cdot PCF_{ij}^{N}}{3} \quad (16)$$

1. Put forward the PSNN positive Logarithmic distance (PLD_{PSNN}) from $PIDA_{PSNN}$ and PSNN negative Logarithmic distance (NLD_{PSNN}) from $NIDA_{PSNN}$:

$$PLD_{PSNN}(A_i, PIDA_{PSNN}) = \sum_{j=1}^{n} cw_j \left(PLD_{PSNN}(A_i, PIDA_{PSNN}^{j})\right)$$

$$= \sum_{j=1}^{n} cw_j \left(\begin{array}{l} \left(CT_{ij}^{N} \times PCT_{ij}^{N}\right) \log \left(\dfrac{\left(CT_{ij}^{N} \times PCT_{ij}^{N}\right)}{avg\left(\left(CT_{ij}^{N} \times PCT_{ij}^{N}\right), \left(CT_{j}^{+N} \times PCT_{j}^{+N}\right)\right)} \right) \\ +\left(CT_{j}^{+N} \times PCT_{j}^{+N}\right) log\left(\dfrac{\left(CT_{j}^{+N} \times PCT_{j}^{+N}\right)}{avg\left(\left(CT_{ij}^{N} \times PCT_{ij}^{N}\right), \left(CT_{j}^{+N} \times PCT_{j}^{+N}\right)\right)} \right) \\ +\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right) log\left(\dfrac{\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right)}{avg\left(\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right), \left(CI_{j}^{+N} \times PCI_{j}^{+N}\right)\right)} \right) \\ +\left(CI_{j}^{+N} \times PCI_{j}^{+N}\right) log\left(\dfrac{\left(CI_{j}^{+N} \times PCI_{j}^{+N}\right)}{avg\left(\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right), \left(CI_{j}^{+N} \times PCI_{j}^{+N}\right)\right)} \right) \\ +\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right) log\left(\dfrac{\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right)}{Avg\left(\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right), \left(CF_{j}^{+N} \times PCF_{j}^{+N}\right)\right)} \right) \\ +\left(CF_{j}^{+N} \times PCF_{j}^{+N}\right) log\left(\dfrac{\left(CF_{j}^{+N} \times PCF_{j}^{+N}\right)}{Avg\left(\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right), \left(CF_{j}^{+N} \times PCF_{j}^{+N}\right)\right)} \right) \end{array} \right) \quad (17)$$

$$NLD_{PSNN}(A_i, NIDA_{PSNN}) = \sum_{j=1}^{n} cw_j \left(NLD_{PSNN}\left(A_i, NIDA_{PSNN}^{j}\right) \right)$$

$$= \sum_{j=1}^{n} cw_j \left(\begin{array}{l} \left(CT_{ij}^{N} \times PCT_{ij}^{N}\right) \log \left(\dfrac{\left(CT_{ij}^{N} \times PCT_{ij}^{N}\right)}{avg\left(\left(CT_{ij}^{N} \times PCT_{ij}^{N}\right), \left(CT_{j}^{-N} \times PCT_{j}^{-N}\right)\right)} \right) \\ +\left(CT_{j}^{-N} \times PCT_{j}^{-N}\right) log\left(\dfrac{\left(CT_{j}^{-N} \times PCT_{j}^{-N}\right)}{avg\left(\left(CT_{ij}^{N} \times PCT_{ij}^{N}\right), \left(CT_{j}^{-N} \times PCT_{j}^{-N}\right)\right)} \right) \\ +\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right) log\left(\dfrac{\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right)}{avg\left(\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right), \left(CI_{j}^{-N} \times PCI_{j}^{-N}\right)\right)} \right) \\ +\left(CI_{j}^{-N} \times PCI_{j}^{-N}\right) log\left(\dfrac{\left(CI_{j}^{-N} \times PCI_{j}^{-N}\right)}{avg\left(\left(CI_{ij}^{N} \times PCI_{ij}^{N}\right), \left(CI_{j}^{-N} \times PCI_{j}^{-N}\right)\right)} \right) \\ +\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right) log\left(\dfrac{\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right)}{Avg\left(\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right), \left(CF_{j}^{-N} \times PCF_{j}^{-N}\right)\right)} \right) \\ +\left(CF_{j}^{-N} \times PCF_{j}^{-N}\right) log\left(\dfrac{\left(CF_{j}^{-N} \times PCF_{j}^{-N}\right)}{Avg\left(\left(CF_{ij}^{N} \times PCF_{ij}^{N}\right), \left(CF_{j}^{-N} \times PCF_{j}^{-N}\right)\right)} \right) \end{array} \right) \quad (18)$$

2. Put forward the PSNN close degree values (PSNNCDV) from PSNNPIDA.

$$PSNNCDV(CA_i, PIDA_{PSNN}) = \frac{PLD_{PSNN}(A_i, PIDA_{PSNN})}{\left(NLD_{PSNN}(A_i, NIDA_{PSNN}) + PLD_{PSNN}(A_i, PIDA_{PSNN})\right)} \quad (19)$$

3. According to $PSNNCDV(A_i, NIDA_{PSNN})$. The lower $PSNNCDV(A_i, PIDA_{PSNN})$ is selected as the better alternative.

4. Validation Procedures

To demonstrate the efficacy of the proposed research methodology, a public case study is carefully chosen, in which healthy older people are monitored and recorded based on battery less wearable sensors.

4.1 Description of Required Aspects

The data of this case study was labeled for multi-class activity recognition tasks. In particular, the motion data was collected by 14 healthy elderly individuals in age intervals from 66 and 86 years old, each was wearing batteryless wearable sensors on top of their clothing at sternum level. The data was inherently and noisy because of the reliance on a passive sensor. This reflects the appropriateness of our experiments as it reflects uncertainty settings. The data contained nine variables namely 'Time in seconds', 'Acceleration reading in G for frontal axis', 'Acceleration reading in G for vertical axis', 'Acceleration reading in G for lateral axis', 'Id of antenna reading sensor', 'Received signal strength indicator (RSSI)', 'Phase', 'Frequency', and 'Label'. The sensory data was labeled with three distinct labels namely 1: sit on bed, 2: sit on chair, 3: lying, 4: ambulating. To get useful insights about the sensory data, we present a concise summary of descriptive statistics in Table 1. The data contains a total of 52482 observations. It could be noted that the RSSI values span from -70 to -38.5, with a mean of -58.431, which indicates signal strength distribution.

The ranges of constituting variables diverge expressively, with some (e.g., time and frequency) spanning large intervals, while others (e.g., frontal axis and vertical axis) are limited to slighter scales. This necessitates scaling or normalization of values to guarantee uniformity across variables for ML models. Additionally, there are no missing values, as evidenced by the consistent count for all variables. The data is well-structured, making it suitable for preprocessing and further analysis, such as predictive modeling and classification tasks.

Table 1. Summarized descriptive statistics for elder people care case study

	Time	frontal axis	vertical axis	lateral axis	antenna ID	RSSI	Phase	Frequency	Label
count	52482	52482	52482	52482	52482	52482	52482	52482	52482
mean	235.002	0.80504	0.3778	0.00771	2.36075	-58.431	3.27591	922.762	2.376
std	148.257	0.39636	0.4689	0.18067	1.26154	4.61122	2.24034	1.69377	0.9418
min	0	-0.7481	-0.5535	-0.4812	1	-70	0	920.25	1
25%	117.98	0.42446	-0.0253	-0.0935	1	-61.5	0.95107	921.25	1
50%	227.665	0.9521	0.1355	-0.0251	3	-57.5	4.0727	922.75	3
75%	325.75	1.128	0.91636	0.06614	4	-56.5	5.4257	924.25	3
max	730.25	1.5032	2.0302	1.2178	4	-38.5	6.2817	925.75	4

Proof-of-concept experimentations are conducted on a Dell workstation operated by Windows 10 64-bit. This workstation was equipped with 64 G RAM and Intel(R) Core(TM) i5-7200U CPU @ 2.50GHz 2.71 GHz. To facilitate the reproducibility of ML experimental results, we provide a summary of training settings, as shown in Table 2.

Table 2. Summary of implementation setups.

Description	Value	Description	Value

Session id	42	Imputation type	simple
Target	Decision	Numeric imputation	mean
Target type	Multiclass	Categorical imputation	mode
Original data shape	(41985, 9)	Fold Generator	StratifiedKFold
Transformed data shape	(41985, 9)	Fold Number	10
Transformed train set shape	(33588, 9)	CPU Jobs	-1
Transformed test set shape	(8397, 9)	Use GPU	FALSE
Numeric features	8	Log Experiment	FALSE
Preprocess	TRUE	Experiment Name	clf-default-name

4.2 *Metrics of Validation*

The process of evaluating the classification performance of different ML algorithms is conducted based on a popular set of evaluation metrics, which are computed based on the elementary components of the confusion matrix: True positive (TP), False Positive (FP), True Negative (TN), and False Negative (FN).

$$\mathbf{Accuracy} = \frac{\mathbf{TP} + \mathbf{TN}}{\mathbf{TP} + \mathbf{TN} + \mathbf{FP} + \mathbf{FN}} \quad (\mathbf{20})$$

$$\text{Precision} = \frac{\text{TP}}{\text{TP} + \text{FP}} \quad (21)$$

$$\mathbf{Recall} = \frac{\mathbf{TP}}{\mathbf{TP+FN}} \quad \mathbf{(22)}$$

$$\mathbf{F_1 - score} = \mathbf{2}\frac{\mathbf{precision \times recall}}{\mathbf{precision + recall}} \quad (23)$$

$$\mathbf{Matthews\ Correlation\ Coefficient\ (MCC)} = \frac{\mathbf{TP \times TN - FP \times FN}}{\sqrt{(\mathbf{TP + FP})(\mathbf{TP + FN})(\mathbf{TN + FP})(\mathbf{TN + FN})}} \quad (24)$$

$$\text{cohen kappa score} = \frac{\text{p}_\text{o} \times \text{p}_\text{e}}{1 - \text{P}_\text{e}} \quad (25)$$

where Po symbolizes the relative observed agreement among raters. Pe symbolizes the hypothetical possibility of agreement occurring by chance.

5. Comparative Analysis and Discussion

In Table 3, we introduce and compare the classification performance of various ML algorithms involved in our framework, according to different evaluation metrics as mentioned before. These metrics aim to detect the strengths and weaknesses of each algorithm for monitoring and caring for elderly people's activities. Also, they give the decision-makers valuable insights into the trade-offs between different performance aspects. With these tabulated results, we aren't only highlighting the erraticism in algorithm performance, but we are informing the decision-makers to select the ML algorithms that can satisfy the needs of elder people's healthcare.

Table 3. Quantitative comparison of the performance of ML for elder people activity recognition

ML algorithms	Accuracy	Recall	Prec.	F1-score	Kappa	MCC
Extreme Gradient Boosting	0.9932	0.9932	0.9932	0.9931	0.9879	0.9879
Light Gradient Boosting Machine	0.9927	0.9927	0.9926	0.9925	0.9869	0.9870
Random Forest Classifier	0.9919	0.9919	0.9919	0.9917	0.9856	0.9856
Extra Trees Classifier	0.9915	0.9915	0.9916	0.9912	0.9848	0.9849
Decision Tree Classifier	0.9883	0.9883	0.9883	0.9883	0.9791	0.9791
Gradient Boosting Classifier	0.9845	0.9845	0.9842	0.9835	0.9723	0.9725
K Neighbors Classifier	0.9820	0.9820	0.9816	0.9814	0.9678	0.9679
Quadratic Discriminant Analysis	0.9548	0.9548	0.9540	0.9541	0.9188	0.9190
Naive Bayes	0.9345	0.9345	0.9360	0.9294	0.8807	0.8833
Logistic Regression	0.9029	0.9029	0.9027	0.8851	0.8213	0.8294
Linear Discriminant Analysis	0.8955	0.8955	0.8883	0.8831	0.8088	0.8131
SVM - Linear Kernel	0.8896	0.8896	0.8759	0.8548	0.7949	0.8087
Ridge Classifier	0.8730	0.8730	0.7860	0.8227	0.7613	0.7795
Ada Boost Classifier	0.7579	0.7579	0.8380	0.7672	0.6042	0.6412

To formulate the above results as a MADM problem, in which healthcare stockholders can rank and select ML algorithms according to their needs, we consider ML algorithms as alternatives, while the performance metrics are considered as the criteria for evaluation. This approach enables a systematic comparison of the models based on multiple performance dimensions simultaneously. With the inherent uncertainty and variability in model performance, we employ TOPSIS based on PSNN- methodology. This enhanced TOPSIS framework leverages neutrosophic logic to effectively handle indeterminacy and imprecision in the criteria evaluations. Based on these comparisons, we have many ML systems for elderly care. Wherein Table 4 provides a concise mapping and definition of both system alternatives and evaluation criteria driven by ML results, which can be later used for our MADM problem.

Table 4. Formulation of alternatives and criteria in MADM

Metrics	**Accuracy**	**Recall**	**Prec.**	**F1-score**	**Kappa**	**MCC**
Criteria	C1	C2	C3	C4	C5	C6

ML algorithms →	**Alternatives**
Extreme Gradient Boosting	MLT1
Light Gradient Boosting Machine	MLT2
Random Forest Classifier	MLT3
Extra Trees Classifier	MLT4
Decision Tree Classifier	MLT5
Gradient Boosting Classifier	MLT6
K Neighbors Classifier	MLT7
Quadratic Discriminant Analysis	MLT8
Naive Bayes	MLT9
Logistic Regression	MLT10
Linear Discriminant Analysis	MLT11
SVM - Linear Kernel	MLT12
Ridge Classifier	MLT13
Ada Boost Classifier	MLT14

Following, quantitative results from Table 3 are encoded into PSNN representation with three components for each criterion, as shown in Table 5. This encoding process involved insights of many ML experts to assess the reliability of each evaluation metric concerning each algorithm, which helped ensure that reliability is determined according to a mixture of expert judgment, and consistency across different criteria. Doing so, the indeterminacy for each metric is calculated as the accompaniment of reliability, using the relationship $I = 1 - Reliability$. The tabulated values can effectively encode and capture the certainty as well as the ambiguity associated with the performance of various ML models, which offer a nuanced framework for decision-making. Given that ML results are inherently normalized, we can refer to Table 5, as the normalized matrix $NCR = \left[NCR_{ij}\right]_{11\times6}$.

Table 5. The normalized decision in the form of PSNNs

Model	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**
MLT1	{0.993 (0.634), 0.567 (0.362), 0.007 (0.004)}	{0.993 (0.822), 0.209 (0.173), 0.007 (0.006)}	{0.993 (0.806), 0.233 (0.189), 0.007 (0.006)}	{0.993 (0.757), 0.311 (0.237), 0.007 (0.005)}	{0.988 (0.812), 0.217 (0.178), 0.012 (0.01)}	{0.988 (0.814), 0.213 (0.176), 0.012 (0.01)}

MLT2	{0.993 (0.648), 0.531 (0.347), 0.007 (0.005)}	{0.993 (0.799), 0.243 (0.195), 0.007 (0.006)}	{0.993 (0.818), 0.214 (0.176), 0.007 (0.006)}	{0.993 (0.764), 0.299 (0.23), 0.007 (0.006)}	{0.987 (0.799), 0.235 (0.19), 0.013 (0.011)}	{0.987 (0.803), 0.229 (0.186), 0.013 (0.011)}
ML3	{0.992 (0.653), 0.52 (0.342), 0.008 (0.005)}	{0.992 (0.797), 0.245 (0.197), 0.008 (0.007)}	{0.992 (0.806), 0.23 (0.187), 0.008 (0.007)}	{0.992 (0.793), 0.251 (0.201), 0.008 (0.007)}	{0.986 (0.784), 0.256 (0.204), 0.014 (0.011)}	{0.986 (0.796), 0.239 (0.193), 0.014 (0.012)}
MLT4	{0.992 (0.662), 0.498 (0.332), 0.008 (0.006)}	{0.992 (0.741), 0.338 (0.252), 0.008 (0.006)}	{0.992 (0.658), 0.508 (0.337), 0.008 (0.006)}	{0.991 (0.71), 0.395 (0.283), 0.009 (0.006)}	{0.985 (0.823), 0.197 (0.164), 0.015 (0.013)}	{0.985 (0.779), 0.265 (0.21), 0.015 (0.012)}
MLT5	{0.988 (0.705), 0.401 (0.286), 0.012 (0.008)}	{0.988 (0.624), 0.583 (0.368), 0.012 (0.007)}	{0.988 (0.698), 0.416 (0.294), 0.012 (0.008)}	{0.988 (0.678), 0.458 (0.314), 0.012 (0.008)}	{0.979 (0.669), 0.463 (0.317), 0.021 (0.014)}	{0.979 (0.656), 0.492 (0.33), 0.021 (0.014)}
MLT6	{0.985 (0.674), 0.462 (0.316), 0.015 (0.011)}	{0.985 (0.635), 0.551 (0.355), 0.015 (0.01)}	{0.984 (0.673), 0.462 (0.316), 0.016 (0.011)}	{0.984 (0.616), 0.598 (0.374), 0.016 (0.01)}	{0.972 (0.644), 0.51 (0.338), 0.028 (0.018)}	{0.973 (0.664), 0.464 (0.317), 0.027 (0.019)}
MLT7	{0.982 (0.666), 0.474 (0.321), 0.018 (0.012)}	{0.982 (0.628), 0.564 (0.361), 0.018 (0.012)}	{0.982 (0.666), 0.474 (0.322), 0.018 (0.012)}	{0.981 (0.634), 0.548 (0.354), 0.019 (0.012)}	{0.968 (0.614), 0.576 (0.365), 0.032 (0.02)}	{0.968 (0.665), 0.456 (0.313), 0.032 (0.022)}
MLT8	{0.955 (0.649), 0.471 (0.32), 0.045 (0.031)}	{0.955 (0.613), 0.558 (0.358), 0.045 (0.029)}	{0.954 (0.616), 0.55 (0.355), 0.046 (0.03)}	{0.954 (0.623), 0.53 (0.347), 0.046 (0.03)}	{0.919 (0.596), 0.542 (0.351), 0.081 (0.053)}	{0.919 (0.62), 0.483 (0.326), 0.081 (0.055)}
MLT9	{0.935 (0.628), 0.488 (0.328), 0.066 (0.044)}	{0.935 (0.605), 0.545 (0.353), 0.066 (0.042)}	{0.936 (0.593), 0.579 (0.367), 0.064 (0.041)}	{0.929 (0.605), 0.537 (0.349), 0.071 (0.046)}	{0.881 (0.57), 0.545 (0.353), 0.119 (0.077)}	{0.883 (0.587), 0.504 (0.335), 0.117 (0.078)}
MLT10	{0.903 (0.616), 0.467 (0.318), 0.097 (0.066)}	{0.903 (0.587), 0.538 (0.35), 0.097 (0.063)}	{0.903 (0.572), 0.577 (0.366), 0.097 (0.062)}	{0.885 (0.589), 0.502 (0.334), 0.115 (0.076)}	{0.821 (0.528), 0.557 (0.358), 0.179 (0.115)}	{0.829 (0.547), 0.516 (0.34), 0.171 (0.113)}
MLT11	{0.896 (0.582), 0.539 (0.35), 0.105 (0.068)}	{0.896 (0.574), 0.561 (0.359), 0.105 (0.067)}	{0.888 (0.589), 0.507 (0.336), 0.112 (0.074)}	{0.883 (0.555), 0.593 (0.372), 0.117 (0.073)}	{0.809 (0.532), 0.519 (0.342), 0.191 (0.126)}	{0.813 (0.542), 0.501 (0.334), 0.187 (0.125)}
MLT12	{0.89 (0.608), 0.463 (0.317), 0.11 (0.075)}	{0.89 (0.592), 0.503 (0.335), 0.11 (0.073)}	{0.876 (0.555), 0.578 (0.366), 0.124 (0.079)}	{0.855 (0.566), 0.509 (0.337), 0.145 (0.096)}	{0.795 (0.523), 0.52 (0.342), 0.205 (0.135)}	{0.809 (0.532), 0.52 (0.342), 0.191 (0.126)}
MLT13	{0.873 (0.65), 0.343 (0.256), 0.127 (0.095)}	{0.873 (0.544), 0.604 (0.376), 0.127 (0.079)}	{0.786 (0.502), 0.566 (0.361), 0.214 (0.137)}	{0.823 (0.576), 0.428 (0.3), 0.177 (0.124)}	{0.761 (0.495), 0.538 (0.35), 0.239 (0.155)}	{0.78 (0.494), 0.579 (0.367), 0.221 (0.14)}
MLT14	{0.758 (0.437), 0.735 (0.424), 0.242 (0.14)}	{0.758 (0.399), 0.899 (0.473), 0.242 (0.127)}	{0.838 (0.556), 0.507 (0.337), 0.162 (0.107)}	{0.767 (0.457), 0.68 (0.405), 0.233 (0.139)}	{0.604 (0.355), 0.702 (0.412), 0.396 (0.233)}	{0.641 (0.402), 0.595 (0.373), 0.359 (0.225)}

Figure 2 represents the results of the PSNS-CRTIC computed to assign weight to each evaluation criterion, which reflects the relative importance of each criterion in causative decision-making according to its variability and impact. Wherein C3 recorded highest value otherwise C4 recorded lowest value

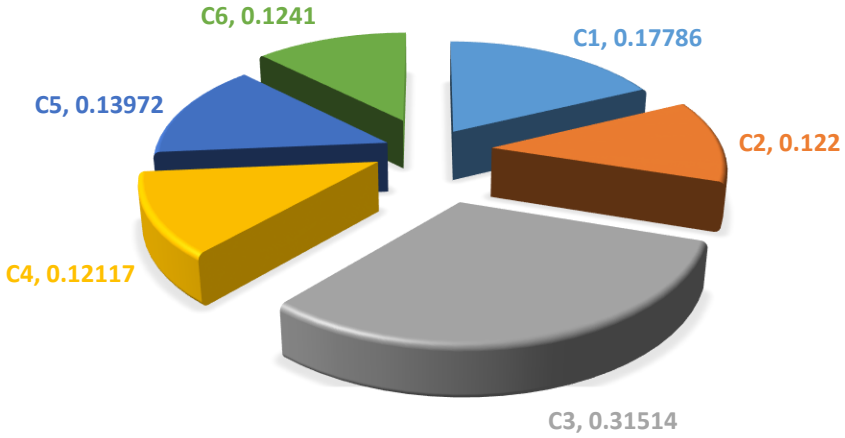

Figure 2. Final Criteria Weights

Table 6 presents the PSNNPIDA and PSNNNIDA derived as part of the decision-making framework. The former represents the optimal values for each criterion, which serve as benchmarks for alternatives. Conversely, the latter signifies the least needed performance values, which provide a reference for the lower bounds of acceptability.

Table 6. The $PIDA_{PSNN}$ and $NIDA_{PSNN}$

	$\mathbf{PIDA_{PSNN}}$	$\mathbf{NIDA_{PSNN}}$
C1	{0.993 (0.634), 0.567 (0.362), 0.007 (0.004)}	{0.758 (0.437), 0.735 (0.424), 0.242 (0.14)}
C2	{0.993 (0.799), 0.243 (0.195), 0.007 (0.006)}	{0.89 (0.592), 0.503 (0.335), 0.11 (0.073)}
C3	{0.993 (0.806), 0.233 (0.189), 0.007 (0.006)}	{0.786 (0.502), 0.566 (0.361), 0.214 (0.137)}
C4	{0.992 (0.793), 0.251 (0.201), 0.008 (0.007)}	{0.823 (0.576), 0.428 (0.3), 0.177 (0.124)}
C5	{0.988 (0.812), 0.217 (0.178), 0.012 (0.01)}	{0.604 (0.355), 0.702 (0.412), 0.396 (0.233)}
C6	{0.988 (0.814), 0.213 (0.176), 0.012 (0.01)}	{0.641 (0.402), 0.595 (0.373), 0.359 (0.225)}

Following, we make use of $PIDA_{PSNN}$ and $NIDA_{PSNN}$ to calculate the $PLD_{PSNN}(MLT_i, PIDA_{PSNN})$ and $NLD_{PSNN}(MLT_i, PIDA_{PSNN})$ presented in Table 7.

Table 7. The $NLD_{PSNN}(MLT_i, NIDA_{PSNN})$ and $PLD_{PSNN}(MLT_i, PIDA_{PSNN})$

Alternatives	$\boldsymbol{NLD_{PSNN}(MLT_i, PIDA_{PSNN})}$	$\boldsymbol{PLD_{PSNN}(MLT_i, PIDA_{PSNN})}$

MLT_1	0.232005	0.000621
MLT_2	0.231966	0.000672
MLT_3	0.226083	0.000653
MLT_4	0.179804	0.026402
MLT_5	0.141492	0.048833
MLT_6	0.125629	0.059494
MLT_7	0.117359	0.063446
MLT_8	0.090704	0.081505
MLT_9	0.071575	0.096659
MLT_{10}	0.049688	0.116274
MLT_{11}	0.044957	0.113560
MLT_{12}	0.041897	0.125661
MLT_{13}	0.041074	0.168411
MLT_{14}	0.026084	0.273210

The ranking of ML alternatives process require compute PSNNCDV based on $PLD_{PSNN}(MLT_i, PIDA_{PSNN})$and $NLD_{PSNN}(MLT_i, PIDA_{PSNN})$, as shown in Table 8. Based on the findings of this Table MLT_1 is the optimal solution for elderly care monitoring.

5.Conclusion

In this study, a robust framework is introduced to take advantage of ML techniques as well as the PSNS to develop a comprehensive methodology for elderly care service while considering the uncertainty and variability of the environment. In this framework, a broad range of ML algorithms evaluated according to different sets of evaluation criteria, in which their results are encoded into the PSNS domain. Then, valuable insight into the importance of each evaluation criterion is obtained by the CRITIC approach to computing the weights of each criterion. Followingly, a customized TOPSIS approach is introduced to select the best alternative in a PSNS-driven MADM methods. Quantitative analysis of public case study sensor-based elderly caring services is performed to legalize the proposed approach. The results demonstrate the promise of our solution, which might have valuable managerial implications for improving the decision-making process.

Table 8. The PSNNCDV based on PSNNPIDA

Alternatives	PSNNCDV	Rank
MLT_1	0.0026701	1
MLT_2	0.0028872	3
MLT_3	0.0028814	2
MLT_4	0.1280348	4
MLT_5	0.2565759	5
MLT_6	0.3213774	6
MLT_7	0.3509097	7
MLT_8	0.473291	8
MLT_9	0.5745505	9
MLT_{10}	0.7006051	10
MLT_{11}	0.7163919	11
MLT_{12}	0.7499547	12
MLT_{13}	0.8039296	13
MLT_{14}	0.9128486	14

Data Availability:

deployed data is available at: https://archive.ics.uci.edu/dataset/427/activity+recognition+with+healthy+older+people+using+a+batteryless+wearable+sensor.

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The “MITSSP” Strategy and its Impact on the Development of Critical Thinking: A Study Based on Neutrosophic Statistics

Flor de María Ureta-Chávez [1], Nélida Angela Beraun-Sánchez [2], Yris Yolanda Bedoya-Campos [3], Wilber Huamanyauri-Cornelio [4], Edgar Augusto Salinas-Loarte [5], Beker Maraza-Vilcanqui [6], Rosaluz Victoria Bardales-Gonzales [7], Arcadio Aguirre-Rojas [8]

[1] Universidad Nacional Hermilio Valdizan-Huánuco-Perú. Email: 46c.ureta.huanuco@gmail.com
[2] Universidad Nacional Intercultural de la Amazonía-Pucallpa-Perú. Email: nberauns@unia.edu.pe
[3] Universidad Nacional Intercultural de la Amazonía-Pucallpa-Perú. Email: ybedoyac@unia.edu.pe
[4] Universidad Nacional Intercultural de la Amazonía-Pucallpa-Perú. Email: huamanyauri.docente@gmail.com
[5] Universidad Nacional de Cañete-Lima-Perú. Email: esalinas@undc.edu.pe
[6] Universidad Nacional Intercultural de la Amazonía-Pucallpa-Perú. Email: bmarazav@unia.edu.pe
[7] Universidad Nacional de Ucayali-Pucallpa-Perú. Email: victoriabardalesgonzales@gmail.com
[8] Universidad Nacional de Ucayali-Pucallpa-Perú. Email: arcadio.philos@gmail.com

Abstract. Nowadays, with the massive use of Information and Communication Technologies and social networks, it is more necessary than ever for citizens to develop critical thinking. Information media such as the Internet are flooded with false ideas, totally or partially. That is why it is necessary to provide young people with critical thinking that allows them to differentiate what is true from what is false, or even to identify dubious information. Based on the scientific method and under the protection of the educational research methodology, this article aims to demonstrate the effectiveness of the MITSSP strategy for the development of critical thinking in secondary school students at the Educational Institution "Héroes de Jactay" located in Huánuco, Peru. We conducted the study with the help of the Mann-Whitney Test in addition to the Neutrosophic Contingency Tables and the calculation of independence. This is because incomplete data were considered when comparing the results of the experimental group and the control group of students from the institution. The incomplete data is because some students did not finish the study, others joined late, and others had a question canceled on the questionnaire, among other cases.

Keywords: Strategy didactics, critical thought, comprehensive training, Neutrosophic Statistics, Neutrosophic Contingency Table, Neutrosophic Chi-square test, Neutrosophic Mann-Whitney test.

1 Introduction

Critical thinking is one of the most frequently mentioned and present forms of thinking in the literature, especially in the field of pedagogy, because it aims to form people with a reflective sense, capable of judging reasonably. This can also be achieved with a critical thinking process that involves a series of interconnected stages. Thus, critical thinking, although it is not synonymous with good thinking, is an essential human phenomenon that allows self-rectification. It is primarily associated with the form of self-evaluation, to then take corrective measures. It is already part of human nature; we all have this critical capacity developed at different levels or degrees. Therefore, these critical thinking skills constitute a current challenge for the educational system and higher education.

Today, university students need the tools and critical vision to face their circumstances, and to question and transform the society in which they live. According to reviews of the state of the art, the acquisition of these tools is still incipient at the university level in Latin American countries.

The above definitions describe critical thinking and competency; however, studies are required to identify strategies that favor its development. In this sense, it is important that teaching integrates instructional designs that foster in students the characteristics of a critical thinker. That is, with an open mind, flexible, prudent, and tolerant; as well as in the analysis of the issue to be debated and the management of arguments in a short and fast time from the points of view for and against. These are the aspects that together favor the development of critical thinking and the management of argumentation from this construct.

Defining critical thinking is complex, as there is no consensus in the academic community about its conceptualization or the skills involved in its development. Today's social and cultural changes require citizens trained with

a critical, open, and flexible mindset in the face of change. Some of the skills that are desired to be achieved are classroom observation, individual interviews, and analysis of student productions.

"Critical Thinking" is therefore a polysemic concept. The plurality of its meanings is directly related to the disciplinary contexts from which it emerges. This is evident in the various nuances and academic communities that promote one or another meaning. Furthermore, the traditions and trajectories from which it emerges are also diverse.

As far as women and their development of critical thinking skills are concerned, it opens up new opportunities to cultivate critical thinking among university students, because when women share and discuss their experiences of injustice based on their gender, a sense of community is strengthened that empowers students to take concrete steps to change their situation through a well-founded life project.

The relationships between critical thinking and meaningful learning are then evident. Critical thinking is understood as an intellectual process that is activated when a person receives information, processes it, interprets it, infers it, and produces knowledge that can be put into practice by showing a positive attitude. In addition, the nature of critical thinking is very complex, so thinking critically involves taking charge of the mind and, therefore, of life, seeking to improve it based on one's criteria.

As an essential competence, critical thinking constitutes a liberating force that allows us to think autonomously, to avoid the shackles that are sometimes imposed on us by doctrines, ideologies, the media, and all the artifacts created by humanity as instruments of manipulation, oppression, and subjugation. That is why humanity is very submissive, obedient, and incapable of claiming its rights; consequently, the greater the knowledge or application of critical thinking, the more active, prudent, and reflective people there will be. The article studies the effectiveness of the strategy known as MITSSP by its acronym in Spanish, where students of the Educational Institution "Héroes de Jactay" in Huánuco, Peru are encouraged and provided with tools.

The test to measure critical thinking allows each student to be given a score on this aspect. Unlike other studies that could be carried out classically, in this work, for the calculations, we took into account all the irregularities that occurred during the study. Those students who did not complete the strategy or who did not answer a question in the tests or some who joined later and did not take the test, among other similar cases, were taken into account in the final calculations. This is possible with the help of Neutrosophic Statistics, which is the extension of classical statistical methods with imprecise interval data, when there is no exact number for the sample or population size, or when there are interval-valued parameters [1-7].

The Neutrosophic Mann-Whitney test was used, which allows for comparing indeterminate data to check the effectiveness of treatment [8-11]. This is a non-parametric statistical test. The comparison is carried out between a control sample and an experimental one. We also applied the calculation of independence between the values of the questionnaire for the students who passed the MITSSP strategy and the questionnaire was also applied to the control group, with the help of the Neutrosophic Contingency Tables [12-17].

This paper is divided into a Materials and Methods section where the basic notions of Neutrosophic Contingency Tables and related topics are explained as well as the Neutrosophic Mann-Whitney test. The next section presents the results of the study. The last section is the Conclusion.

2 Materials and Methods

This section contains the basics of Neutrosophic Statistics, the algebra of neutrosophic numbers, the Neutrosophic Mann-Whitney test, and Neutrosophic Contingency Tables.

2.1 Basic Notions on Neutrosophic Statistics

Neutrosophic statistics refers to a set of data, such that the data or a part of it is indeterminate to some degree, and to the methods used to analyze these data [5].

In classical statistics, all data is unambiguous; this distinguishes neutrosophic statistics from classical statistics. In numerous instances, when the indeterminacy is null, neutrosophic statistics align with classical statistics. Neutrosophic measurement is applicable for quantifying indeterminate data. Neutrosophic statistical methods will enable the interpretation and organization of neutrosophic data, which may possess indeterminacy, to uncover underlying patterns. Numerous methodologies can be employed in neutrosophic statistics.

In *neutrosophic probability*, indeterminacy differs from randomness. Classical statistics focuses exclusively on randomness, whereas neutrosophic statistics addresses both randomness and, notably, indeterminacy.

Neutrosophic descriptive statistics include all methodologies for summarizing and delineating the attributes of neutrosophic numerical data. Due to the indeterminacies inherent in neutrosophic numerical data, neutrosophic line plots and neutrosophic histograms are represented in three-dimensional space, in contrast to the two-dimensional space utilized in classical statistics. The third dimension, beside the Cartesian XOY system, is that of indeterminacy (I). From ambiguous pictorial data, we can derive indeterminate neutrosophic information.*Neutrosophic data* are data containing some indeterminacy. In a similar way to classical statistics, it can be classified as:

- *Discrete neutrosophic data*, if the values are isolated points; for example $6 + I_1$, where $I_1 \in [0,1]$,7, $26 + I_2$, where $I_2 \in [3,5]$;

Continuous neutrosophic data, if the values form one or more intervals, for instance: [0, 0.8] or [0.1, 1.0] (i.e., uncertain which one).

Other classification:

- *Quantitative (numerical) neutrosophic data*; for instance, a value within the interval [7, 9] (exact value unknown), or specific numbers such as 43, 57, 67, or 69 (exact values unknown);

- and *Qualitative (categorical) neutrosophic data*; for example: blue or red (exact value unknown), as well as white, black, green, or yellow (exact values unknown).

Additionally, we can categorize neutrosophic data into two types:

-*Univariate neutrosophic data*, which comprises observations on a single neutrosophic characteristic.

-*Multivariate neutrosophic* data, which consists of observations on two or more attributes. Specifically, we refer to bivariate neutrosophic data and trivariate neutrosophic data.

A neutrosophic sample is a selected subset of a population that exhibits some degree of indeterminacy, either regarding certain individuals (who may not fully belong to the population under investigation or may only partially belong) or concerning the subset as a whole.

Classical samples yield exact information, but neutrosophic samples offer ambiguous or incomplete data. One may assert that any sample qualifies as a neutrosophic sample, as its indeterminacy might be regarded as equivalent to zero.

Neutrosophic survey results are those that encompass indeterminacy. A neutrosophic population is one whose membership is indeterminate, meaning it is uncertain whether some individuals belong to the population or not. For example, as in the neutrosophic set, a generic element x belongs to the neutrosophic population M as follows, $x(t, i, f) \in M$ which means: x is t% in the population M, f% of x is not in the population M, while i% membership of x in M is indeterminate (unknown, unclear, neutral: neither in the population nor outside) .

2.2 Neutrosophic Numbers

Definition 1: ([5, 16]) A *neutrosophic number* N is defined as a number as follows:

$N = d + I$ (1)

Where d is called the *determined part* and I is called the *indeterminate part*.

Given that $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ are two neutrosophic numbers, some operations between them are defined as follows:

$N_1 + N_2 = a_1 + a_1 + (b_1 + b_2)I$ (Addition);

$N_1 - N_2 = a_1 - a_1 + (b_1 - b_2)I$ (Difference),

$N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$ (Product),

$\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I$ (Division).

Smarandache also defined types of truth, indeterminacy, and falsity in a symbolic way beyond the T, I, and F. He called this refinement, where T is divided into $T_1, T_2, \dots, T_p$; I is split into $I_1, I_2, \dots, I_q$; and F is split into $F_1, F_2, \dots, F_r$, which depend on the problem at hand. Specifically, he extended the numbers of the form given in Equation 1, to represent the *Refined Neutrosophic Numbers*.

Definition 2: ([5, 16]) Given $I_1, I_2, \dots, I_q$, with $q \geq 1$, a *Refined Neutrosophic Number* is obtained from the previous set as $N_q = a + b_1 I_1 + b_2 I_2 + \cdots + b_q I_q$, where a is the determinate part and $b_j I_j$ ($j = 1,2, \dots, q$) are the indeterminate parts, such that $a, b_1, b_2, \dots, b_q$ are real or complex numbers.

Some properties that are fulfilled are those shown below:

1. $mI_k + nI_k = (m + n)I_k$,
2. $0I_k = 0$,
3. $I_k^n = I_k$,
4. I_k / I_k = undefined,

5. I_jI_k with j ≠ k is defined depending on the problem being addressed.

Calculation with Neutrosophic Numbers is equivalent to interval calculation. If we substitute $I = [c, d]$, where $c, d \in \mathbb{R}$, and $c \leq d$ in the neutrosophic number $N = a + bI$, then we obtain its equivalent in interval form $N = [a, a] + b[c, d] = [a + bc, a + bd]$.

2.3 Neutrosophic Mann Whitney Test

The Mann-Whitney test, as its classic variant, is a non-parametric test that allows two samples to be compared as to whether or not they belong to the same population. It is used to demonstrate the effectiveness of a treatment, where two samples are compared, one applied to a control group and another to an experimental group that receives the treatment. This test does not require that the variable correspond to a specific distribution. Below we write the details of the test [16].

1. Merge all the observations from different samples and deal with them as one sample.
2. Calculate the score, accuracy, and certainty of each observation.
3. Compare and rank these observations.
4. Give the ranked observations ranks from 1 to n and if we have two equal observations then we average their ranks.
5. Compute Mann Whitney test statistical pairwise based on ranked data using steps 1-4 using the formula:
   $Z_N = \frac{U_N - \bar{U}_N}{\sigma_{U_N}}$
   Where:
   $\bar{U}_N = \frac{n_1 n_2}{2}$
   $\sigma_{U_N} = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}}$
   $U_N = min\left(n_1 n_2 + \frac{n_1(n_2+1)}{2} - (R_1)_N, n_1 n_2 + \frac{n_2(n_1+1)}{2} - (R_2)_N\right)$
   Where:
   n_i: number of observations in the ith group,
   $N = \sum n_i$: number of observations in all samples,
   R_i: sum of ranks for the ith group.
6. If $|Z_N| < Z_{1-\frac{\alpha}{2}}$ then two compared samples are drawn from the same population and otherwise samples are drawn from different populations.

2.4 Neutrosophic Contingency Tables

A neutrosophic contingency table is a contingency table where the data contained in it are indeterminate values, either in the form of intervals or in the form of neutrosophic numbers. The following independence statistic is applied to them with the help of the Chi-square distribution ([18, 19]):

Step 1. The neutrosophic null hypothesis H_{0N} is stated such that both variables are independent versus the neutrosophic alternative hypothesis H_{1N} such that both variables are dependent.

Step 2. The significance α-level is specified.
Step 3. The neutrosophic values of the expected frequency are calculated:

$$\chi^2_N = \sum \frac{(O_{i_N} - E_{i_N})^2}{E_{i_N}} \qquad (2)$$

Step 4. The critical value χ^2_c is selected from the contingency table with a degree of freedom $(r-1) \times (c-1)$ of α as the significance level.

Step 5. Reject H_{0N} if χ^2_N is larger than the critical value. This means that the lower value of χ^2_N is greater than χ^2_c, or χ^2_c is between the lower and upper values. Otherwise, do not reject H_{0N}.

3 Results

The MITSSP in Spanish for "Model for Interaction in Working in Learning Situations" strategy is a pedagogical approach designed to foster critical thinking and active participation of students. It focuses on promoting interaction between students and teachers, as well as the development of critical skills through various activities.

The MITSSP model is based on several key components:

Motivation: To encourage students' interest and curiosity towards the topics covered.

Interaction: Promote collaborative work and dialogue among students, allowing them to share ideas and perspectives.

Teamwork: Develop social and collaborative skills by working in groups.

Learning situations: Create relevant contexts where students can apply their knowledge and skills.

Critical thinking: Stimulate reflection, analysis, and evaluation of information to make informed decisions.

In summary, the MITSSP strategy seeks to create a dynamic educational environment that fosters the development of critical thinking through interaction and collaboration among students.

The analysis presents the information collected from the sessions implemented in the experimental group (4th "A"), using the strategies addressed as theoretical support. The information collected by the other control group (4th "B") is also presented.

For data collection, a test was used to measure critical thinking, which consists of 20 questions, distributed in three dimensions, namely, literal, inferential, and evaluative. The literal level is presented in questions 1, 2, 3, 4, and 5, whose answers are data. The inferential level is present in questions 6, 7, 8, 17, 18, and 19 whose answers are analysis and deductions according to the readings presented. The evaluative level is presented in questions 9, 10, 11, 12, 13, 14, 15, 16, and 20 whose answers are opinions, but take into account a level of argumentation. After applying the test to measure critical thinking, the evaluation rubric was used, with which the count was made for each student taking into account the indicators of each of the dimensions.

For the study we developed, we first applied the Neutrosophic Mann-Whitney test to compare the results of the experimental group with the control group. In general, the data with which we worked were neutrosophic numbers of the form $a + bI$, where $I = [0, 1]$. If $b = 0$ it is an exact number, while $b \neq 0$ implies that there is some indeterminacy. To achieve greater accuracy, we took into account students who did not finish the strategy or joined late, among other cases as indicated above. For these cases, they were assigned a non-exact score in $[a, b]$, which can be represented as $a + bI$.

The other important detail to keep in mind is the comparison between two neutrosophic numbers, such that it is considered $a_1 + b_1 I \leq a_2 + b_2 I$ if $a_1 \leq a_2$ and $b_1 \leq b_2$. If on the other hand $a_2 + b_2 I \not\leq a_1 + b_1 I$ according to the previous criterion, then both values are classified as incomparable and the following is taken as a comparison criterion:

$a_1 + b_1 I \leq a_2 + b_2 I$ if $a_1 \leq a_2$.

In the study, 31 students from group 4th A of the institution were given the exam after going through the MITSSP strategy. On the other hand, students from group 4th B, consisting of 26 students, followed the course that is usually taught.

The results were as follows:

$\bar{U}_N = 403,$

$\sigma_{U_N} = 61.87,$

$R_1 = 861.5,$

$R_2 = 1347,$

$U_N = -125,$

$Z_N = -8.534.$

$|Z_N| > 0.3289$, therefore the null hypothesis is rejected and it is determined that both samples belong to different populations.

The second test consists of a neutrosophic contingency table, as shown in Table 1.

Grade and Section	Results			
	Very low	Low	Regular	High
4th A	0	0	$3I$	$25 + 3I$
4th B	0	$12 + 8I$	$6I$	0

Table 1. Neutrosophic Contingency Table on the results of both groups evaluated in nominal variables.

Note that in Table 1, indeterminate results were obtained because cases with incomplete evaluation were included.

The result of the Chi-square statistic is $\chi^2_N = [133.7990, 155.4764]$. When this result is compared with the theoretical value of the Chi-square with 3 degrees of freedom, we have: $\chi^2_N = [133.7990, 155.4764] > \chi^2_{0.95}(3) = 7.81$. Therefore, the null hypothesis of equality between both groups is rejected.

Now let us compare the results regarding the gender of the students in the experimental group. We use the contingency table expressed in Table 2.

Gender	Results			
	Very low	Low	Regular	High
Male	0	0	I	$11 + I$
Female	0	0	$2I$	$17 + I$

Table 2. Neutrosophic Contingency Table on the results regarding gender in the experimental group expressed in nominal variables.

The result is $\chi^2_N = [76.1754, 82.386]$ which is evidently interpreted as rejecting the null hypothesis. This is why better results are observed in girls than in boys, which is an interesting reason for analysis to consider.

Conclusion

The development of critical thinking is crucial in a person's life and must be promoted in all areas. For this reason, when diagnosing critical thinking skills, there are only two valid options: it is in development or it is under observation. The argument is based on the foundations established by psychology and neuroscience. Human beings develop the foundations of critical thinking between the ages of 9 and 10 on average when they apply formal logical thinking in an incipient manner. This development is more noticeable at the age of 13 to 14, but this process does not end until death.

Currently, there are difficulties for teachers to apply relevant teaching strategies to achieve learning in students. According to the results of the control group and the experimental group, the experimental group has a significantly higher level than the control group. Therefore, it is concluded that there is effectiveness when applying the MITSSP strategy in the development of critical thinking. Additionally, the values of the Chi-square test were calculated and it was determined that girls have better final results than boys. This is an issue to consider in future studies.

To carry out the study and reach the aforementioned conclusions we used the neutrosophic statistic, where the data are expressed in interval form or equivalently in neutrosophic form numbers. First, we used the Neutrosophic Mann-Whitney test and then the Chi-square test for Neutrosophic Contingency Tables. These methods allow us to take into account inaccurate data, due to incomplete test results, instead of discarding incomplete data.

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A Hybrid Approach for Educational Development Assessment in Higher Education with Trapezoidal Fuzzy Neutrosophic Analysis: Towards Optimized Decision-Making and Strategic Improvement

Bin Ge*
Wuxi Vocational College of Science and Technology, Wuxi, 214028, Jiangsu, China
*Corresponding author, E-mail: 13771188342@163.com

Abstract: The rapid advancement of technology has significantly driven social development, leading to transformative changes across various sectors of society. As a result, there are increasing demands for higher standards of management in the education sector. Consequently, reforming education management has become an urgent necessity. In China, higher education plays a pivotal role as the primary platform for cultivating and supplying talent to society. With the emergence of new challenges and heightened societal expectations, ensuring high-quality teaching and implementing effective management reforms in higher education have become critical priorities. The evaluation of innovative development models in higher education management can be approached through multiple-attribute decision-making (MADM) techniques. Among the recent advancements in this domain, the MULTIMOORA method and average models have been applied to address MADM problems. To better represent the inherent uncertainty in decision-making processes, trapezoidal fuzzy neutrosophic sets (TFNSs) have emerged as an effective tool. These sets enable a more nuanced expression of uncertain information, making them particularly suitable for evaluating innovative development modes in higher education management. This study proposes a novel decision-making framework: the trapezoidal fuzzy neutrosophic number MULTIMOORA (TFNN-MULTIMOORA) model. Additionally, the average technique is employed to determine attribute weights within the TFNS framework. To validate the effectiveness and applicability of the proposed TFNN-MULTIMOORA technique, a numerical example focusing on the evaluation of innovative development modes in higher education management is presented, along with comparative analyses.

Keywords: multiple-attribute decision-making (MADM); TFNSs; MULTIMOORA model; innovative development modes evaluation

1. Introduction

Education management refers to the process in which educators organize and coordinate educational teams to effectively utilize human, financial, material, and informational resources within the education system. By leveraging various favorable conditions, the aim is to efficiently achieve educational management goals [1-3]. From a macro perspective, education management encompasses a series of activities where the state organizes, coordinates, and controls the broader education system. For universities, it involves the strategic adjustment of internal resources and educational affairs to align all driving factors, ensuring they are positioned appropriately while maximizing their strengths. Currently, China is experiencing a new phase of comprehensive development across multiple sectors [4, 5]. The rapid advancement of information technology has accelerated the optimization and transformation of industries, driving an increasing demand for highly skilled talent from universities. This has significantly increased the complexity of talent cultivation, presenting universities with unprecedented challenges. As a result, it has become imperative for universities to further refine and enhance their education management models to ensure the smooth operation of various institutional functions. Moreover, these models must enable universities to effectively implement management and educational directives. To address these challenges, it is crucial to thoroughly analyze innovative trends and development pathways in higher education management. This analysis must take into account the unique demands of the current era [6]. These demands reflect broader societal progress, and as institutions deeply embedded in and influenced by societal changes, universities are naturally compelled to adapt to new requirements. They must respond proactively by optimizing and innovating their management practices to remain relevant and effective in this dynamic context. In conclusion, the evolving demands of the modern era require universities to rethink and innovate their approaches to education management. By doing so, they can ensure their systems and practices align with societal progress, meeting the growing complexities of talent cultivation while maintaining the effective operation of their institutions [7]. At present, China's society has entered a state of knowledge explosion, and various emerging concepts are constantly emerging. The ideological consciousness of college students is inherently susceptible to interference from complex social environments[8]. Combined with the continuous upgrading of the education level of the general public, the competition for future employment among college students has become increasingly severe. In light with this, universities should not only seek more living space for their own development, but also consider the quality of students' future employment and entrepreneurship. And university education management is a direct factor that affects the quality of education and management in universities[9]. So in order to cater to the new situation and enhance the vitality of

development of universities in the new era, universities must innovate the mechanisms of educational management forms[10, 11]. Only by actively adapting to the development of the times and optimizing themselves, can universities unleash higher quality industry attractiveness and attract more high-quality and high-quality talents to enter the school. If, in the face of new situations, the management mode of higher education still follows the outdated traditional mode, the quality of talent cultivation and development strategies on campus will not be able to adapt to the new form of society, and the competitiveness of the times will be lost. Therefore, from any perspective, in the new social development pattern, the management mode of higher education must innovate its mechanism, which is an inevitable direction and choice to adapt to the times[12-14]. The actual quality of higher education management models can not only represent the external image of universities, but also directly affect the operational quality of internal education, teaching, and management work[15-17]. The management of higher education consists of two parts: management system and management methods. If the management system is not perfect and cannot be set according to the specific situation of teaching and learning on campus, it will lead to significant management gaps or loopholes in education management, making teaching and daily work unable to proceed smoothly[18, 19]. If the management methods used by managers lack strength or fail to pay attention to humanistic care for the managed, it will cause dissatisfaction among the managed and further reduce the stability of the daily operation of universities[20, 21].

1.1 Research gaps

In short, the lack of innovation in the management mode of higher education will make it difficult for managers to control the fundamental stability of education and daily business, and universities will not be able to operate normally[22, 23]. Therefore, from the perspective of self-optimization and quality assurance in universities, the mode of education management should be optimized[24, 25]. Improving management quality is an essential step and a critical measure to ensure the sustainable and secure development of universities. The quality of education management within universities directly influences not only the overall education quality but also the effective management of students and faculty. Therefore, it is essential for universities to fully recognize the pivotal role of education management and prioritize its continuous improvement. Chinese universities are currently navigating a new phase of development characterized by challenges and opportunities in areas such as talent cultivation, education internationalization, advancements in educational technology, and the industrialization of education.

1.2 Motivation

To effectively address these challenges, higher education management must undergo significant reforms. This includes rethinking traditional management concepts,

strengthening the skills and awareness of management teams, and integrating practical considerations to optimize and innovate evaluation and management systems.

To align universities' operations across various dimensions with the evolving demands of the modern era, structural transformations are necessary. These transformations require a concerted effort to enhance the quality and efficiency of university management while simultaneously elevating educational standards. By doing so, universities can better meet the complex, multifaceted requirements of today's rapidly changing environment. Ultimately, this comprehensive approach will enable universities to adapt to new circumstances, foster continuous improvement, and strengthen their overall competitiveness. By prioritizing the optimization of management systems and embracing innovation, universities can ensure their development remains aligned with societal progress while maintaining their critical role in talent cultivation and knowledge advancement.

1.3 Contribution of this study

The innovative development modes evaluation in higher education management could be put forward MADM. Recently, MULTIMOORA [26,27] was utilized to solve MADM. The TFNSs [28] are put forward better technique for expressing uncertain information during the innovative development modes evaluation in higher education management. In this study, the TFNN-MULTIMOORA model with TFNSs. Finally, numerical example for innovative development modes evaluation in higher education management is put forward with different comparisons to proof the TFNN-MULTIMOORA approach.

Thus, the key contributions of TFNN-MULTIMOORA model are summarized: (1) MULTIMOORA model is enhanced under TFNSs; (2) Finally, numerical example for innovative development modes evaluation in higher education management is put forward with different comparisons to verify TFNN-MULTIMOORA technique.

1.4 Organization of this study

The framework of such study is put forward: Sect. 2 lists the TFNNSs. Sect. 3 put forward the MADM technique in light with TFNN-MULTIMOORA model. Sect. 4 put forward the TFNN-MULTIMOORA through numerical study for innovative development modes evaluation in higher education management with some comparisons. Conclusions are dealt with in Sect. 5.

2. Preliminaries

Ye [28] put forward the TFNSs.

Definition 1[28]**.** The TFNSs $b\eta$ is:

$$b\eta = \left\{ \left(\theta, b\phi_{b\eta}(\theta), b\varphi_{b\eta}(\theta), b\gamma_{b\eta}(\theta) \right) \middle| \theta \in \Theta \right\} \tag{1}$$

where $b\phi_{b\eta}(\theta), b\varphi_{b\eta}(\theta), b\gamma_{b\eta}(\theta)$ is truth-membership (TM), indeterminacy-membership (IM) and falsity-membership (FM) which is dealt with trapezoidal fuzzy numbers (TFNs).

$$b\phi_{b\eta}(\theta)=\left(b\phi_{b\eta}^{L}(\theta),b\phi_{b\eta}^{M}(\theta),b\phi_{b\eta}^{U}(\theta),b\phi_{b\eta}^{R}(\theta)\right) \quad (2)$$

$$b\varphi_{b\eta}(\theta)=\left(b\varphi_{b\eta}^{L}(\theta),b\varphi_{b\eta}^{M}(\theta),b\varphi_{b\eta}^{U}(\theta),b\varphi_{b\eta}^{R}(\theta)\right) \quad (3)$$

$$b\gamma_{b\eta}(\theta)=\left(b\gamma_{b\eta}^{L}(\theta),b\gamma_{b\eta}^{M}(\theta),b\gamma_{b\eta}^{U}(\theta),b\gamma_{b\eta}^{R}(\theta)\right) \quad (4)$$

where $0\le b\phi_{b\eta}^{L}(\theta)\le b\phi_{b\eta}^{M}(\theta)\le b\phi_{b\eta}^{U}(\theta)\le b\phi_{b\eta}^{R}(\theta)\le 1$,

$0\le b\varphi_{b\eta}^{L}(\theta)\le b\varphi_{b\eta}^{M}(\theta)\le b\varphi_{b\eta}^{U}(\theta)\le b\varphi_{b\eta}^{R}(\theta)\le 1$,

$0\le b\gamma_{b\eta}^{L}(\theta)\le b\gamma_{b\eta}^{M}(\theta)\le b\gamma_{b\eta}^{U}(\theta)\le b\gamma_{b\eta}^{R}(\theta)\le 1$.

For convenience, $b\eta=\left\{\left(b\phi^{L},b\phi^{M},b\phi^{U},b\phi^{R}\right),\left(b\varphi^{L},b\varphi^{M},b\varphi^{U},b\varphi^{R}\right),\left(b\gamma^{L},b\gamma^{M},b\gamma^{U},b\gamma^{R}\right)\right\}$ is a TFNN which meets condition $0\le b\phi^{R}+b\varphi^{R}+b\gamma^{R}\le 3$.

In order to fuse the TFNNs, Ye [28] dealt with novel operations on the TFNNs $b\eta$, $b\eta_1$ and $b\eta_2$.

Definition 2[28]. Let $b\eta_1=\left\{\left(b\phi_1^{L},b\phi_1^{M},b\phi_1^{U},b\phi_1^{R}\right),\left(b\varphi_1^{L},b\varphi_1^{M},b\varphi_1^{U},b\varphi_1^{R}\right),\left(b\gamma_1^{L},b\gamma_1^{M},b\gamma_1^{U},b\gamma_1^{R}\right)\right\}$,

$b\eta_2=\left\{\left(b\phi_2^{L},b\phi_2^{M},b\phi_2^{U},b\phi_2^{R}\right),\left(b\varphi_2^{L},b\varphi_2^{M},b\varphi_2^{U},b\varphi_2^{R}\right),\left(b\gamma_2^{L},b\gamma_2^{M},b\gamma_2^{U},b\gamma_2^{R}\right)\right\}$ and $b\eta=\left\{\left(b\phi^{L},b\phi^{M},b\phi^{U},b\phi^{R}\right),\left(b\varphi^{L},b\varphi^{M},b\varphi^{U},b\varphi^{R}\right),\left(b\gamma^{L},b\gamma^{M},b\gamma^{U},b\gamma^{R}\right)\right\}$, the novel operations of TFNNs are put forward:

$$(1)\ b\eta_1\oplus b\eta_2=\left\{\begin{array}{l}\left(b\phi_1^{L}+b\phi_2^{L}-b\phi_1^{L}b\phi_2^{L},b\phi_1^{M}+b\phi_2^{M}-b\phi_1^{M}b\phi_2^{M},b\phi_1^{U}+b\phi_2^{U}-b\phi_1^{U}b\phi_2^{U},b\phi_1^{R}+b\phi_2^{R}-b\phi_1^{R}b\phi_2^{R}\right),\\ \left(b\varphi_1^{L}b\varphi_2^{L},b\varphi_1^{M}b\varphi_2^{M},b\varphi_1^{U}b\varphi_2^{U},b\varphi_1^{R}b\varphi_2^{R}\right),\\ \left(b\gamma_1^{L}b\gamma_2^{L},b\gamma_1^{M}b\gamma_2^{M},b\gamma_1^{U}b\gamma_2^{U}\right)\end{array}\right\};$$

$$(2)\ b\eta_1\otimes b\eta_2=\left\{\begin{array}{l}\left(b\phi_1^{L}b\phi_2^{L},b\phi_1^{M}b\phi_2^{M},b\phi_1^{U}b\phi_2^{U},b\phi_1^{R}b\phi_2^{R}\right),\\ \left(b\varphi_1^{L}+b\varphi_2^{L}-b\varphi_1^{L}b\varphi_2^{L},b\varphi_1^{M}+b\varphi_2^{M}-b\varphi_1^{M}b\varphi_2^{M},b\varphi_1^{U}+b\varphi_2^{U}-b\varphi_1^{U}b\varphi_2^{U},b\varphi_1^{R}+b\varphi_2^{R}-b\varphi_1^{R}b\varphi_2^{R}\right),\\ \left(b\gamma_1^{L}+b\gamma_2^{L}-b\gamma_1^{L}b\gamma_2^{L},b\gamma_1^{M}+b\gamma_2^{M}-b\gamma_1^{M}b\gamma_2^{M},b\gamma_1^{U}+b\gamma_2^{U}-b\gamma_1^{U}b\gamma_2^{U},b\gamma_1^{R}+b\gamma_2^{R}-b\gamma_1^{R}b\gamma_2^{R}\right)\end{array}\right\};$$

$$(3)\ \xi\times b\eta=\left\{\begin{array}{l}\left(1-\left(1-b\phi^{L}\right)^{\xi},1-\left(1-b\phi^{M}\right)^{\xi},1-\left(1-b\phi^{U}\right)^{\xi}\right),\\ \left(\left(b\varphi^{L}\right)^{\xi},\left(b\varphi^{M}\right)^{\xi},\left(b\varphi^{U}\right)^{\xi}\right),\\ \left(\left(b\gamma^{L}\right)^{\lambda\xi},\left(b\gamma^{M}\right)^{\xi},\left(b\gamma^{U}\right)^{\xi}\right)\end{array}\right\},\xi>0;$$

$$(4)\ b\eta^{\xi}=\left\{\begin{array}{l}\left(\left(b\phi^{L}\right)^{\xi},\left(b\phi^{M}\right)^{\xi},\left(b\phi^{U}\right)^{\xi},\left(b\phi^{R}\right)^{\xi}\right),\\ \left(1-\left(1-b\varphi^{L}\right)^{\xi},1-\left(1-b\varphi^{M}\right)^{\xi},1-\left(1-b\varphi^{U}\right)^{\xi},1-\left(1-b\varphi^{R}\right)^{\xi}\right),\\ \left(1-\left(1-b\gamma^{L}\right)^{\xi},1-\left(1-b\gamma^{M}\right)^{\xi},1-\left(1-b\gamma^{U}\right)^{\xi},1-\left(1-b\gamma^{R}\right)^{\xi}\right)\end{array}\right\},\xi>0.$$

Then, these operation laws have different properties.

$$(1)b\eta_1\oplus b\eta_2=b\eta_2\oplus b\eta_1, b\eta_1\otimes b\eta_2=b\eta_2\otimes b\eta_1,\left(\left(b\eta_1\right)^{\xi_1}\right)^{\xi_2}=\left(b\eta_1\right)^{\xi_1\xi_2};\quad(5)$$

$$(2)\ \xi\left(b\eta_1\oplus b\eta_2\right)=\xi b\eta_1\oplus\xi b\eta_2,\left(b\eta_1\otimes b\eta_2\right)^{\xi}=\left(b\eta_1\right)^{\xi}\otimes\left(b\eta_2\right)^{\xi};\quad(6)$$

$$(3)\ \xi_1 b\eta_1\oplus\xi_2 b\eta_1=\left(\xi_1+\xi_2\right)b\eta_1,\left(b\eta_1\right)^{\xi_1}\otimes\left(b\eta_1\right)^{\xi_2}=\left(b\eta_1\right)^{(\xi_1+\xi_2)}.\quad(7)$$

Ye [28] put forward the TFNN Euclid distance (TFNNED).

Definition 4[28]. Let $b\eta_1=\left\{\left(b\phi_1^L,b\phi_1^M,b\phi_1^U,b\phi_1^R\right),\left(b\varphi_1^L,b\varphi_1^M,b\varphi_1^U,b\varphi_1^R\right),\left(b\gamma_1^L,b\gamma_1^M,b\gamma_1^U,b\gamma_1^R\right)\right\}$, $b\eta_2=\left\{\left(b\phi_2^L,b\phi_2^M,b\phi_2^U,b\phi_2^R\right),\left(b\varphi_2^L,b\varphi_2^M,b\varphi_2^U,b\varphi_2^R\right),\left(b\gamma_2^L,b\gamma_2^M,b\gamma_2^U,b\gamma_2^R\right)\right\}$, the TFNNED is constructed:

$$TFNNED\left(b\eta_1,b\eta_2\right)=\sqrt{\frac{1}{12}\left(\begin{array}{l}\left|b\phi_1^L-b\phi_2^L\right|^2+\left|b\phi_1^M-b\phi_2^M\right|^2+\left|b\phi_1^U-b\phi_2^U\right|^2+\left|b\phi_1^R-b\phi_2^R\right|^2\\ +\left|b\varphi_1^L-b\varphi_2^L\right|^2+\left|b\varphi_1^M-b\varphi_2^M\right|^2+\left|b\varphi_1^U-b\varphi_2^U\right|^2+\left|b\varphi_1^R-b\varphi_2^R\right|^2\\ +\left|b\gamma_1^L-b\gamma_2^L\right|^2+\left|b\gamma_1^M-b\gamma_2^M\right|^2+\left|b\gamma_1^U-b\gamma_2^U\right|^2+\left|b\gamma_1^R-b\gamma_2^R\right|^2\end{array}\right)}\quad(8)$$

Ye [28] put forward the TFNN score values (TFNNSV) and TFNN accuracy values (TFNNAV) in order to compare two TFNNs.

Definition 5[28]. Let $b\eta=\left\{\left(b\phi^L,b\phi^M,b\phi^U,b\phi^R\right),\left(b\varphi^L,b\varphi^M,b\varphi^U,b\varphi^R\right),\left(b\gamma^L,b\gamma^M,b\gamma^U,b\gamma^R\right)\right\}$, the TFNNSV and TFNNAV are put forward:

$$TFNNSV\left(b\eta\right)=\frac{1}{12}\left[\begin{array}{l}8+\left(b\phi^L+b\phi^M+b\phi^U+b\phi^R\right)-\left(b\varphi^L+b\varphi^M+b\varphi^U+b\varphi^R\right)\\ -\left(b\gamma^L+b\gamma^M+b\gamma^U+b\gamma^R\right)\end{array}\right],$$

$$TFNNSV\left(b\eta\right)\in\left[0,1\right]\quad(9)$$

$$TFNNAV\left(b\eta\right)=\frac{1}{4}\left[\left(b\phi^L+b\phi^M+b\phi^U+b\phi^R\right)-\left(b\gamma^L+b\gamma^M+b\gamma^U+b\gamma^R\right)\right],$$

$$TFNNAV\left(b\eta\right)\in\left[-1,1\right]\quad(10)$$

For TFNNs $x\eta_1$ and $x\eta_2$, then

(1) *if* $TFNNSV\left(b\eta_1\right) \text{p}\ TFNNSV\left(b\eta_2\right)$, *then* $b\eta_1 \text{p}\ b\eta_2$;

(2) *if* $TFNNSV\left(b\eta_1\right) = TFNNSV\left(b\eta_2\right), TFNNAV\left(b\eta_1\right) \text{p}\ TFNNAV\left(b\eta_2\right)$, *then* $b\eta_1 \text{p}\ b\eta_2$;

(3) *if* $TFNNSV\left(b\eta_1\right) = TFNNSV\left(b\eta_2\right), TFNNAV\left(b\eta_1\right) = TFNNAV\left(b\eta_2\right)$, *then* $b\eta_1 = b\eta_2$.

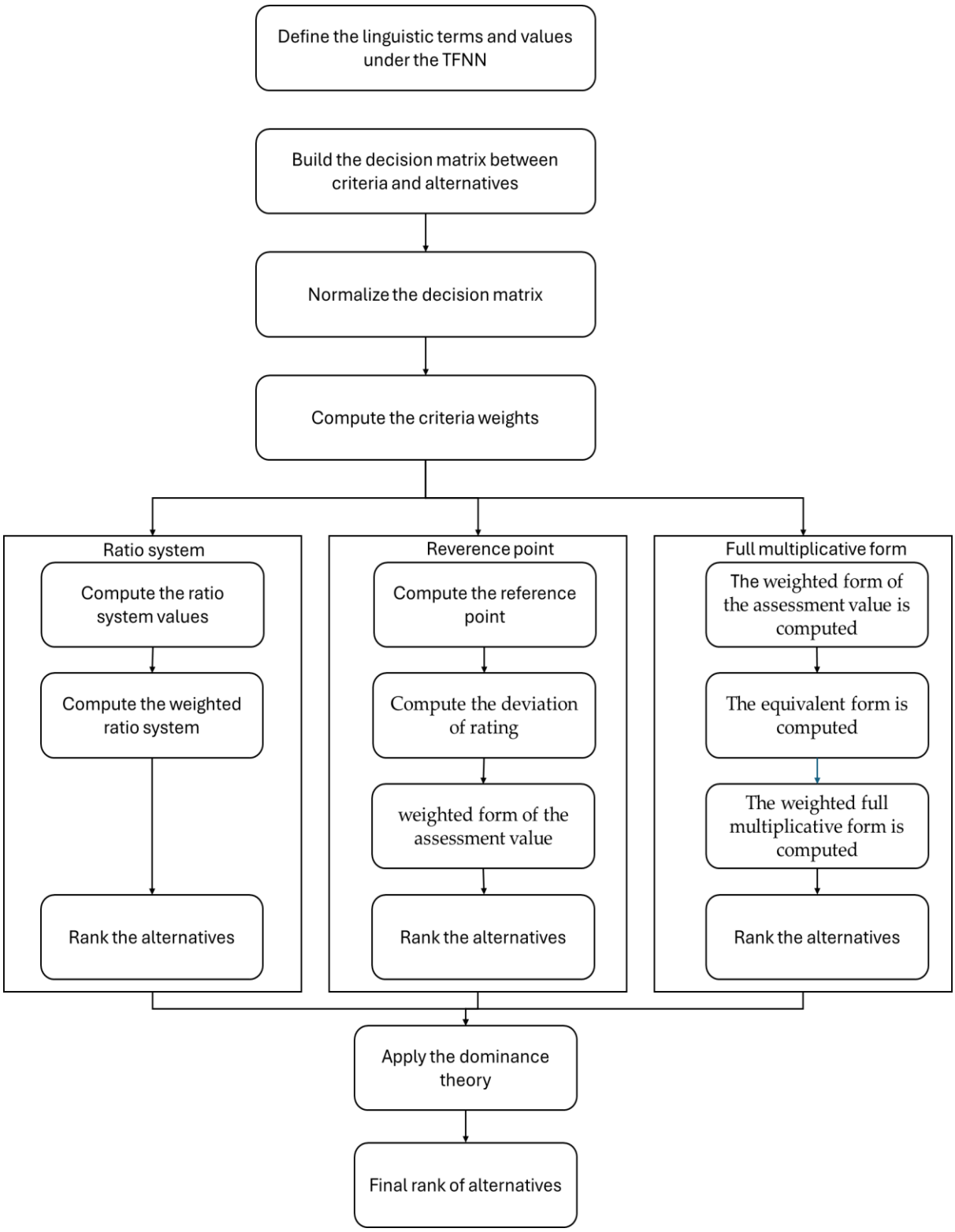
Define the linguistic terms and values under the TFNN
Build the decision matrix between criteria and alternatives
Normalize the decision matrix
Compute the criteria weights
Ratio system
Compute the ratio system values
Compute the weighted ratio system
Rank the alternatives
Reverence point
Compute the reference point
Compute the deviation of rating
weighted form of the assessment value
Rank the alternatives
Full multiplicative form
The weighted form of the assessment value is computed
The equivalent form is computed
The weighted full multiplicative form is computed
Rank the alternatives
Apply the dominance theory
Final rank of alternatives

Figure 1. The steps of the TFNN-MULTIMOORA.

3. TFNN-MULTIMOORA approach for MADM

The reference point technique and the ratio system are the two components of the MOORA method, which was first presented by Brauers and Zavadskas. The MOORA approach was expanded by Brauers and Zavadskas to consider the entire multiplicative form. The whole multiplicative form and the MOORA subordinate portions are included in the updated method, which is called MULTIMOORA. Figure 1 shows the steps of the proposed method. The MULTIMOORA method starts with the decision matrix X with a set of criteria and alternatives:

The decision matrix is normalized to obtain the comparable and dimensionless rating. The normalization method is a comparison between the criteria and alternatives. The equation below is used to normalize the decision matrix as:

$$L_{ij} = \frac{x_{ij}}{\left[\sum_{i=1}^{m} x_{ij}^2\right]^{1/2}} \tag{11}$$

The criteria weights are computed using the average method.

$$W = \left[w_j\right]_{1\times n} \tag{12}$$

$$\sum_{j=1}^{n} w_j = 1 \tag{13}$$

3.1 The ratio system

To compute the ratio system the normalized values are added for positive criteria and subtracted from negative criteria as:

$$t_i = \sum_{j=1}^{g} L_{ij} - \sum_{j=g+1}^{n} L_{ij} \tag{14}$$

Where g refers to the number of positive criteria and (n-g) refers to the negative criteria.

The weighted form of the ratio system is computed by using the criteria weights as:

$$t_i^w = \sum_{j=1}^{g} L_{ij} w_j - \sum_{j=g+1}^{n} L_{ij}\, w_j \tag{15}$$

The best alternative in the ratio system can be computed as the highest score in the weighted normalized values.

3.2 The reference point method

This method is started with the maximal criteria reference point. The maximal objective reference point can be computed as:

$$r_j = \begin{cases} \max L_{ij}, j \le g \\ \min L_{ij}, j \le g \end{cases} \tag{15}$$

Compute the deviation of rating from the reference point as:

$$d_i = \max\left|r_j - L_{ij}\right| \tag{16}$$

The weighted form of the assessment value of the reference point method can be computed as:

$$z_i = \max\left|w_j r_j - w_j L_{ij}\right| \tag{17}$$

The best alternative form this approach can be computed based on the minimum value of the assessment values.

3.3 The full multiplicative form

The third part in the MULTIMOORA methodology is a full multiplicative form. The assessment value can be computed as:

$$Q_i = \frac{\prod_{j=1}^{g}(x_{ij})}{\prod_{j=g+1}^{n}(x_{ij})} \quad (18)$$

The equivalent form can be computed as:

$$F_i = \frac{\prod_{j=1}^{g}(L_{ij})}{\prod_{j=g+1}^{n}(L_{ij})} \quad (19)$$

The weighted full multiplicative form can be computed as:

$$S_i = \frac{\prod_{j=1}^{g}(L_{ij})^{w_j}}{\prod_{j=g+1}^{n}(L_{ij})^{w_j}} \quad (20)$$

The best alternative can be computed by the maximum assessment value

3.4 Final rank of the alternatives by the dominance theory

The dominance theory used serval ranks and applied dominance to obtain the final rank of alternatives. The final rank can be computed by the combined all three ranks.

4. Application in decision making problem

This section shows the illustrative example of the case study to compute the criteria weights and rank the alternatives. This section shows the results of the MULTIMOORA method under the trapezoidal fuzzy neutrosophic numbers to deal with the vague and uncertain information in the decision-making process.

4.1 Case Study

Universities are the primary battleground for education, with teaching as their central activity, yet the importance of management cannot be overlooked. University education management involves the organization of educational teams by administrators to ensure the efficient and scientific allocation of teaching resources. This process serves as the backbone for achieving higher education goals. As a driving force for educational development and a guarantee of educational quality, effective management is indispensable to the growth and success of universities. Higher education management exhibits comprehensive, social, and directional characteristics, all of which are critical to supporting talent development. Each era brings its own priorities, and in the current age of rapidly advancing information technology, particularly with the rise of the Internet, colleges and universities worldwide are increasingly emphasizing education management. However, the reform and innovation of college education management face numerous challenges and obstacles. Firstly, the complexity of higher education management stems from its involvement of multiple stakeholders and intricate processes. This often results in lengthy work cycles and lags in management efficiency. Problems arising in the management process are difficult to address promptly, leading to delays and inefficiencies. Secondly, the digital age has empowered individuals with equal rights, granting students—the central participants in education management—greater autonomy and more distinct personalities. This shift makes education management more complex than

ever before. Additionally, the emergence of new technologies has introduced advanced tools for teaching and management, rendering traditional methods increasingly obsolete and necessitating systematic updates and upgrades. Higher education management today faces pressing challenges that demand immediate solutions. This encompasses not only teaching management but also student, faculty, and administrative management. In teaching management, universities must align with the demands of the digital era by leveraging online resources to enrich teaching content and enhance the appeal of education. Upgrading management tools and building digital teaching platforms can significantly improve teaching efficiency. For student management, universities should cater to students' diverse developmental needs, granting them more choices and decision-making power. This approach fosters greater interest in learning and supports a more dynamic and engaging management model. In faculty management, strengthening the teaching workforce is essential. Universities should attract exceptional talent, implement effective teacher assessment mechanisms, and create an environment that maximizes teaching potential, ultimately enhancing teaching quality. With regard to administrative management, universities can adopt practices from the corporate world, analyzing the pros and cons of centralized and decentralized management to build an efficient, streamlined administrative system. As a cornerstone of national talent development, higher education is a driving force for a nation's long-term progress. Therefore, higher education management plays a vital role in maintaining and improving the quality of education. In the face of a rapidly evolving social landscape, it is crucial for higher education management to adapt to the times through comprehensive transformation. This includes upgrading teaching tools and content, tailoring management practices to students' needs, and fostering innovation in management approaches. Ultimately, a fully modernized and innovative university education management system is essential to meet the demands of the present era and ensure the long-term success of higher education institutions. The innovative development modes evaluation in higher education management is MADM. Seven higher education management innovative development modes are assessed through employing ten attributes.

Table 1. The linguistic terms of the TFNN

	Terms	TFNN
A	Very Very Low	((0.10,0.12,0.14,0.15),(0.85,0.87,0.90,0.95),(0.05,0.07,0.10,0.15))
B	Very Low	((0.15,0.17,0.19,0.20),(0.75,0.80,0.83,0.85),(0.15,0.17,0.19,0.20))
C	Low	((0.12,0.12,0.24,0.25),(0.67,0.69,0.73,0.75),(0.20,0.21,0.24,0.25))
D	Medium Low	((0.25,0.27,0.29,0.30),(0.55,0.57,0.63,0.65),(0.25,0.27,0.29,0.30))
E	Medium	((0.50,0.53,0.54,0.55),(0.45,0.47,0.52,0.55),(0.30,0.35,0.37,0.40))
F	Medium High	((0.55,0.57,0.63,0.65),(0.25,0.27,0.29,0.30),(0.25,0.27,0.29,0.30))
G	High	((0.65,0.70,0.73,0.75),(0.20,0.21,0.23,0.25),(0.20,0.21,0.23,0.25))
H	Very High	((0.75,0.80,0.83,0.85),(0.15,0.17,0.19,0.20),(0.15,0.16,0.19,0.20))
I	Very Very High	((0.85,0.89,0.93,0.95),(0.10,0.11,0.14,0.15),(0.05,0.10,0.13,0.15))

4.2 Solving the illustrative example

The aim of this study is to compute the criteria weights and rank the alternatives to select the best one. Three experts are evaluated the criteria and alternatives using the TFNN as shown in Table 1. Then we evaluate the criteria and alternatives as shown in Table 2 by using the TFNN. Then we apply the score function to obtain the crip values. Then we combined these matrices into one matrix.

Table 2. The TFNN information.

D_1	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))
C_2	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))
C_3	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))
C_4	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))
C_5	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))
C_6	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))
C_7	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))
C_8	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))
C_9	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))
C_{10}	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))

D_2	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))
C_2	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))
C_3	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))
C_4	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))
C_5	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))
C_6	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))
C_7	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))
C_8	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))
C_9	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))
C_{10}	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))

D_3	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))
C_2	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))
C_3	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))
C_4	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))
C_5	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))
C_6	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.55,0.57,0.63,0.65),(0.25, 0.27,0.29,0.30),(0.25,0.27,0. 29,0.30))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))
C_7	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))
C_8	((0.75,0.80,0.83,0.85),(0.15, 0.17,0.19,0.20),(0.15,0.16,0. 19,0.20))	((0.50,0.53,0.54,0.55),(0.45, 0.47,0.52,0.55),(0.30,0.35,0. 37,0.40))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))
C_9	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))	((0.65,0.70,0.73,0.75),(0.20, 0.21,0.23,0.25),(0.20,0.21,0. 23,0.25))	((0.25,0.27,0.29,0.30),(0.55, 0.57,0.63,0.65),(0.25,0.27,0. 29,0.30))	((0.12,0.12,0.24,0.25),(0.67, 0.69,0.73,0.75),(0.20,0.21,0. 24,0.25))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.15,0.17,0.19,0.20),(0.75, 0.80,0.83,0.85),(0.15,0.17,0. 19,0.20))	((0.10,0.12,0.14,0.15),(0.85, 0.87,0.90,0.95),(0.05,0.07,0. 10,0.15))

Then we computed the criteria weighs as shown in Table 3.

Table 3. The weights of criteria.

Criteria	Notation	Weight	Rank
C_1	W_1	0.0872	1
C_2	W_2	0.096372	3
C_3	W_3	0.111414	10

C_4	W_4	0.099728	6
C_5	W_5	0.104492	9
C_6	W_6	0.103815	8
C_7	W_7	0.099325	5
C_8	W_8	0.099023	4
C_9	W_9	0.09607	2
C_{10}	W_{10}	0.102562	7

Eq. (11) is used to normalize the decision matrix as shown in Table 4.

Table 4. The normalization matrix

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.422902	0.435584	0.401674	0.367214	0.356462	0.329169	0.315936
C_2	0.375273	0.3714	0.313777	0.330483	0.289082	0.27746	0.594142
C_3	0.273911	0.519859	0.321152	0.342336	0.282173	0.437876	0.40377
C_4	0.331223	0.270936	0.500735	0.31491	0.294105	0.382762	0.483713
C_5	0.387415	0.358076	0.38164	0.336591	0.3657	0.422992	0.387415
C_6	0.424724	0.307775	0.397397	0.40805	0.40805	0.358723	0.324449
C_7	0.539694	0.38628	0.297524	0.385564	0.339754	0.334267	0.308499
C_8	0.580357	0.396601	0.366566	0.314301	0.322815	0.297983	0.282374
C_9	0.2797	0.516932	0.48203	0.401977	0.295564	0.308255	0.2797
C_{10}	0.356579	0.305738	0.31402	0.311719	0.355199	0.372453	0.564545

4.2.1 The ratio system

Eq. (14) is used to compute the ratio system values. Then Eq. (15) is used to compute the weight sum of ratio system as shown in Table 5.

Table 5. The weight sum ratio system.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.036877	0.037983	0.035026	0.032021	0.031083	0.028703	0.02755
C_2	0.036166	0.035792	0.030239	0.031849	0.027859	0.026739	0.057258
C_3	0.030517	0.05792	0.035781	0.038141	0.031438	0.048786	0.044986
C_4	0.033032	0.02702	0.049937	0.031405	0.02933	0.038172	0.048239
C_5	0.040482	0.037416	0.039878	0.035171	0.038213	0.044199	0.040482
C_6	0.044093	0.031952	0.041256	0.042362	0.042362	0.037241	0.033683
C_7	0.053605	0.038367	0.029552	0.038296	0.033746	0.033201	0.030642
C_8	0.057469	0.039273	0.036299	0.031123	0.031966	0.029507	0.027962
C_9	0.026871	0.049662	0.046309	0.038618	0.028395	0.029614	0.026871
C_{10}	0.036572	0.031357	0.032207	0.031971	0.03643	0.0382	0.057901

4.2.2 The reference point method

Then we compute the reference point. Then we compute the deviation of rating from the reference point. Then we computed the weight for the assessment values as shown in Table 6.

Table 6. The weighted assessment values.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	9.64E-05	0	0.000258	0.00052	0.000602	0.000809	0.00091
C_2	0.002033	0.002069	0.002604	0.002449	0.002833	0.002941	0
C_3	0.003053	0	0.002467	0.002204	0.00295	0.001018	0.001441
C_4	0.001686	0.002285	0	0.001848	0.002055	0.001173	0.000169
C_5	0.000388	0.000709	0.000452	0.000943	0.000626	0	0.000388
C_6	0	0.00126	0.000295	0.00018	0.00018	0.000711	0.001081
C_7	0	0.001514	0.002389	0.001521	0.001973	0.002027	0.002281
C_8	0	0.001802	0.002096	0.002609	0.002525	0.002769	0.002922
C_9	0.00219	0	0.000322	0.001061	0.002043	0.001926	0.00219

C_{10}	0.002188	0.002722	0.002635	0.002659	0.002202	0.002021	0

4.2.3 The full multiplicative form

Then we compute the full multiplicate form as shown in Table 7.

Table 7. The full multiplicative form.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.927701	0.930095	0.923545	0.916349	0.913978	0.907651	0.90441
C_2	0.90987	0.90896	0.894311	0.898793	0.887274	0.883772	0.951063
C_3	0.865649	0.929706	0.881131	0.887425	0.86852	0.912098	0.903895
C_4	0.895659	0.877892	0.933346	0.891159	0.885106	0.908671	0.930132
C_5	0.905666	0.898244	0.904245	0.892455	0.900223	0.914018	0.905666
C_6	0.914939	0.884853	0.908644	0.911142	0.911142	0.899037	0.889712
C_7	0.94058	0.909848	0.886559	0.909681	0.898324	0.896872	0.889754
C_8	0.947546	0.912489	0.905401	0.891714	0.894077	0.887018	0.882305
C_9	0.884797	0.938576	0.932294	0.916169	0.889499	0.893099	0.884797
C_{10}	0.899639	0.885556	0.887987	0.887318	0.899281	0.903666	0.943048

4.2.4 Final rank of the alternatives by the dominance theory

Then we applied the dominance theory to obtain the final rank of the alternatives as shown in Table 8.

Table 8. The final rank of alternatives.

Criteria	Ratio system	Reference point	Full multiplication form	Final rank
A_1	7	1	7	7
A_2	5	5	5	5
A_3	4	7	4	4
A_4	2	6	2	2
A_5	1	2	1	1
A_6	3	3	3	3
A_7	6	4	6	6

4.3. Comparative analysis

To solve MCGDM problems in an incomplete, ambiguous, and unpredictable environment, numerous esteemed academics working on intelligent systems, artificial intelligence, and soft computing have created innovative decision-making ideas. Our expanded TFNN-MULTIMOORA approach and other decision-making techniques were compared in this study in order to examine each of their features with regard to application domains, criterion types, calculation operators employed, and fundamental concepts.

Then, TFNN-MULTIMOORA approach is compared TFNN-VIKOR technique [29], , TFNN-CE technique [30], TFNN-GRA model [31] and TFNN-MABAC model [32]. The order is expressed in Table 9.

Table 9. The comparative analysis results.

Criteria	TFNN-MABAC technique	TFNN-GRA model	TFNN-VIKOR technique	TFNN-CE method	Proposed
A_1	7	7	7	7	7
A_2	5	5	4	5	5

A_3	4	4	5	4	4
A_4	2	3	2	2	2
A_5	1	1	1	1	1
A_6	3	2	3	3	3
A_7	6	6	6	6	6

5. Conclusion

To achieve sustainable and stable improvement in the quality of higher education, it is essential to continuously innovate and refine the management modes of higher education. Currently, with the rapid advancement of the economy and technology, China's higher education management faces significant challenges. These challenges include outdated educational goals and content, insufficient emphasis on humanistic care in educational methods, and a lack of scientific rigor in management systems. As the evolving societal and technological landscape presents increasingly complex demands, it is imperative to reshape and improve the management modes of higher education to address these issues effectively. In response to these challenges, higher education management must innovate by aligning its management content with the context of the times. This requires a rethinking of traditional approaches, actively improving the work quality of management teams, and reshaping the evaluation systems within education management. Furthermore, the internal management system of higher education should be rebuilt using scientific thinking as the foundation and supervision as a guiding principle. By addressing these aspects through both internal logic and external oversight, universities can optimize and innovate their management modes to better meet modern requirements. The evaluation of innovative development modes in higher education management can be approached through MADM. This methodology allows for the systematic assessment of diverse factors influencing higher education management. Recently, the MULTIMOORA approach and average models have been utilized to address MADM problems in this context. These methods provide robust tools for analyzing and prioritizing decision-making criteria, particularly when dealing with complex and multi-dimensional issues. However, the uncertainty inherent in higher education management requires advanced techniques to enhance decision-making. To address this, TFNSs have emerged as a more effective framework for handling uncertain data in the evaluation of innovative development modes in higher education management. In this study, a TFNN-MULTIMOORA model is proposed. To validate the effectiveness of the TFNN-MULTIMOORA model, a numerical example is presented, along with comparative analyses to demonstrate the model's advantages over existing methods.

The key advantages of the TFNN-MULTIMOORA approach in addressing innovative development modes evaluation in higher education management are as follows:

- Handling Uncertainty: The TFNN-MULTIMOORA approach effectively incorporates uncertainty into the evaluation process. It not only provides a robust mechanism for

addressing the ambiguity and vagueness inherent in higher education management but also captures the shape similarity degree during the evaluation of innovative development modes. This ensures a more accurate and realistic representation of the decision-making environment.

- The uncertainty information is solved by using the neutrosophic numbers.

In conclusion, the proposed TFNN-MULTIMOORA approach represents a significant advancement in the evaluation of management modes in higher education. By addressing uncertainty, leveraging advanced techniques, and integrating multiple methodologies, this approach provides a robust, efficient, and scientifically sound framework for optimizing higher education management. As universities continue to face complex and evolving challenges, the TFNN-MULTIMOORA method offers a valuable tool for ensuring the continuous improvement and innovation of management practices in higher education.

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Energy of a neutrosophic soft set and its applications to multi-criteria decision-making problems

Zeynep Başer[1], Vakkas Uluçay[2,*]

[1]Department of Mathematics, Kilis 7 Aralık University, 79000 Kilis, Turkey; zeynep.baser@kilis.edu.tr

[2]Department of Mathematics, Kilis 7 Aralık University, 79000 Kilis, Turkey; vakkas.ulucay@kilis.edu.tr

*Correspondence: vakkas.ulucay@kilis.edu.tr

Abstract. In our paper, we continue the study of the theory of neutrosophic soft sets and their properties. Based on the singular values of the corresponding matrix, we define the energy of a neutrosophic soft set, as well as the lower energy of a neutrosophic soft set and the upper energy of a neutrosophic soft set, allowing us to introduce an effective method for decision-making. To do this, we first write a neutrosophic soft set as a matrix and multiply it by its transpose with respect to the given product and the given norm t, s. Then, we consider the limits of the defined energies, which are essentially non-negative numerical values. The paper demonstrates through examples how the introduced method can be successfully applied to many problems containing uncertainties.

Keywords: neutrosophic soft set; upper energy; lower energy; decision-making

1. Introduction

In 1978, I. Gutman [1] defined The energy of the graph G is computed as the total of the absolute values of the eigenvalues of the adjacency matrix associated with G. Namely, given the eigenvalues $\theta_1 \geq \theta_2 \geq \ldots \theta_m \geq 0$ of the adjacency matrix of a graph G, the energy of G;

$$E(G) = \sum_{i=1}^{m} |\theta_i|$$

denoted by $E(G)$.

For studies on energy, the following can be mentioned based on the Survey of Graph Energies [2] article by Ivan Gutman and Boris Furtula [3] About 20 years later, energy has become a very favorite field of study. The idea of defining energy in this way and calling it energy comes from quantum chemistry. The other reason is that the right-hand side of the equation is

independent of the labeling of the eigenvalues. So there is no need for $\theta_1 \geq \theta_2 \geq \ldots \theta_m \geq 0$. Since graph energy is a symmetric function of the eigenvalues of the graph, it is among the most frequently analyzed functions. The study of the energies of graphs is very active today. More than a hundred energy variants have been proposed on various graphs based on matrices other than the adjacency matrix. There are more than 1000 publications on this topic and these publications have found various fields of application. Beyond graph energy: Norms of graphs and matrices [4], energy of matrices [5], energy of pythagorean fuzzy graphs with applications [6], on incidence energy of graphs [7], energy of some graphs [8–11]. In these studies, lower and upper bounds for energies are found and various applications are given.

Zadeh [12] introduced fuzzy set theory in 1965 to tackle difficulties associated with uncertainty. Since then, fuzzy sets and fuzzy logic have been thoroughly investigated by researchers aiming to address a wide range of real-world problems with ambiguity and uncertainty. Atanassov (1986) [13] extended this framework by introducing intuitionistic fuzzy sets. Subsequently, Smarandache [14] developed the theory of neutrosophic sets, which provides a framework for dealing with issues involving imprecise, indeterminate, and inconsistent data. Neutrosophic sets generalize classical and fuzzy sets and are applied in diverse areas such as decision-making, control theory, medicine, topology, and other practical scenarios. The concept of soft set theory was introduced by Molodtsov [15] in 1999 as a new mathematical approach for managing uncertainties. His soft sets offer a novel method for addressing uncertainty through parameter analysis. Maji et al. [16, 17] later expanded this theory to include fuzzy soft sets and neutrosophic soft sets. Recently, research in neutrosophic theory has gained momentum, with studies focusing on the extension of neutrosophic sets [18–21], neutrosophic algebra [22], decision-making processes [23–27], and neutrosophic environments [28–31].
Studies on energies are not limited to graphs, and energy have also been defined on fuzzy sets. [32] can be given as an example.
Previous literature has not proposed an energy for neutrosophic soft sets by integrating their lower and upper energies. Additionally, there has been no discussion regarding energy in neutrosophic soft sets. The aim of the current research is to propose a new method for a new approach energy using a neutrosophic soft set by in multi criteria decision-making problems. However, when information is endowed with correct, uncertain, and incorrect data, neutrosophic soft sets become a suitable representation for decision-making. The energy defined through this integration is used to solve problems characterized by correct, uncertain, and incorrect data.

The remainder of the paper is organized as follow: In section 2, we gave a brief introduction of key definitions and propositions. In section 3, we introduced a new method for a new approach energy using a neutrosophic soft set and also we discussed their desirable properties.

In section 4, we develop the algorithm for MCDM. In section 5, an illustrative example is developed to describe the effectiveness of the developed method. In section 6, we gave a comparison with some existing methods. Finally, a conclusion is made in Section 7.

2. Preliminaries

Definition 2.1. [33] An eigenvector of a $n \times n$ matrix $\mathcal{N}$ is nonzero vector x such that $\mathcal{N}x = cx$ for some scalar c. A scalar c is called an eigenvalue of $\mathcal{N}$ if there is nontrivial solution x of $\mathcal{N}x = cx$, such an x is called an eigenvector corresponding to c.

Definition 2.2. [14] A neutrosophic sets $\mathcal{K}$ on the universe of discourse $\mathcal{E}$ is defined as: $\mathcal{K} = \{\langle x, (\alpha_{\mathcal{K}}(x), \beta_{\mathcal{K}}(x), \gamma_{\mathcal{K}}(x))\rangle : x \in \mathcal{E}, \alpha_{\mathcal{K}}(x), \beta_{\mathcal{K}}(x), \gamma_{\mathcal{K}}(x) \in [0,1]\}$. There is no restriction on the sum of $\alpha_{\mathcal{K}}(x)$, $\beta_{\mathcal{K}}(x)$ and $\gamma_{\mathcal{K}}(x)$, so $^-0 \leq \alpha_{\mathcal{K}}(x) + \beta_{\mathcal{K}}(x) + \gamma_{\mathcal{K}}(x) \leq 3^+$.

Definition 2.3. [36] Let $\mathcal{U}$ be a universe set and $\mathcal{E}$ be a set of parameters. Consider $\mathcal{A} \subseteq \mathcal{E}$. Let $\mathcal{N}\int\mathcal{U}$ denotes the set of all neutrosophic sets of $\mathcal{U}$. The collection $(\Phi, \mathcal{A})$ is termed to be the neutrosophic soft set over $\mathcal{U}$, where Φ is a mapping given by $\Phi : \mathcal{A} \to \mathcal{N}\int\mathcal{U}$.

Definition 2.4. [37] Let $\mathcal{N} = \{x, \{< u, \alpha_{\phi_{\mathcal{N}}(x)}(u), \beta_{\phi_{\mathcal{N}}(x)}(u), \gamma_{\phi_{\mathcal{N}}}(u) >: x \in U\} : x \in E\}$ be a neutrosophic soft set. Then the complement of a neutrosophic soft set $\mathcal{N}$ denoted by $\mathcal{N}^{\mathcal{C}}$ and is defined by

$$\mathcal{N}^{\mathcal{C}} = \{(x, \{< u, \gamma_{\phi_{\mathcal{N}}(x)}(u), 1 - \beta_{\phi_{\mathcal{N}}(x)}(u), \alpha_{\phi_{\mathcal{N}}(x)}(u) >: x \in \mathcal{U}\} : x \in \mathcal{E}\}$$

Definition 2.5. [37] Let $\mathcal{N}$ be a neutrosophic soft set over $\mathcal{N}(U)$. Then a subset of $\mathcal{N}(U) \times \mathcal{E}$ is uniquely defined by

$$\mathcal{R}_{\mathcal{N}} = \{(\phi_{\mathcal{N}(x)}, x) : x \in \mathcal{E}, \phi_{\mathcal{N}(x)} \in \mathcal{N}(\mathcal{U})\}$$

which is called a relation form of $(\mathcal{N}, \mathcal{E})$. The characteristic function of $\mathcal{R}_{\mathcal{N}}$ is written by

$$\theta_{\mathcal{R}_{\mathcal{N}}} : \mathcal{N}(\mathcal{U}) \times \mathcal{E} \to [0,1] \times [0,1] \times [0,1], \quad \theta_{\mathcal{R}_{\mathcal{N}}}(u,x) = (\alpha_{\phi_{\mathcal{N}}(x)}(u), \beta_{\phi_{\mathcal{N}}(x)}(u), \gamma_{\phi_{\mathcal{N}}(x)}(u))$$

where $\alpha_{\phi_{\mathcal{N}}(x)}(u)$, $\beta_{\phi_{\mathcal{N}}(x)}(u)$, $\gamma_{\phi_{\mathcal{N}}(x)}(u)$ is the truth-membership, indeterminacy-membership and falsity-membership of $u \in \mathcal{U}$, respectively.

Definition 2.6. [37] The transpose of square neutrosophic soft matrix $[\omega_{ij}]$ of order $m \times n$ is another square neutrosophic soft matrix of order $n \times m$ obtained from $[\omega_{ij}]$ by interchanging its rows and columns. It is denoted by $[\omega_{ij}^T]$. Therefore the neutrosophic soft set associated with $[\omega_{ij}^T]$ becomes a new neutrosophic soft set over the same universe and over the same set of parameters.

Definition 2.7. [32] The energy of a fuzzy soft set $\Gamma_{\mathcal{A}}$, denoted as $\xi(\Gamma_{\mathcal{A}})$, is defined as $\xi(\Gamma_{\mathcal{A}}) = \sum_{i=1}^{m} \theta_i$ where $\theta_1 \geq \theta_2 \geq \ldots \theta_m \geq 0$ are the singular values of the matrix $\mathcal{A}$ corresponding to the fuzzy soft set $\Gamma_{\mathcal{A}}$.

3. Energy of A Neutrosophic Soft Set

Definition 3.1. [37] Let $\mathcal{U} = \{u_1, \ldots, u_m\}$, $\mathcal{E} = \{x_1, \ldots, x_n\}$ and $\mathcal{N}$ be a neutrosophic soft set over $\mathcal{N}(\mathcal{U})$. Then

$\mathcal{R}_\mathcal{N}$	$\phi_\mathcal{N}(x_1)$	$\phi_\mathcal{N}(x_2)$	$\ldots$	$\phi_\mathcal{N}(x_n)$
u_1	$\theta_{\mathcal{R}_\mathcal{N}}(u_1, x_1)$	$\theta_{\mathcal{R}_\mathcal{N}}(u_1, x_2)$	$\ldots$	$\theta_{\mathcal{R}_\mathcal{N}}(u_1, x_n)$
u_2	$\theta_{\mathcal{R}_\mathcal{N}}(u_2, x_1)$	$\theta_{\mathcal{R}_\mathcal{N}}(u_2, x_2)$	$\ldots$	$\theta_{\mathcal{R}_\mathcal{N}}(u_2, x_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\theta_{\mathcal{R}_\mathcal{N}}(u_m, x_1)$	$\theta_{\mathcal{R}_\mathcal{N}}(u_m, x_2)$	$\ldots$	$\theta_{\mathcal{R}_\mathcal{N}}(u_m, x_m)$

If $\omega_{ij} = \theta_{\mathcal{R}_\mathcal{N}}(u_i, x_j)$, we can define a matrix:

$$[\omega_{ij}] = \begin{bmatrix} \omega_{11} & \omega_{12} & \ldots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \ldots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{1m} & \omega_{m2} & \ldots & \omega_{mn} \end{bmatrix}$$

such that $\omega_{ij} = (\alpha_{\phi_\mathcal{N}(x_j)}(u_i), \beta_{\phi_\mathcal{N}(x_j)}(u_i), \gamma_{\phi_\mathcal{N}(x_j)}(u_i)) = (\alpha^a_{ij}, \beta^a_{ij}, \gamma^a_{ij})$ which is called a $m \times n$ *neutrosophic soft matrix* (or namely NS-matrix) of the neutrosophic soft set $\mathcal{N}$ over $\mathcal{N}(\mathcal{U})$.

Example 3.2. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $\mathcal{E} = \{x_1, x_2, x_3\}$. $\mathcal{N}$ be a neutrosophic soft set over neutrosophic as:

$$\begin{aligned} \mathcal{N} = \Big\{ & (x_1, \{< u_1, (0.3, 0.5, 0.8) >, < u_2, (0.7, 0.4, 0.2) >, < u_3, (0.3, 0.6, 0.1) >\}), \\ & (x_2, \{< u_1, (0.4, 0.3, 0.9) >, < u_2, (0.8, 0.5, 0.2) >, < u_3, (0.5, 0.2, 0.4) >\}), \\ & (x_3, \{< u_1, (0.9, 0.4, 0.2) >, < u_2, (0.8, 0.1, 0.5) >, < u_3, (0.4, 0.6, 0.3) >\}) \Big\} \end{aligned}$$

Then, the NS-matrix $[\omega_{ij}]$ is written by

$$[\omega_{ij}] = \begin{bmatrix} (0.3, 0.5, 0.8) & (0.7, 0.4, 0.2) & (0.3, 0.6, 0.1) \\ (0.4, 0.3, 0.9) & (0.8, 0.5, 0.2) & (0.5, 0.2, 0.4) \\ (0.9, 0.4, 0.2) & (0.8, 0.1, 0.5) & (0.4, 0.6, 0.3) \end{bmatrix}$$

Definition 3.3. Let $[\omega_{ij}] \in \mathcal{N}_{m\times n}$, $[\vartheta_{jq}] \in \mathcal{N}_{n\times k}$. Then AND-product of $[\omega_{ij}]$ and $[\vartheta_{jq}]$ is defined by

$$\wedge : \mathcal{N}_{m\times n} \times \mathcal{N}_{n\times k} \to \mathcal{N}_{m\times k} \qquad [\omega_{ij}] \wedge [\vartheta_{jq}] = [\eta_{iq}] = (\alpha'_{iq}, \beta'_{iq}, \gamma'_{iq})$$

where

$$\alpha'_{iq} = max(t(\alpha^{\omega}_{i1}, \alpha^{\vartheta}_{1q}), t(\alpha^{\omega}_{i2}, \alpha^{\vartheta}_{2q}), \ldots, t(\alpha^{\omega}_{in}, \alpha^{\vartheta}_{nq})),$$
$$\beta'_{iq} = min(s(\beta^{\omega}_{i1}, \beta^{\vartheta}_{1q}), s(\beta^{\omega}_{i2}, \beta^{\vartheta}_{2q}), \ldots, s(\beta^{\omega}_{in}, \beta^{\vartheta}_{nq})) \text{ and}$$
$$\gamma'_{iq} = min(s(\gamma^{\omega}_{i1}, \gamma^{\vartheta}_{1q}), s(\gamma^{\omega}_{i2}, \gamma^{\vartheta}_{2q}), \ldots, s(\gamma^{\omega}_{in}, \gamma^{\vartheta}_{nq}))$$

such that $1 \leq i \leq m$ and $1 \leq q \leq n$.

Example 3.4. Consider Example 3.2:

$$[\omega_{ij}] = \begin{bmatrix} (0.3, 0.5, 0.8) & (0.7, 0.4, 0.2) & (0.3, 0.6, 0.1) \\ (0.4, 0.3, 0.9) & (0.8, 0.5, 0.2) & (0.5, 0.2, 0.4) \\ (0.9, 0.4, 0.2) & (0.8, 0.1, 0.5) & (0.4, 0.6, 0.3) \end{bmatrix}$$

$$[\omega^T_{ij} = \vartheta_{jq}] = \begin{bmatrix} (0.3, 0.5, 0.8) & (0.4, 0.3, 0.9) & (0.9, 0.4, 0.2) \\ (0.7, 0.4, 0.2) & (0.8, 0.5, 0.2) & (0.8, 0.1, 0.5) \\ (0.3, 0.6, 0.1) & (0.5, 0.2, 0.4) & (0.4, 0.6, 0.3) \end{bmatrix}$$

and for t= einstein product and s= einstein sum;

$$[\omega_{ij}] \wedge [\vartheta_{jq}] = [\Lambda] = \begin{bmatrix} (0.449, 0.689, 0.198) & (0.528, 0.695, 0.384) & (0.528, 0.480, 0.388) \\ (0.528, 0.695, 0.384) & (0.615, 0.384, 0.384) & (0.615, 0.471, 0.201) \\ (0.528, 0.480, 0.388) & (0.615, 0.571, 0.625) & (0.801, 0.198, 0.384) \end{bmatrix}$$

Definition 3.5. Let $[\omega_{ij}] \in \mathcal{N}_{m\times n}$, $[\vartheta_{jq}] \in \mathcal{N}_{n\times k}$. Then OR-product of $[\omega_{ij}]$ and $[\vartheta_{jq}]$ is defined by

$$\vee : \mathcal{N}_{m\times n} \times \mathcal{N}_{n\times k} \to \mathcal{N}_{m\times k} \qquad [\omega_{ij}] \vee [\vartheta_{jq}] = [\eta_{iq}] = (\alpha^c_{iq}, \beta^c_{iq}, \gamma^c_{iq})$$

where

$$\alpha'_{iq} = min(s(\alpha^{\omega}_{i1}, \alpha^{\vartheta}_{1q}), s(\alpha^{\omega}_{i2}, \alpha^{\vartheta}_{2q}), \ldots, s(\alpha^{\omega}_{in}, \alpha^{\vartheta}_{nq})),$$
$$\beta'_{iq} = max(t(\beta^{\omega}_{i1}, \beta^{\vartheta}_{1q}), t(\beta^{\omega}_{i2}, \beta^{\vartheta}_{2q}), \ldots, t(\beta^{\omega}_{in}, \beta^{b}_{nq})) \text{ and}$$
$$\gamma'_{iq} = max(t(\gamma^{\omega}_{i1}, \gamma^{\vartheta}_{1q}), t(\gamma^{\omega}_{i2}, \gamma^{\vartheta}_{2q}), \ldots, t(\gamma^{\omega}_{in}, \gamma^{\vartheta}_{nq}))$$

such that $1 \leq i \leq m$ and $1 \leq q \leq n$.

Example 3.6. Again consider Example 3.2:

$$[\omega_{ij}] = \begin{bmatrix} (0.3, 0.5, 0.8) & (0.7, 0.4, 0.2) & (0.3, 0.6, 0.1) \\ (0.4, 0.3, 0.9) & (0.8, 0.5, 0.2) & (0.5, 0.2, 0.4) \\ (0.9, 0.4, 0.2) & (0.8, 0.1, 0.5) & (0.4, 0.6, 0.3) \end{bmatrix}$$

$$[\omega^T_{ij} = \vartheta_{jq}] = \begin{bmatrix} (0.3, 0.5, 0.8) & (0.4, 0.3, 0.9) & (0.9, 0.4, 0.2) \\ (0.7, 0.4, 0.2) & (0.8, 0.5, 0.2) & (0.8, 0.1, 0.5) \\ (0.3, 0.6, 0.1) & (0.5, 0.2, 0.4) & (0.4, 0.6, 0.3) \end{bmatrix}$$

and for t= einstein product and s= einstein sum;

$$[\omega_{ij}] \vee [\vartheta_{jq}] = [\nu] = \begin{bmatrix} (0.550, 0.310, 0.615) & (0.625, 0.153, 0.705) & (0.625, 0.310, 0.137) \\ (0.625, 0.153, 0.705) & (0.689, 0.201, 0.801) & (0.750, 0.090, 0.166) \\ (0.625, 0.310, 0.137) & (0.750, 0.0909, 0.166) & (0.689, 0.310, 0.201) \end{bmatrix}$$

Proposition 3.7. *Let* $[\omega_{ij}] \in \mathcal{N}_{m\times n}$, $[\vartheta_{jq}] \in \mathcal{N}_{n\times k}$. *Then,*

(1) $([\omega_{ij}] \vee [\vartheta_{jq}])^c = [\omega_{ij}]^c \wedge [\vartheta_{iq}]^c$

(2) $([\omega_{ij}] \wedge [\vartheta_{iq}])^c = [\omega_{ij}]^c \vee [\vartheta_{iq}]^c$

Definition 3.8. Let $\mathcal{N} = \{(x, \{< u, \alpha_{\phi_{\mathcal{N}}(x)}(u), \beta_{\phi_{\mathcal{N}}(x)}(u), \gamma_{\phi N_{(x)}}(u) >: x \in U\} : x \in \mathcal{E}\}$ be a neutrosophic soft set. Then *score function* of $\mathcal{N}$, is denoted by $S^{\mathcal{N}}_{energy}$, is defined as;

$$S^{\mathcal{N}}_{energy} = \frac{\alpha_{\phi_{\mathcal{N}}(x)}(u) - \beta_{\phi_{\mathcal{N}}(x)}(u) - \gamma_{\phi_{\mathcal{N}}(x)}(u)}{\alpha_{\phi_{\mathcal{N}}(x)}(u) + \gamma_{\phi_{\mathcal{N}}(x)}(u)}$$

Example 3.9. Score function of Example 3.4:

$$S^{\Lambda}_{energy} = \begin{bmatrix} -0.676 & -0.604 & -0.371 \\ -0.604 & -0.153 & -0.069 \\ -0.371 & -0.468 & 0.184 \end{bmatrix}$$

and score function of Example 3.6:

$$S^{\nu}_{energy} = \begin{bmatrix} -0.321 & -0.176 & 0.231 \\ -0.176 & -0.209 & 0.537 \\ 0.231 & 0.537 & 0.201 \end{bmatrix}$$

Definition 3.10. The *upper energy* of a neutrosophic soft set $\mathcal{N}(\mathcal{U})$, denoted as $\xi^+(\mathcal{N})$ is defined as $\xi^+(\mathcal{N}) = \sum_{i=1}^{m} |_{\Delta^+}\theta_i^+|$ where $_{\Delta^+}\theta_i$ are eigenvalues of the matrix Δ^+ corresponding to the matrix obtained by $\wedge$-product the neutrosophic matrix and its transpose with the score function.

Example 3.11. To determine upper energy of Example 3.2 from Example 3.9 eigenvalues of S^{Λ}_{energy};

$$_{\Lambda^+}\theta_1^+ = 1 \quad _{\Lambda^+}\theta_2^+ = 0.2 \quad _{\Lambda^+}\theta_3^+ = 0.3$$

and sum of eigenvalues ;

$$\xi^+(\mathcal{N}) = \sum_{i=1}^{3} {}_{\Lambda^+}\theta_i^+ = {}_{\Lambda^+}\theta_1^+ + {}_{\Lambda^+}\theta_2^+ + {}_{\Lambda^+}\theta_3^+ = 1 + 0.2 + 0.3 = 1.5$$

therefore upper energy of Example 3.2 is 1.5.

Definition 3.12. The *lower energy* of a neutrosophic soft set $\mathcal{N}(\mathcal{U})$, denoted as $\xi^-(\mathcal{N})$ is defined as $\xi^-(\mathcal{N}) = \sum_{i=1}^{m} |_{\Delta^-}\theta_i^-|$ where θ_i are eigenvalues of the matrix Δ^- corresponding to the matrix obtained by $\vee$-product the neutrosophic matrix and its transpose with the score function.

Example 3.13. To determine lower energy of Example 3.2 from Example 3.9 eigenvalues of S^{ν}_{energy};

$$_{\nu^-}\theta_1^- = 0.8 \quad _{\nu^-}\theta_2^- = 0.3 \quad _{\nu^-}\theta_3^- = 0.1$$

and sum of eigenvalues ;

$$\xi^{-}(\mathcal{N}) = \sum_{i=1}^{3} {}_{\nu^-}\theta_i^- = {}_{\nu^-}\theta_1^- + {}_{\nu^-}\theta_2^- + {}_{\nu^-}\theta_3^- = 0.8 + 0.3 + 0.1 = 1.2$$

therefore lower energy of Example 3.2 is 1.2.

Definition 3.14. The energy of a neutrosophic soft set $\mathcal{N}(\mathcal{U})$, denoted as $\xi(\mathcal{N})$ is defined as $\xi(\mathcal{N}) = \xi^{+}(\mathcal{N}) - \xi^{-}(\mathcal{N})$.

Example 3.15. From Example 3.11 and Example 3.13 energy of given matrix in Example 3.2 is:

$$\xi(\mathcal{N}) = \xi^{+}(\mathcal{N}) - \xi^{-}(\mathcal{N}) = 1.5 - 1.2 = 0.3$$

4. Algorithm of Proposed Decision-Making

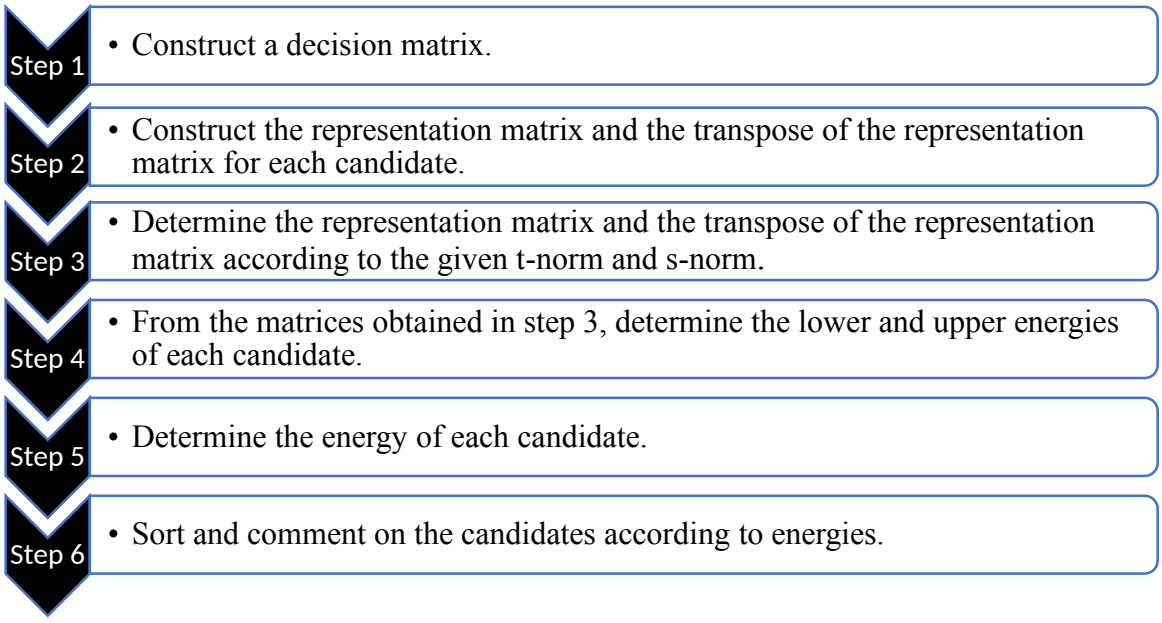
Step 1
• Construct a decision matrix.
Step 2
• Construct the representation matrix and the transpose of the representation matrix for each candidate.
Step 3
• Determine the representation matrix and the transpose of the representation matrix according to the given t-norm and s-norm.
Step 4
• From the matrices obtained in step 3, determine the lower and upper energies of each candidate.
Step 5
• Determine the energy of each candidate.
Step 6
• Sort and comment on the candidates according to energies.

FIGURE 1. Flowchart of Proposed Decision-Making

Algorithm

Step 1: Construct a decision matrix.

Step 2: Construct the representation matrix and the transpose of the representation matrix for each candidate. For construct representation matrix of candidate 1 remove first column and row. For construct representation matrix of candidate 2 remove second column and row, and so on.

Step 3 Determine the representation matrix and the transpose of the representation matrix according to the given t-norm and s-norm.

Step 4 From the matrices obtained in step 3, determine the lower and upper energies of each candidate.

Step 5 Determine the energy of each candidate.

Step 6 Sort and comment on the candidates according to energies.

5. Application of Proposed Decision-Making

Example 5.1. A company plans to invest in a new artificial intelligence (AI) to enhance its existing products. Let $\mathcal{U} = \{u_1, u_2, u_3\}$ be the candidate set for the artificial intelligence program. The set of parameters to be considered is $\mathcal{E} = \{x_1, x_2, x_3\}$. For $i = 1, 2, 3$ the parameters x_i represent, in order, security, "budget, and capacity which is represented in the form of single-valued neutrosophic soft set; (In the following example, we use Einstein product.)

$$\begin{aligned}\mathcal{N} =\Big\{ &(x_1, \{< u_1, (0.3, 0.4, 0.2) >, < u_2, (0.7, 0.3, 0.2) >, < u_3, (0.2, 0.1, 1.0) >\}),\\ &(x_2, \{< u_1, (0.1, 0.6, 0.5) >, < u_2, (0.9, 0.2, 0.5) >, < u_3, (0.2, 0.2, 1.0) >\}),\\ &(x_3, \{< u_1, (0.1, 0.8, 0.4) >, < u_2, (0.5, 0.4, 0.2) >, < u_3, (1.0, 0.2, 0.1) >\})\Big\}\end{aligned}$$

and the representation of neutrosophic matrix

$$[\Omega_{ij}] = \begin{bmatrix} (0.3, 0.4, 0.2) & (0.1, 0.6, 0.5) & (0.1, 0.8, 0.4) \\ (0.7, 0.3, 0.2) & (0.9, 0.2, 0.5) & (0.5, 0.4, 0.2) \\ (0.2, 0.1, 1.0) & (0.2, 0.2, 1.0) & (1.0, 0.2, 0.1) \end{bmatrix}$$

for candidate 1

$$\chi_1 = \begin{bmatrix} (0.9, 0.2, 0.5) & (0.5, 0.4, 0.2) \\ (0.2, 0.2, 1.0) & (1.0, 0.2, 0.1) \end{bmatrix}$$

and

$$\chi_1^T = \begin{bmatrix} (0.9, 0.2, 0.5) & (0.2, 0.2, 1.0) \\ (0.5, 0.4, 0.2) & (1.0, 0.2, 0.1) \end{bmatrix}$$

$\wedge$-product of this matrices

$$\chi_1 \wedge \chi_1^T = \chi_1^+ = \begin{bmatrix} (0.801, 0.384, 0.384) & (0.501, 0.384, 0.294) \\ (0.501, 0.384, 0.294) & (1.0, 0.384, 0.198) \end{bmatrix}$$

score function of this matrices

$$S_{energy}^{\chi_1^+} = \begin{bmatrix} 0.027 & -0.225 \\ -0.225 & 0.348 \end{bmatrix}$$

therefore eigenvalues of this matrix;

$$_{\chi_1^+}\theta_1^+ = 0.4 \quad _{\chi_1^+}\theta_2^+ = 0.1$$

so upper energy for candidate 1

$$\xi^{+}(\chi_1) = {}_{\chi_1^{+}}\theta_1^{+} + {}_{\chi_1^{+}}\theta_2^{+} = 0.4 + 0.1 = 0.5$$

$\vee$-product of this matrices

$$\chi_1 \vee \chi_1^{T} = \chi_1^{-} = \begin{bmatrix} (0.8, 0.117, 0.2) & (0.932, 0.054, 0.5) \\ (0.932, 0.054, 0.5) & (0.384, 0.024, 1.0) \end{bmatrix}$$

score function of this matrices

$$S_{energy}^{\chi_1^{-}} = \begin{bmatrix} 0.482 & 0.264 \\ 0.264 & -0.462 \end{bmatrix}$$

therefore eigenvalues of this matrix;

$${}_{\chi_1^{-}}\theta_1^{-} = 0.7 \quad {}_{\chi_1^{-}}\theta_2^{-} = 0.7$$

so lower energy for candidate 1

$$\xi^{-}(\chi_1) = {}_{\chi_1^{-}}\theta_1^{-} + {}_{\chi_1^{-}}\theta_2^{-} = 0.7 + 0.7 = 1.4$$

hence the energy of candidate 1:

$$\xi(\chi_1) = \xi^{+}(\chi_1) - \xi^{-}(\chi_1) = 0.5 - 1.4 = -0.9$$

and now for candidate 2

$$\chi_2 = \begin{bmatrix} (0.3, 0.4, 0.2) & (0.1, 0.8, 0.4) \\ (0.2, 0.1, 1.0) & (1.0, 0.2, 0.1) \end{bmatrix}$$

and

$$\chi_2^{T} = \begin{bmatrix} (0.3, 0.4, 0.2) & (0.2, 0.1, 1.0) \\ (0.1, 0.8, 0.4) & (1.0, 0.2, 0.1) \end{bmatrix}$$

$\wedge$-product of this matrices

$$\chi_2 \wedge \chi_2^{T} = \chi_2^{+} = \begin{bmatrix} (0.060, 0.689, 0.384) & (0.1, 0.480, 0.480) \\ (0.1, 0.480, 0.480) & (1.0, 0.198, 0.198) \end{bmatrix}$$

score function of this matrices

$$S_{energy}^{\chi_2^{+}} = \begin{bmatrix} -2.278 & -1.483 \\ -1.483 & 0.504 \end{bmatrix}$$

therefore eigenvalues of this matrix;

$${}_{\chi_2^{+}}\theta_1^{+} = 0.7 \quad {}_{\chi_2^{+}}\theta_2^{+} = 0.1$$

so upper energy for candidate 2

$$\xi^{+}(\chi_2) = {}_{\chi_2^{+}}\theta_1^{+} + {}_{\chi_2^{+}}\theta_2^{+} = 0.7 + 0.1 = 0.8$$

$\vee$-product of this matrices

$$\chi_2 \vee \chi_2^T = \chi_2^- = \begin{bmatrix} (0.198, 0.615, 0.117) & (0.471, 0.137, 0.2) \\ (0.471, 0.137, 0.2) & (0.384, 0.024, 1.0) \end{bmatrix}$$

score function of this matrices

$$S_{energy}^{\chi_2^-} = \begin{bmatrix} -1.694 & 0.199 \\ 0.199 & -0.462 \end{bmatrix}$$

therefore eigenvalues of this matrix;

$$_{\chi_2^-}\theta_1^- = 1.0 \quad _{\chi_2^-}\theta_2^- = 0.3$$

so lower energy for candidate 2

$$\xi^-(\chi_2) = {}_{\chi_2^-}\theta_1^- + {}_{\chi_2^-}\theta_2^- = 1.0 + 0.3 = 1.3$$

hence the energy of candidate 2 :

$$\xi(\chi_2) = \xi^+(\chi_2) - \xi^-(\chi_2) = 0.8 - 1.3 = -0.5$$

and now for candidate 3 :

$$\chi_3 = \begin{bmatrix} (0.3, 0.4, 0.2) & (0.1, 0.6, 0.5) \\ (0.7, 0.3, 0.2) & (0.9, 0.2, 0.5) \end{bmatrix}$$

and

$$\chi_3^T = \begin{bmatrix} (0.3, 0.4, 0.2) & (0.7, 0.3, 0.2) \\ (0.1, 0.6, 0.5) & (0.9, 0.2, 0.5) \end{bmatrix}$$

$\wedge$-product of this matrices

$$\chi_3 \wedge \chi_3^T = \chi_3^+ = \begin{bmatrix} (0.060, 0.689, 0.384) & (0.173, 0.625, 0.384) \\ (0.173, 0.625, 0.384) & (0.890, 0.384, 0.384) \end{bmatrix}$$

score function of this matrices

$$S_{energy}^{\chi_3^+} = \begin{bmatrix} -2.278 & -1.497 \\ -1.497 & 0.094 \end{bmatrix}$$

therefore eigenvalues of this matrix;

$$_{\chi_3^+}\theta_1^+ = 1.7 \quad _{\chi_3^+}\theta_2^+ = 0.3$$

so upper energy for candidate 3

$$\xi^+(\chi_3) = {}_{\chi_3^+}\theta_1^+ + {}_{\chi_3^+}\theta_2^+ = 1.7 + 0.3 = 2.0$$

$\vee$-product of this matrices

$$\chi_3 \vee \chi_3^T = \chi_3^- = \begin{bmatrix} (0.198, 0.310, 0.2) & (0.864, 0.090, 0.2) \\ (0.826, 0.090, 0.2) & (0.939, 0.060, 0.2) \end{bmatrix}$$

score function of this matrices

$$S_{energy}^{\chi_3^-} = \begin{bmatrix} -0.784 & 0.521 \\ 0.521 & 0.595 \end{bmatrix}$$

therefore eigenvalues of this matrix;

$$_{\chi_3^-}\theta_1^- = 0.7 \quad _{\chi_3^-}\theta_2^- = 0.1$$

so lower energy for candidate 3

$$\xi^-(\chi_3) = {}_{\chi_3^-}\theta_1^- + {}_{\chi_3^-}\theta_2^- = 0.7 + 0.1 = 0.8$$

hence the energy of candidate 3:

$$\xi(\chi_3) = \xi^+(\chi_3) - \xi^-(\chi_3) = 2.0 - 0.8 = 1.2$$

and so

$$u_3 > u_1 > u_2$$

6. **Comparison and Analysis Discussion**

Nearly all new properties or characteristics of neutrosophic soft sets have spurred further research aimed at establishing comparisons, exploring additional generalizations, proving new properties, or developing applications related to those properties. Alongside the many applications of neutrosophic soft set theory, this paper introduces additional parameters that characterize the nature of neutrosophic soft sets. This contribution aims to advance the application of neutrosophic soft set theory, similar to how graph energy functions in graph theory. The above defined energy of a neutrosophic soft set are the only tool to solve this kind of information and help decision analyst to make a decision. To demonstrate the effectiveness of the proposed method, a comparison of the decision-making results of the MCDM methods based on energy of existing presented by Mudric et al. [32] is carried out, as shown in Table 1.

Table 1. The results from the different operators

Methods	Ranking of alternatives	The optimal alternative
The method in [32]	$U_3 > U_4 > U_2 > U_1$,	U_3
Proposed Method	$U_4 > U_3 > U_2 > U_1$,	U_4

7. Conclusion

Neutrosophic Soft Sets play a very important role in the decision-making process. In addition to numerous applications of the theory of neutrosophic soft sets, this paper defines additional parameters characterizing the nature of neutrosophic soft sets, contributing to further advancements in the application of the theory of neutrosophic soft sets, similar to the role of graph energy in graph theory. A neutrosophic soft energy model provides more precision, flexibility, and compatibility to the system as compared to the classical and fuzzy model. Therefore, developed method have been applied to many real-life decision-making problems. The research significance and value of this paper are highlighted below:

(1) The upper and the lower energies based on neutrosophic soft sets were defined.
(2) In this article presented a new MCDM method for upper and lower energies based on neutrosophic soft sets.
(3) And then, developed method have been applied to a real-life decision-making problems.

Further a numerical example is developed to illustrate the flexibility and applicability of the proposed method. In further work, it is necessary and meaningful to extend the energy of a neutrosophic soft set (1) neutrosophic soft triplet, (2) Interval-valued neutrosophic soft set, and (3) Quadruple neutrosophic theory, and (4) extension of neutrosophic numbers.

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Operators for Uncertain Over/Under/Off-Sets/-Logics/-Probabilities/-Statistics

Florentin Smarandache[1]*

[1]* University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

Abstract: In this paper we propose for the first time the operators (intersection/conjunction and union/disjunction) for the Uncertain Over/Under/Off-Sets/-Logics/-Probabilities-Statistics. Practical applications of Fuzzy Over/Under/Off-Sets/-Logics/-Probabilities-Statistics are presented. Similar applications can be designed for any fuzzy-extensions Over/Under/Off-Sets/-Logics/-Probabilities-Statistics.

Keywords: Norm, OverNorm, UnderNorm, Offnorm; Conorm, OverConorm, UnderConorm, OffConorm.

1. Introduction

The Neutrosophic Set was extended for the first time by Smarandache [2-6] in the year 2007 to **Neutrosophic OverSet** (when some neutrosophic component is > 1), since we observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1; and to **Neutrosophic UnderSet** (when some neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0; and to the **Neutrosophic OffSet** (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and other neutrosophic component < 0).
Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively **Neutrosophic Over-/Under-/Off- Logic / Measure / Probability / Statistics** etc.

They resulted from practical applications when such degrees $[\varphi, \psi]$ are outside the box (outside the interval [0, 1]), or $\varphi \leq 0 < 1 \leq \psi$.

"Over" means degree > 1, "Under" means degree < 0, while "Off" means both types of degrees > 1 and < 0.

But "uncertain" sets/logics/measures/probabilities/statistics we mean, in general, all kinds of Fuzzy, and fuzzy-extensions (Neutrosophic, Plithogenic, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy, Picture Fuzzy, Ternary Fuzzy, Pythagorean Fuzzy, Atanassov's Intuitionistic Fuzzy, Fermatean Fuzzy, Spherical Fuzzy, n-HyperSpherical Neutrosophic, q-Rung Orthopair Fuzzy, Refined Neutrosophic, and so on).

In this paper we extend the t-norm to the over/under/off-norm, and similarly we extend the t-conorm to the over/under/off-conorm.

2. Annotation

Unlike the previous versions of the Uncertain Over/Under/Off-Sets/-Logics/-Probabilities papers and books [2 – 6], where the non-classical interval was denoted by [ψ, φ], we now propose to use [φ, ψ], since the letter φ is ahead of the letter ψ in the Greek alphabet.

All other ideas remain the same.

3. Classical Triangular Norms and Conorms

By classical triangular norms and conorms [1] we mean those norms and conorms whose domains and codomains are equal to [0, 1], as for example in fuzzy theories.

While, by **Over/Under/Off Triangular Norms and Conorms** [2 – 6], we mean those norms and conforms whose domains and codomains are different from [0, 1].

3.1. Definition of Classical Triangular Norm

A **triangular norm** (abbreviation **t-norm**) is a binary operation T on the interval [0, 1] satisfying the following conditions:

$T: [0,1] \times [0,1] \to [0,1]$.

For any $(x, y) \in [0,1] \times [0,1]$, one has:

$T(x, y) = T(y, x)$ (commutativity)

$T\big(x, T(y, z)\big) = T(T(x, y), z)$ (associativity)

$y \leq z \Rightarrow T(x, y) \leq T(x, z)$ (monotonicity)

$T(x, 1) = x$ (neutral element 1)

3.1.1. Examples of t-norms

$T_M(x, y) = \min(x, y)$ (minimum or Gödel t-norm)

$T_P(x, y) = x \cdot y$ (product t-norm)

$T_L(x, y) = max(x + y - 1, 0)$ (Lukasiewicz t-norm)

3.2. Definition of Classical Triangular Conorm

A dual notion to the triangular norm is the **triangular conorm** (abbreviation **t-conorm**, also **s-norm**).

$S: [0,1] \times [0,1] \to [0,1]$.

And for all $(x, y) \in [0,1] \times [0,1]$, one has:

$S(x, y) = S(y, x)$ (commutativity)

$S\big(x, S(y, z)\big) = S(S(x, y), z)$ (associativity)

$y \leq z \Rightarrow S(x, y) \leq S(x, z)$ (monotonicity)

$S(x, 0) = x$ (neutral element 0)

S has 0 (zero) as neutral element, instead of 1, but all other conditions remain unchanged.

3.2.1. Examples of t-conorms

$S_M(x, y) = max(x, y)$ (maximum, or Gödel t-conorm)

$S_P(x, y) = x + y - x \cdot y$ (product t-conorm, or probabilistic sum)

$S_L(x, y) = max(x + y, 1)$ (Lukasiewicz t-conorm, or bounded sum)

4. Over/Under/Off-Norm and Over/Under/Off-Conorm

We extend the classical triangular norm and conorm, whose domain and codomain is [0, 1], to:

— overnorm and overconorm, when their domain and codomain are $[0, \psi]$, with $\psi > 1$;

— undernorm and underconorm, when their domain and codomain are [φ, 1], with $\varphi < 0$;

— offnorm and offconorm, when their domain and codomain are [φ, ψ], where $\varphi < 0 < 1 < \psi$.

4.1. Definition of Triangular OffNorm

A **Triangular OffNorm** (abbreviation **t-offnorm**) is a binary operation
$T_{OffNorm}: [\varphi, \psi] \times [\varphi, \psi] \to [\varphi, \psi]$, $\varphi < 0 < 1 < \psi$,
so that, for all $(x, y) \in [\varphi, \psi] \times [\varphi, \psi]$ one has:
$T_{OffNorm}(x, y) = T_{OffNorm}(y, x)$, commutativity;
$T_{OffNorm}\left(x, T_{OffNorm}(y, z)\right) = T_{OffNorm}\left(T_{OffNorm}(x, y), z\right)$, associativity;
If $y \leq z$, then $T_{OffNorm}(x, y) \leq T_{OffNorm}(x, z)$, monotonicity;
$T_{OffNorm}(x, \psi) = x$, or the neutral element is ψ (the largest one).

4.1.1. Examples of Triangular OffNorm

$T_{OffNorm}(x, y) = min(x, y)$, which is the extended Gödel t-norm from the unitary interval [0, 1] to the non-unitary interval [φ, ψ], for $\varphi < 0 < 1 < \psi$.

Proof

The commutativity, associativity, and monotonicity of the *min* operator are obvious on any real interval.

While the neutral element is the largest element in the given interval, in this case the neutral element is ψ (the biggest one).

4.2. Definition of Triangular OverNorm

It is like the Triangular OffNorm, but on the interval [0, ψ], ψ>1.
$T_{OverNorm}: [0, \psi] \times [0, \psi] \to [0, \psi]$, where $\psi > 1$.
The axioms of commutativity, associativity, monotonicity, and neutral element are verified:
$T_{OverNorm}(x, y) = T_{OverNorm}(y, x)$, commutativity;
$T_{OverNorm}\left(x, T_{OverNorm}(y, z)\right) = T_{OverNorm}(T_{OverNorm}(x, y), z)$, associativity;
If $y \leq z$, then $T_{OverNorm}(x, y) \leq T_{OverNorm}(x, z)$, monotonicity;
$T_{OverNorm}(x, \psi) = x$, or the neutral element is ψ.

4.2.1. Examples of Triangular OverNorm

$T_{OverNorm}(x, y) = min\ (x, y)$

4.3. Definition of Triangular UnderNorm

It is similar to the Triangular OffNorm, but on the interval [φ, 1], $\varphi < 0$.
$T_{UnderNorm}: [\varphi, 1] \times [\varphi, 1] \to [\varphi, 1]$, where $\varphi > 0$.
For any $(x, y) \in [\varphi, 1] \times [\varphi, 1]$,
the axioms of commutativity, associativity, monotonicity, and neutral element are verified.
$T_{UnderNorm}(x, y) = T_{UnderNorm}(y, x)$, commutativity;
$T_{UnderNorm}\left(x, T_{UnderNorm}(y, z)\right) = T_{UnderNorm}(T_{UnderNorm}(x, y), z)$, associativity;
If $y \leq z$, then $T_{UnderNorm}(x, y) \leq T_{UnderNorm}(x, z)$, monotonicity;
$T_{UnderNorm}(x, 1) = x$, or the neutral element is 1.

4.3.1. Examples of Triangular UnderNorm

$T_{UnderNorm}(x, y) = min\ (x, y)$

4.4. Definition of Triangular OffConorm

Similarly to the classical Triangular Conorm, the Triangular OffConorm is the dual of the Triangular OffNorm.

$T_{OffConorm}: [\varphi, \psi] \times [\varphi, \psi] \to [\varphi, \psi]$, where $\varphi < 0 < 1 < \psi$,

and for any $(x, y) \in [\varphi, \psi] \times [\varphi, \psi]$ one has:

$S_{OffConorm}(x, y) = S_{OffConorm}(y, x)$, commutativity;

$S_{OffConorm}\left(x, S_{OffConorm}(y, z)\right) = S_{OffConorm}\left(S_{OffConorm}(x, y), z\right)$, associativity;

If $y \leq z$, then $S_{OffConorm}(x, y) \leq S_{OffConorm}(x, z)$, monotonicity;

$S_{OffConorm}(x, \varphi) = x$, the neutral element is the smallest element φ.

4.4.1. Examples of Triangular OffConorm

$S_{OffConorm}(x, y) = max\ (x, y)$, which is another extended Gödel t-conorm.

Commutativity, associativity, and monotonicity of the max operator are valid on any real interval.

But as neutral element one has now φ (the smallest element of the real interval $[\varphi, \psi]$, instead of 0 (zero) as in the classical S_{Conorm}.

4.5. Definition of Triangular OverConorm

It is like the Triangular OffConorm, but on the interval $[0, \psi]$, $\psi > 1$.

$T_{OverConorm}: [0, \psi] \times [0, \psi] \to [0, \psi]$, where $\psi > 1$.

and for any $(x, y) \in [0, \psi] \times [0, \psi]$,

the axioms of commutativity, associativity, monotonicity, and neutral element are verified:

$S_{OverConorm}(x, y) = S_{OverConorm}(y, x)$, commutativity

$S_{OverConorm}\left(x, S_{OverConorm}(y, z)\right) = S_{OverConorm}(S_{OverConorm}(x, y), z)$, associativity;

If $y \leq z$, then $S_{OverConorm}(x, y) \leq S_{OverConorm}(x, z)$, monotonicity;

$S_{OverConorm}(x, 0) = x$, or the neutral element is 0 (zero).

4.5.1. Example of Triangular OverConorm

$S_{OverConorm}(x, y) = max\ (x, y)$

4.6. Definition of Triangular UnderConorm

It is alike the Triangular OffConorm, but on the interval $[\varphi, 1]$, $\varphi < 0$.

$T_{UnderConorm}: [\varphi, 1] \times [\varphi, 1] \to [\varphi, 1]$, $\varphi > 0$

and for any $(x, y) \in [\varphi, 1] \times [\varphi, 1]$,

the same axioms are verified:

$S_{UnderConorm}(x, y) = S_{UnderConorm}(y, x)$, commutativity;

$S_{UnderConorm}\left(x, S_{UnderConorm}(y, z)\right) = S_{UnderConorm}(S_{UnderConorm}(x, y), z)$, associativity;

If $y \leq z$, then $S_{UnderConorm}(x, y) \leq S_{UnderConorm}(x, z)$, monotonicity;

$S_{UnderConorm}(x, \varphi) = x$, or the neutral element is φ (the smallest one).

4.6.1. Example of Triangular UnderConorm

$S_{UnderConorm}(x, y) = max(x, y)$

5. Practical Applications of OffNorm and OffConorm

5.1. Application of the Fuzzy OverSet

In a math class, the professor gives a test to the students with 10 mandatory questions. For each question answered correctly, 10 points are awarded.

However, considering that there are several very gifted students who participate in regional Mathematics Olympiads, the teacher adds an additional difficult question, which is optional.

Correctly solving the first ten mathematical problems, a student gets $10 \times 10 = 100$ points, equivalent to the grade A.

We recall that the grades are the following, in ascending order:

F < D < C < B < A,

or

{F, D, C, B, A}

They mean, failing if a student gets F or D, and passing if he gets C, B, or A.

That are respectively corresponding numerically approximately to:

{0, 0.25, 0.50, 0.75, 1}, or F(0), D(0.25), C(0.50), B(0.75), A(1),

where, of course, 0 = 0% meaning 0 points out of 100 points, 0.25 = 25% or 25 points out of 100 points, etc.

But for students who also solve the 11th problem, the grade A^+ is awarded, which is higher than A, since:

11 questions × 10 points = 110 points = A^+, which is > 100 points = A.

110 points out 100 required points is 110/100 =1.1 degree corresponding to A^+.

Therefore, the grading scale was extended from the classical unit interval [0, 1] to the non-unit interval [0, 1.1].

Therefore, the degree 1.1 of A^+ is bigger than the interval [0, 1], that's why we have an OverSet.

Application of the Fuzzy OverLogic

Let be the proposition:

P = "The student Marcella gets a passing grade."

What may it be the OverTruth-value of this proposition?

Passing means getting a score between 50% and 110%, or in between [0.5, 1.1].

Thus, the OverTruth-value is the whole interval [0.5, 1.1], while in the classical logic the truth-value is smaller, in the whole interval [0.5, 1].

Application of the Fuzzy OverProbability

What is the OverProbability that Marcella passes the test (getting A^+ is included)?

Another one: what is the OverProbability that a student gets an A^+ (degree > 1)?

Passing means getting a score between 50% and 110%, or in between [0.5, 1.1] out of [0, 1.1], hence: $\frac{1.1-0.5}{1.1-0} = \frac{0.6}{1.1} \approx 0.55$.

It is worth noting that in the classic case (no-OverProbability, or no A^+ on the grading scale), when the domain and codomain are just [0, 1], the classical probability result is in general smaller:

$$\frac{1-0.5}{1-0} = \frac{0.5}{1} = 0.50.$$

Application of the Fuzzy OverStatistics

The OverStatistics, in addition to the classical statistics, takes into account the OverProbability events as well. For example, the number of students that get over 100% to the test.

5.1.1. Numerical Fuzzy Examples

$$T_{OverNorm}:[0,1.1]\times[0,1.1]\to[0,1.1]$$

$$T_{OverNorm}(0.4,1.1)=\min\{0.4,1.1\}=0.4$$

$$S_{OverConorm}:[0,1.1]\times[0,1.1]\to[0,1.1]$$

$$S_{OverConorm}(0.4,1.1)=\max\{0.4,1.1\}=1.1$$

5.2. Application of the Fuzzy UnderSet

In an entrance exam at the University X, students must pass several tests. On each test, they score between 0% - 100% points, or a degree between [0, 1]. The lowest value is 0 (zero) point. Even if a student gets a zero on a test, he is not eliminated, he still has a chance of being admitted if he gets higher marks on the other tests and, of course, other students get lower marks.

Student Sean is caught *cheating* and is then *expelled* from the competition, therefore he has no more chance. So, the verdict of "expelled from the competition" should have a degree less than zero.

Therefore, we now deal with the subset $\varphi\cup[0,1]$, where $\varphi<0$, φ represents the degree of being expelled. As a numerical example, $\varphi=-1$, and the subset is $\{-1\}\cup[0,1]$.

$$T_{UnderNorm}:(\{-1\}\cup[0,1])\times(\{-1\}\cup[0,1])\to(\{-1\}\cup[0,1])$$

$$S_{UnderNorm}:(\{-1\}\cup[0,1])\times(\{-1\}\cup[0,1])\to(\{-1\}\cup[0,1])$$

Application of the Fuzzy UnderLogic

Similarly, let be the proposition:

P = "The student Marcella gets a passing grade."

What may it be the Fuzzy UnderTruth-value of this proposition?

Passing means getting a score between 50% and 100%, or in between [0.5, 1].

Thus, the Fuzzy OverTruth-value is the interval [0.5, 1], the same as in the classical logic.

Application of the Fuzzy UnderProbability

In the classical probability, what is the probability that the student John gets 23 points out of 100 total points? It is: one favorite possibility (23 points), out of one hundred and one total possible possibilities:

{0, 1, 2, 3, 4, 5, …, 23, 24, …, 100}

i.e. $\frac{1}{101}=0.00990099...$

In the Fuzzy UnderProbability, the probability that the student John gets 23 points out of 100 points is:

one favorite possibility (23 points), but out of one hundred two total possible possibilities:

{-1, 0, 1, 2, 3, 4, 5, …, 23, 24, …, 100}

i.e. $\frac{1}{102}=0.00980392...$

which is smaller than the classical probability.

What is the Fuzzy UnderProbability that the student Raymond gets zero points, or is expelled from the competition (i.e. the student gets the degree -1)? Two favorable cases out of 102 total possible cases:

it is: $\frac{2}{102}=0.01960784...$

Application of the Fuzzy UnderStatistics

The Fuzzy UnderStatistics, in addition to the classical statistics, considers the UnderProbability events as well. For example, the number of students that get below 0% points (expelled) from the test, etc.

5.3. Application of the Fuzzy OffSet

We retake the previous practical examples and combine them:

In a math class, the professor gives a test to the students with 10 mandatory questions. For each question answered correctly, 10 points are awarded.

However, considering that there are several very gifted students who participate in regional Mathematical Olympiads, the examiners add an additional difficult question, which is optional, and assign 10 points as well for correctly solving it.

If a student is caught cheating, he is expelled from the school. Similarly, expelled means less than 0 (zero) points.

The domain and range are now, for example, [-1.2, 1.1], where the degree -1.2 means expelled and the degree 1.1 means having solved the 11th bonus problem as well.

The lower (φ) and upper (ψ) margins of the Fuzzy OffSet/OffLogic/OffProbability/OffStatistics domain and codomain [φ, ψ] are established by the experts according to the application where they are used in.

Application of the Fuzzy OffLogic

Find the Fuzzy OffTruth-value of the proposition, P = "Miguel does get an A+".

It means, Miguel gets between (1.0, 1.1] to the test, out of [-1.2, 1.1],

or $$\frac{1.1-1.0}{1.1-(-1.2)}=\frac{0.1}{2.3}=0.043...$$

Application of the Fuzzy OffProbability

What is the Fuzzy OffProbability that Marcella gets expelled?

It means, Marcella gets between [-1.2, 0) to the test, out of [-1.2, 1.1],

$$\frac{|0-(-1.2)|}{1.1-(-1.2)}=\frac{1.2}{2.3}=0.521...$$

Application of the Fuzzy OffStatistics

The Fuzzy OffStatistics, in addition to the classical statistics, consider the Fuzzy OffProbability events as well. For example, the statistics about the students that get over 100% or less than 0% to the test.

6. Future Research

Better designs are need for the Fuzzy (and other fuzzy-extensions) Over/Under/Off-Norms and Over/Under/Off-Conorms, possibly using continuous functions, or extensions of the classical Product t-norm / t-conorm and Lukasiewicz t-norm/t-conorm, satisfying of course the required axioms (commutativity, associativity, monotonicity, and the existence of a neutral element). And introducing more operators, for example: complement/negation, inclusion/implication, and equality/equivalence.

The readers are welcome to contribute to developing this new field of study about fuzzy/intuitionistic_fuzzy/neutrosophic/plithogenic and other fuzzy-extensions membership/truth, nonmembership/falsehood, indeterminacy/neutrality etc. degrees that are outside the interval [0, 1].

Practical applications of *Fuzzy* Over/Under/Off-Sets/-Logics/-Probability/-Statistics have been presented in this paper, while applications for the *Neutrosophic* Over/Under/Off-Sets/-Logics/-Probability/-Statistics have been previously published in the years 2007 - 2016 in articles and books [2-6].

Similar applications can be designed for any fuzzy-extension Over/Under/Off-Sets/-Logics/-Probabilities-Statistics and the readers are invited to contribute them.

7. Conclusions

For the first time we introduced now the operators (intersection, union) of the Fuzzy Off/Over/Under Sets/Logics and used them in real life applications.

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Neutrosophic Statistical Interval Methods for Reliability Analysis: A Failure Censored LifeTest with Repetitive Group Sampling Plan

S. Jayalakshmi [1], S. Vijilamery[2]

[1]Assistant Professor ,[2] Research Scholar;

[12],Department of statistics, Bharathiar University, Coimbatore, TamilNadu,India

[1] statjayalakshmi16@gmail.com,[2] vijilamerysrsj@gmail.com

*Correspondence: statjayalakshmi16@gmail.com;

Abstract. The Reliability Sampling Plans based on Failure-Censored data assume that all failure data is reliable, clear, and definitive, as noted in the literature.If the failure data measurements are unclear, indistinct, and unsure in a practical sense, we opt to adopt a new form of reliability theory known as Neutrosophic Reliability Theory. In this paper deals with Repetitive Group Sampling Plan based on failure-censored life tests using the Neutrosophic Statistical Interval Method. This plan assumes that failure times follow a Neutrosophic Generalized Exponential Distribution and also consider the Neutrosophic plan parameters for a given producer's risk and a given consumer's risk. We present the tables and results for various levels of Neutrosophic plan parameterization. The numerical illustrations of real data study provide further clarity, making it easier to interpret the concept.

Keywords: Neutrosophic Generalized Exponential Distribution; Failure Censored Reliability Lifetest; Acceptable Quality Level; Limiting Quality Level

1. **Introduction**

Quality monitoring and control of products are carried out through Statistical Quality Control (SQC). It is used to ensure that product quality is met or exceeded by measuring and analysing data gathered during manufacturing or service delivery. Statistical Quality Control can be divided into two major areas: one is process control and another one is product control.

- Process control, Process requirements and performance changes may be detected using them over time for the monitoring of manufacturing processes.
- Product control techniques focus on testing the end product for defects. It is also called acceptance sampling or sampling inspection, are used to draw inferences about how lots of manufactured goods are being handled, including acceptance or rejection of an individual lot or group of lots, based on specified rules.

Industrial sectors place a high priority on product reliability and consistency of performance. Using a Reliability Sampling Plan, We can ensure the product meets standards based on the Lifetime of products. When there is a certain number of failures that are observed during a life test, Time-censoring is used to suppress this number of failures followed by a predetermined period of time. Time-censoring involves terminating the test after a predetermined period of time.In failure-censoring, which is also referred as Type II censoring, the test is terminated once a certain number of failures have occurred or after a predetermined number of test hours have been completed. Both types of censoring help reduce the cost of testing and minimize the testing time.Schneider(1989) [1] developed prediction models using failure-censored variables. Seo and Yum (1993) [2]developed a method of failure censoring life test failures in exponential distributions. A failure-censored reliability sampling plan can be developed when items should be tested in sets of a fixed size, according to Balasooriya (1995) [3].It assumes that all failure data are determinant. In real life situation, failure data measurement is unclear, uncertainty and indeterminate. Neutrosophic Statistics are useful in such situations and uncertainty or indeterminacy associated with data may be handled more effectively in a flexible manner. Smarandache(2014) [4] introduced the concept of Neutrosophic statistics. Using Neutrosophic statistics, Aslam and Arif(2018) [5] have been able to test grouped products for the Weibull distribution. Neutrosophic statistics developed by Aslam and Raza(2019) [6] are intended to sample the loss function of a process.Utilizing Neutrosophic statistics, Aslam (2019) [7] introduced a novel method based on Neutrosophic statistics to assess roughness coefficients of rock joints. Chen (2017) [8] presented a technique for representing a rock joint's roughness coefficient via the neutrosophic interval. Aslam (2018) [9] investigated the Neutrosophic statistical interval methodology. Gupta and Kundu (2007) [10] formulated the Generalized Exponential Distribution.

According to Aslam(2019) [11], Neutrosophic statistical intervals are used to develop acceptance sampling plans. Isik and Kaya(2022) [12] has designed Single and Double Sampling Plan based on Neutrosophic Statistics.A method for Sudden Death Testing utilizing Repetitive Sampling within the Neutrosophic Statistical Interval Method was created by Aslam et al. (2020). [13].

This paper deals with Repetitive Group sampling plan based on Failure Censored Reliabilty

life test using Neutrosophic Statistical Interval method.Alhasan(2021) [14] introduced a concept of Neutrosophic Reliability Theory. Rao et.al(2023) [15] have developed a Neutrosophic Generalized Exponential Distribution. Sherman(1965) [16]has introduced the Sampling Plan of Repetitive Sampling Group Plan. It improves resource management by enabling more effective resource allocation. Additionally, it aids in decreasing bias during the sampling process by inspecting the items multiple times.

Based on literature review using single sampling plans and failure censored lifetest in Neutrosophic Statistics, we did not came across any studies on the Repetitive Group sampling plan using Neutrosophic Statistical Interval method for failure censored Reliability lifetest. In this article, we will explain how the plan was created utilizing Repetitive Group sampling plan within the neutrosophic statistical interval method. The purpose of this study is used to optimize the neutrosophic plan parameters of the proposed sampling plan for the two specified points: one is Acceptable quality level (AQL) and other one is limiting quality level (LQL) levels. The tables and results are used to explain the data taken from the industrial sectors. In this method, various applications are used like in engineering, economics, medical and other fields. Aslam (2019) [17] has developed a new failure-censored reliability test for Neutrosophic Weibull distributions. In a single sampling plan with failure-censored reliability, inspection costs are high, and we need to remove a large number of test units.

The proposed study aims to minimize the degree of censoring so that we can reduce the number of tests removed as much as possible while minimizing the amount of time and cost involved in inspections. According to Jun et al. (2010) [18], a repetitive group sampling plan under a failure-censored reliability life test can result in a substantial reduction in failures when compared with single sampling plans. Therefore, we develop the Repetitive Group Sampling Plan with a Failure-Censored Reliability Life test using the Neutrosophic Statistical Interval Method. For the proposed study, the life test follows a Neutrosophic Generalized Exponential Distribution.

Using these results, the proposed study can optimize the sampling size, plan parameters, and minimize inspection time and costs.Finally, we compare the results from the proposed sampling plan to those under classical statistics to calculate its effectiveness.

2. **Nomenclature/Notation**

$x_i \rightarrow$ random sample

$\theta \rightarrow$ Scale parameter of Generalized Exponential Distribution

$\delta \rightarrow$ Shape parameter of Generalized Exponential Distribution

$\theta_N \rightarrow$ Neutrosophic Scale parameter of Generalized Exponential Distribution

$\delta_N \rightarrow$ Neutrosophic Shape parameter of Generalized Exponential Distribution

$x_{N_i} \rightarrow$ Neutrosophic random sample

$T_{N_i} \rightarrow$ Neutrosophic random sample of time

$L \rightarrow$ Lowest Specification Limit

$n_N \rightarrow$ Neutrosophic random sample size

$r_N \rightarrow$ performs test until occurs failures

$k_{a_N} \rightarrow$ Neutrosophic acceptance region

$k_{r_N} \rightarrow$ Neutrosophic rejection region

$(ASN)_N \rightarrow$ Neutrosophic Average Sample Number

$AQL \rightarrow$ Acceptable Quality Level

$LQL \rightarrow$ Limiting Quality Level

$\alpha \rightarrow$ Producer's risk

$\beta \rightarrow$ Consumer's risk

$v_N \rightarrow$ Neutrosophic Quantity Statistic

3. **Basic Terminologies**

Definition 3.1. Neutrosophic Statistics

Neutrosophic Statistics pertains to a collection of data where the data or a portion of it possesses some level of indeterminacy, along with techniques employed to examine the data. In Classical Statistics, all data are fixed; this differentiates neutrosophic statistics from classical statistics.

Definition 3.2. Classical Reliability

Reliability function: Let T denote the lifetime of a system, the reliability of that system at the point in time t , that $R(T) = (T > t)$, it is called the reliability at the time t , and we can define it as the probability that the time at which the system could fail is greater than t. The reliability by cumulative distribution function for a random variable as:

$$R(t) = \int_t^{\infty} f(t)dt = 1 - P(\mathcal{T} > t) = 1 - F(t). \tag{1}$$

Definition 3.3. Neutrosophic Reliability

Let T denote a Neutrosophic random variable that signifies the time of system failure, while t represents the duration of this system's operation. We established reliability based on Neutrosophic probability in the following manner:

$$NR(t) = \int_t^{\infty} f_N(t)dt = NP(\mathcal{T} > t), \tag{2}$$

where there can exist an interval, set, or Neutrosophic number that may have some indeterminacy, and t represents the Neutrosophic probability distribution. Then, $f_N(t)$ represents the

Neutrosophic reliability concerning a Neutrosophic probability distribution. The Neutrosophic reliability by cumulative distribution function for a random variable as:

$$R_N(t) = \int_t^{\infty} f_N(t)dt = 1 - P_N(\mathcal{T}_{\mathcal{N}} > t) = 1 - F_N(t). \tag{3}$$

Definition 3.4. Failure Censored Reliability Life Test
The Failure Censored Reliability test involves placing the items on the test simultaneously and continuing the test until a specified number of failures are observed.

4. **Life Time Distribution**

4.1. *Generalized Exponential Distribution*

Probabilistic statistics concerns the distribution of time intervals between events in a continuous, independent Poisson process occurring at a steady average rate, referred to as the exponential distribution. We select a random sample of n items from a generalized exponential distribution , then arrange the samples in ascending order, beginning with the smallest observation, before testing the experiment. A life test experiment is potentially concluded when all r failures have been observed recorded after the initial failures. The primary advantages of possibly halting before completing all n observations arise from the observations occurring in a particular organizing and reaching a conclusion more quickly or with fewer observations than a method that requires examining all the items under evaluation. Consequently, this study concentrates on failure-censored data, where in n items undergo testing, and the experiment concludes when a specified number of items ,like $r(< n)$, have failed.

Let x_j, where j=1,2...n, be a continuous random variable that adheres to the Generalized Exponential distribution, characterized by the shape parameter δ and the scale parameter θ. The Cumulative Distribution Function of the Generalized Exponential distribution is

$$F(x) = \left(1 - \exp\left\{-\frac{x}{\theta}\right\}\right)^{\delta} \text{x} > 0, \delta > 0, v > 0 \tag{4}$$

The Probability Density Function of Generalized Exponential distribution is,

$$f(x) = \frac{\delta}{\theta}\left(1 - e^{-x/\theta}\right)^{\delta-1} \cdot e^{-x/\theta}, \text{x} > 0, \delta > 0, v > 0 \tag{5}$$

The information gained from this kind of testing is crucial when working with expensive, advanced products. To provide a more precise depiction of time until failure, the Generalized Exponential Distribution is utilized in place of the frequently used exponential distribution. While including the Generalized Exponential Distribution in life testing modeling makes modeling and estimation more complicated, it provides a more precise fit to life data compared to the exponential distribution because of its flexibility.

4.2. *Neutrosophic Generalized Exponential Distribution*

The Neutrosophic Generalized Exponential Distribution is defined as the following: If $x_{N_i} \in [x_L, x_U] = i = 1; 2; 3; ...; n_N$ is a random sample whose distribution follows the Neutrosophic Generalized Exponential Distribution with Neutrosophic scale parameters $\theta_N \in [\theta_L, \theta_U]$ and a Neutrosophic shape parameter $\delta_N \in [\delta_L, \delta_U]$. The Neutrosophic Fuzzy Generalized Exponential Distribution with the Probability Density function is defined as follows:

$$\begin{aligned} f(x_N) = \frac{\delta_N}{\theta_N}\left(1 - e^{-x_N/\theta_N}\right)^{\delta_N - 1} e^{-x_N/\theta_N}, \\ \theta_N > 0, \theta_N \in \{\theta_L, \theta_U\}, \delta_N \in \{\delta_L, \delta_U\} \end{aligned} \tag{6}$$

The Neutrosophic Cumulative Distribution Function is

$$F(x_N) = \left(1 - e^{-x_N/\theta_N}\right)^{\delta_N} \tag{7}$$

Neutrosophic Survival function:

$$S(x_N) = 1 - \left(1 - e^{-x_N/\theta_N}\right)^{\delta_N} \tag{8}$$

Neutrosophic Hazard function:

$$h(x_N) = \frac{\frac{\delta_N}{\theta_N}\left(1 - e^{-x_N/\theta_N}\right)^{\delta_N - 1} e^{-x_N/\theta_N}}{1 - \left(1 - e^{-x_N/\theta_N}\right)^{\delta_N}} \tag{9}$$

5. **Design of the Proposed Plan**

Let us assume that $T_{Ni} \in \{T_L, T_U\} = i = 1; 2; 3; ...; n_N$ be a random sample follows the Neutrosophic Generalized exponential distribution having a Neutrosophic scale parameter $\theta_N \in \{\theta_L, \theta_U\}$ and Neutrosophic shape parameter $\delta_N \in \{\delta_L, \delta_U\}$, then following the Neutrosophic Probability Density Function is

$$\begin{aligned} f(T_N) = \frac{\delta_N}{\theta_N}\left(1 - e^{-t_N/\theta_N}\right)^{\delta_N - 1} e^{-t_N/\theta_N}, \\ \theta_N > 0, \theta_N \in \{\theta_L, \theta_U\}, \delta_N \in \{\delta_L, \delta_U\} \end{aligned} \tag{10}$$

The Neutrosophic Cumulative Distribution function is

$$F(T_N) = \left(1 - e^{-t_N/\theta_N}\right)^{\delta_N} \tag{11}$$

When L is the lowest specification limit, then a product outside of this range is considered defective. As a result, item at time L is unreliable, as provided by

$$P_N = p_{r_N}(T_N < L) = F_N(L) \tag{12}$$

Then the following equation 11, this formula can be used to calculate the fraction of nonconforming units before time L:

$$P_N = 1 - \left(1 - e^{-L/\theta_N}\right)^{\delta_N} \tag{13}$$

From equation 13, we have

$$w_N = -\ln(1 - P_N) = (L/\theta_N)^{\delta_N}, \quad \theta_N \in \{\theta_L, \theta_U\} \tag{14}$$

6. **Operating Procedure of Proposed Plan**

We suggest the following sampling plan under the Neutrosophic Statistical Interval Method based upon the information above.

- Step 1: Randomly take a size of sample $n_N \in [n_L, n_U]$ from a lot.
- Step 2: If we have $n_N \in [n_L, n_U]$ items for inspection and carry out tests up to r_N, where $n_N \in [n_L, n_U]$ and $r_N \in [r_L, r_U]$, the outcome is r_N, with $r_N \in \{r_L, r_U\}$.
- Step 3: Let X_{Nj} be the j^{th} failure time ($j = 1, 2, 3, \ldots, r_N$). Calculate the statistic value

$$v_N = \sum_{i=1}^{r_N} \left(X_{N(j)}\right)^{\delta_N} + (n_N - r_N)\left(X_{N(j)}\right)^{\delta_N}; \tag{15}$$

$$v_N \in \{v_L, v_U\}.$$

- Step 4: The lot will be accepted if $v_N < k_{a_N} L^{(\delta_N)}$; $v_N < k_{r_N} L^{(\delta_N)}$ where $k_{a_N} \in [k_{a_L}, k_{a_U}]$ is a Neutrosophic Fuzzy acceptance number and the lot will be rejected if $v_N \geq k_{a_N} L^{(\delta_N)}$; $v_N \geq k_{r_N} L^{(\delta_N)}$ where $k_{aN} \in [k_{aL}, k_{aU}]$, $k_{a_N} \in [k_{rL}, k_{rU}]$, is a Neutrosophic Fuzzy rejection number. Otherwise, the procedure will move to Step 1.

To implement the proposed plan, the values of Neutrosophic Fuzzy parameters $r_N \in [r_L, r_U]$, $k_{a_N} \in [k_{a_L}, k_{a_U}]$ and $k_{r_N} \in [k_{r_L}, k_{r_U}]$ should be known. Based on this assumption,

$$v_N = \sum_{i=1}^{r_N} \left(X_{N(j)}\right)^{\delta_N} + (n_N - r_N)\left(X_{N(j)}\right)^{\delta_N};$$

$$v_N \in \{v_L, v_U\}$$

has a Neutrosophic Chi-square distribution associated with failure censored reliability v_N with a Neutrosophic degree of freedom $2r_N$.Using Chi-square distribution, we can calculate Neutrosophic Fuzzy Operating Characteristics for a Single Sampling Plan as follows:

$$P_a(p_N) = 1 - G_{2r_N}(2k_{a_N} w_N); k_{a_N} \in [k_{a_L}, k_{a_U}] \tag{16}$$

The probability of rejecting a lot under the Neutrosophic Statistical Interval Method is calculated as follows:

$$P_r(p_N) = G_{2r_N}(2k_{rN} w_N); k_{r_N} \in [k_{r_L}, k_{r_U}] \tag{17}$$

The Neutrosophic Operating Characteristic of Repetitive Group Sampling Plan is derived for the proposed sampling plan and given as follows:

$$L(P_N) = \frac{P_a(p_N)}{P_a(p_N) + P_r(p_N)} \tag{18}$$

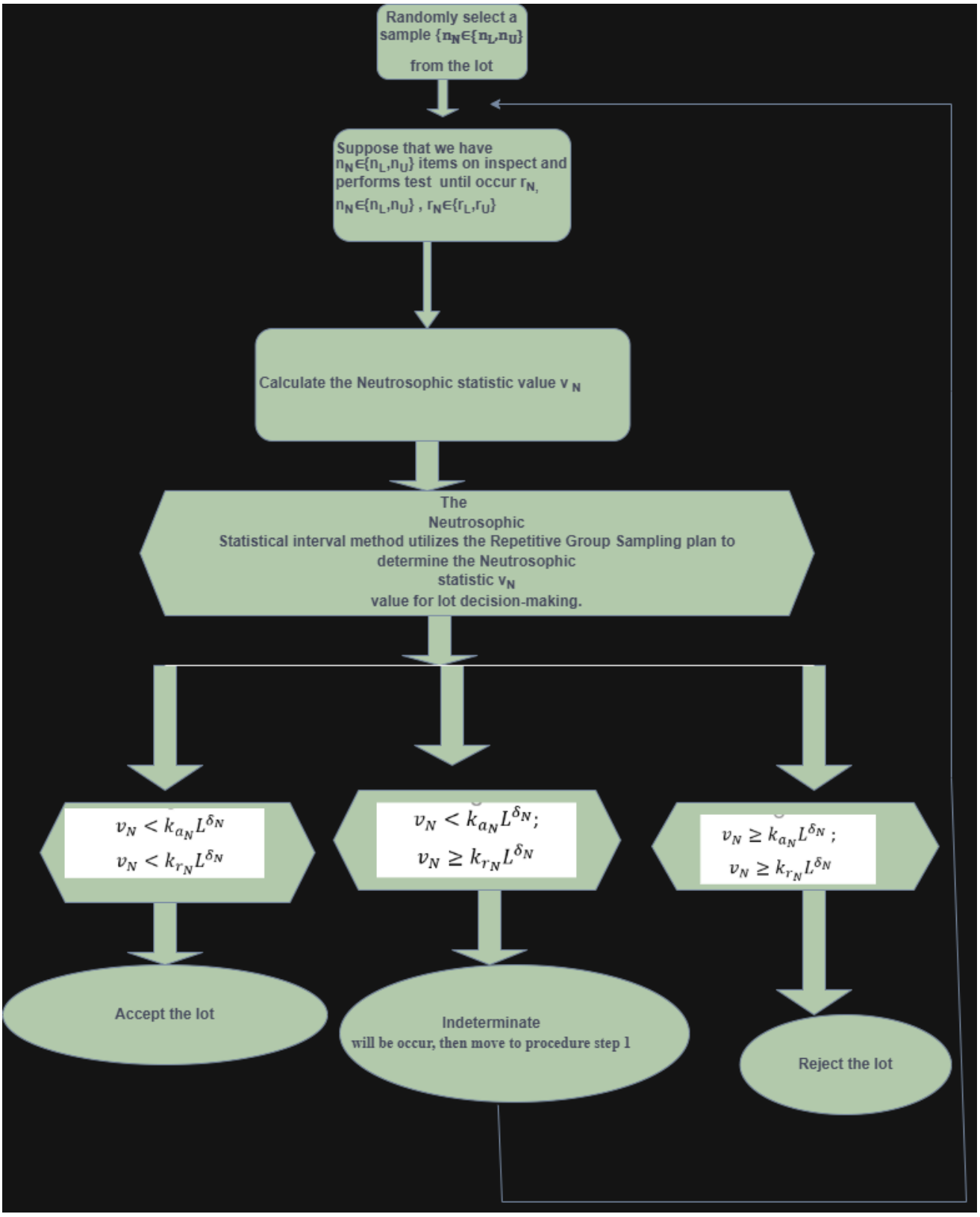
Randomly select a sample $\{n_N \in \{n_L, n_U\}$ from the lot
Suppose that we have $n_N \in \{n_L, n_U\}$ items on inspect and performs test until occur r_N, $n_N \in \{n_L, n_U\}$, $r_N \in \{r_L, r_U\}$
Calculate the Neutrosophic statistic value v_N
The Neutrosophic Statistical interval method utilizes the Repetitive Group Sampling plan to determine the Neutrosophic statistic v_N value for lot decision-making.
$v_N < k_{a_N} L^{\delta_N}$
$v_N < k_{r_N} L^{\delta_N}$
$v_N < k_{a_N} L^{\delta_N}$;
$v_N \geq k_{r_N} L^{\delta_N}$
$v_N \geq k_{a_N} L^{\delta_N}$;
$v_N \geq k_{r_N} L^{\delta_N}$
Accept the lot
Indeterminate will be occur, then move to procedure step 1
Reject the lot

FIGURE 1. Flow chart for proposed plan

$$L\left(P_N\right)=\frac{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)}{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)+G_{2r_N}\left(2k_{r_N}w_N\right)} \tag{19}$$

According to the suggested plan, the Neutrosophic Average Sample Number (NASN) will be calculated as follows:

$$ASN_N\left(P_N\right)=\frac{n_N}{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)+G_{2r_N}\left(2k_{r_N}w_N\right)} \tag{20}$$

The Neutrosophic plan parameters are α,β,AQL, and LQL respectively. It comprises the producer's risk, the consumer's risk, and the acceptable and limiting quality levels. A Neutrosophic fuzzy non-linear optimization grid search technique will be employed using the Neutrosophic Statistical Interval Method to establish the parameters for the suggested Failure Censored Life test plan:

$$\begin{gathered} Minimize ASN_N \in [ASN_L, ASN_U] \\ L\left(C_{AQL}\right)= \\ \frac{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)}{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)+G_{2r_N}\left(2k_{r_N}w_N\right)} \leq 1-\alpha \\ L\left(C_{LQL}\right)= \\ \frac{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)}{\left(1-G_{2r_N}\left(2k_{a_N}w_N\right)\right)+G_{2r_N}\left(2k_{r_N}w_N\right)} \geq \beta \end{gathered} \tag{21}$$

7. **Algorithm**

Failure Censored Lifetest based on Neutrosophic Statistical Interval model designing :

Step 1. Choose the random sample from the given lot

Step 2. Inspect the items based on the proposed sampling plan and continue the test until occurs failures of inspection time of product.

Step 3. Compute the Failure Censored Reliability lifetest based on Neutrosophic quantity statistic v_N and Make a decision of accept and reject the product.

A non-linear optimization procedure under Neutrosophic Statistical Interval Method determines the plan parameters $k_{a_N}\in[k_{a_L},k_{a_U}]$, $k_{r_N}\in[k_{r_L},k_{r_U}]$ and $r_N\in[r_L,r_U]$.

It gives the multiple combinations that satisfy the constraints provided during the simulation process. The Neutrosophic plan parameter combinations with the minimum of Neutrosophic Average Sample Number are chosen among all of these combinations. The following tables1,2,3 shows the various Neutrosophic parametric values obtained based on $(\alpha=0.10,\beta=0.10),(\alpha=0.05,\beta=0.05)$ and $(\alpha=0.05,\beta=0.10)$ From the given table results are,

(1) For the fixed values of Neutrosophic parameters, $r_N\in r_L,r_U$ and $k_{a_N}\in\{k_{a_L},k_{a_U}\}$,$k_{r_N}\in\{k_{r_L},k_{r_U}\}$ decreases.

(2) For the fixed values of Neutrosophic parameters, $r_N\in\{r_L,r_U\}$ and $k_{a_N}\in\{k_{a_L},k_{a_U}\}$,$k_{r_N}\in\{k_{r_L},k_{r_U}\}$ decreases as LQL increases.

TABLE 1. Proposed Sampling plan parameter when α=0.10,β=0.10

AQL	LQL	r_N	ASN_N	k_{a_N}	k_{r_N}	$L(C_{AQL})$	$L(C_{(LQL)})$
0.01	0.035	[3,4]	[4.00,4.42]	[200.9,221.5]	[149.39,167.39]	[0.89,0.89]	[0.03,0.05]
0.01	0.05	[2,3]	[2.55,3.13]	[108.15,127.53]	[45.68,107.76]	[0.90,0.89]	[0.03,0.04]
0.01	0.1	[1,2]	[1.24,2.06]	[37.59,61.87]	[12.45,61.87]	[0.90,0.90]	[0.02.0.01]
0.01	0.2	[1,2]	[1.01,2.05]	[12.05,45.6]	[9.94,34.98]	[0.90,0.94]	[0.07,0.0004]
0.01	0.3	[1,2]	[1.00,2.00]	[6.86,11.14]	[8.98,9.08]	[0.91,0.99]	[0.08,0.100]
0.02	0.06	[4,6]	[4.87,6.05]	[129.33,147.56]	[80.19,142.63]	[0.90,0.92]	[0.05,0.10]
0.02	0.09	[3,4]	[3.23,5.03]	[69.56,119.56]	[52.85,117.14]	[0.89,0.90]	[0.04,0.01]
0.02	0.1	[2,3]	[2.43,4.10]	[49.45,92.02]	[23.34,92.02]	[0.90,0.90]	[0.04,0.01]
0.02	0.2	[1,2]	[1.27,3.00]	[17.30,30.44]	[4.04,28.9]	[0.89,0.90]	[0.03,0.03]
0.02	0.5	[1,2]	[1.00,2.00]	[4.27,5.97]	[2.04,3.76]	[0.90,0.97]	[0.05,0.10]
0.05	0.15	[5,7]	[5.33,7.83]	[56.42,91.69]	[46.28,73.76]	[0.89,0.89]	[0.05,0.008]
0.05	0.2	[4,6]	[4.03,6.09]	[35.22,53.98]	[33.23,49.88]	[0.90,0.95]	[0.04,0.02]
0.05	0.25	[3,5]	[3.30,4.99]	[29.40,30.01]	[20.54,28.21]	[0.89,0.98]	[0.01,0.07]
0.05	0.35	[2,4]	[2.03,3.96]	[11.66,23.34]	[10.47,18.76]	[0.89,0.98]	[0.04,0.01]
0.05	0.5	[1,3]	[1.05,3.00]	[3.60,7.75]	[1.92,7.03]	[0.90,0.99]	[0.10,0.10]

TABLE 2. Proposed Sampling plan parameters when α=0.05,β=0.05

AQL	LQL	r_N	ASN_N	k_{a_N}	k_{r_N}	$L(C_{AQL})$	$L(C_{(LQL)})$
0.01	0.035	[4,6]	[4.79,6.27]	[232.14,296.80]	[167.77,243.95]	[0.95,0.95]	[0.05,0.05]
0.01	0.05	[3,5]	[4.79,6.27]	[127.69,178.87]	[79.44,197.79]	[0.95,0.96]	[0.05,0.04]
0.01	0.1	[2,3]	[2.07,4.00]	[46.10,94.64]	[32.84,91.6]	[0.95,0.98]	[0.05,0.01]
0.01	0.2	[1,3]	[1.11,3.00]	[15.39,30.64]	[4.23,13.98]	[0.95,0.99]	[0.05,0.05]
0.01	0.3	[1,2]	[1.04,2.00]	[8.97,13.12]	[4.23,9.89]	[0.95,0.99]	[0.04,0.05]
0.02	0.06	[6,8]	[6.61,8.30]	[174.04,215.07]	[130.69,180.73]	[0.95,0.96]	[0.05,0.04]
0.02	0.09	[5,6]	[5.04,5.92]	[95.43,109.20]	[91.29,122.41]	[0.95,0.96]	[0.05,0.04]
0.02	0.1	[4,5]	[4.05,5.02]	[72.93,86.72]	[68.45,72.57]	[0.95,0.97]	[0.04,0.04]
0.02	0.2	[2,4]	[2.03,4.01]	[21.43,34.44]	[17.83,27.9]	[0.95,0.99]	[0.05,0.04]
0.02	0.5	[1,2]	[1.03,2.01]	[4.49,6.79]	[2.55,3.56]	[0.95,0.96]	[0.05,0.05]
0.05	0.15	[7,9]	[7.24,9.05]	[72.93,87.69]	[64.94,86.08]	[0.94,0.99]	[0.05,0.04]
0.05	0.2	[6,7]	[6.03,7.03]	[45.91,51.78]	[44.88,49.78]	[0.94,0.96]	[0.05,0.05]
0.05	0.25	[5,6]	[4.91,6.05]	[29.5,35.29]	[23.06,29.21]	[0.95,0.99]	[0.05,0.05]
0.05	0.35	[3,5]	[2.99,5.00]	[14.55,21.06]	[16.28,18.76]	[0.95,0.99]	[0.05,0.04]
0.05	0.5	[2,3]	[2.01,3.02]	[6.84,8.98]	[6.08,7.03]	[0.95,0.99]	[0.05,0.05]

TABLE 3. Proposed Sampling plan parameters when α=0.05,β=0.10

AQL	LQL	r_N	ASN_N	k_{a_N}	k_{r_N}	$L(C_{AQL})$	$L(C_{(LQL)})$
0.01	0.035	[4,6]	[4.51,6.04]	[205.8,269.3]	[137.7,260.32]	[0.95,0.95]	[0.10,0.08]
0.01	0.05	[3,5]	[3.19,5.04]	[113.16,193.67]	[79.44,182.22]	[0.95,0.96]	[0.08,0.03]
0.01	0.1	[2,3]	[2.03,4.13]	[38.9,146.8]	[32.84,114.83]	[0.95,0.96]	[0.08,0.0001]
0.01	0.2	[1,3]	[1.07,3.00]	[12.05,74.1]	[4.23,73.99]	[0.95,0.96]	[0.09,0]
0.01	0.3	[1,2]	[1.07,2.01]	[12.05,38.9]	[4.23,36.21]	[0.95,0.95]	[0.02,0.009]
0.02	0.06	[6,8]	[6.30,8.27]	[156.15,212.94]	[130.69,185.73]	[0.95,0.96]	[0.09,0.01]
0.02	0.09	[5,6]	[5.02,6.05]	[94.03,129.18]	[91.29,122.41]	[0.96,0.96]	[0.06,0.02]
0.02	0.1	[4,5]	[4.04,5.03]	[73.2,96.94]	[68.45,92.57]	[0.95,0.95]	[0.05,0]
0.02	0.2	[2,4]	[2.02,4.04]	[20.39,68.11]	[17.83,63.37]	[0.95,0.95]	[0.06,0]
0.02	0.5	[1,2]	[1.02,2.01]	[3.606,19.6]	[2.55,18.00]	[0.95,0.95]	[0.09,0]
0.05	0.15	[7,9]	[7.07,9.07]	[67.71,91.5]	[64.94,88.08]	[0.95,0.95]	[0.08,0.04]
0.05	0.2	[6,7]	[6.18,7.09]	[58.26,65.26]	[51.69,60.88]	[0.95,0.95]	[0.01,0.01]
0.05	0.25	[5,6]	[5.02,6.04]	[40.21,50.21]	[39.02,48.21]	[0.95,0.95]	[0.01,0.004]
0.05	0.35	[3,5]	[3.01,5.06]	[17.82,36.31]	[16.28,36.31]	[0.95,0.95]	[0.01,0]
0.05	0.5	[2,3]	[1.90,3.02]	[1.32,14.53]	[7.09,13.088]	[0.95,0.95]	[0.03,0.002]

TABLE 4. Dataset of failure times for the air conditioning system

Failure times data on the air cooling system in airplane
23, 261, 87, 7, [121, 125] 14, 62, 47, 225, [71, 74] ,
246, 21,42, 20, 5, 12, 120, [11, 13] 3, 14,
71, 11, 14, 11, 16,90, 1, 16, [52, 55], 95

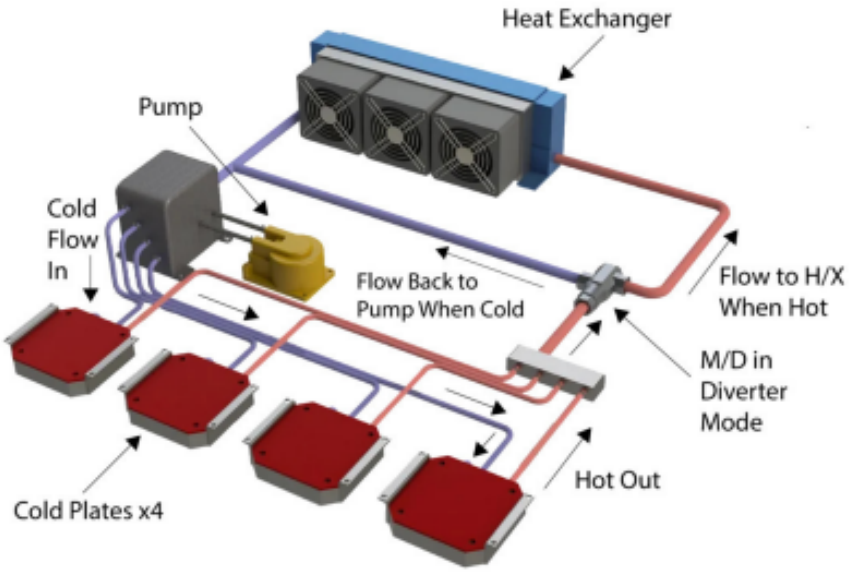

FIGURE 2. Example for Cooler system in an aeroplane

8. **Real Data Study**

In this study involving real data, we focus on assessing applications according to the proposed sampling plan utilizing the Neutrosophic Generalized Exponential distribution. It was used to analyze the real dataset, which was taken from khan et.al(2021) [19]. This dataset has been used by several authors in their studies, including the following:Linhart, Zucchini(1986) [20],Magalhães et.al(2020) [21].The implementation of the proposed plan indicates the failure occurrences of the air cooling system utilized in an aircraft. Figure 2 represents an example of air coolers. The experiment was conducted and only the failure time of the products was recorded. Table 4 represents the failure time of given data sets. To check the normality of data using Empirical Cumulative Distribution Function plot, Histogram and Q-Q Plots. From figure 3 , 4 and 5 represents the figure of the empirical cumulative distribution function, the histogram and the Q-Q plots give the results of the Neutrosophic Generalized Exponential distribution and its goodness of fit in the data sets.

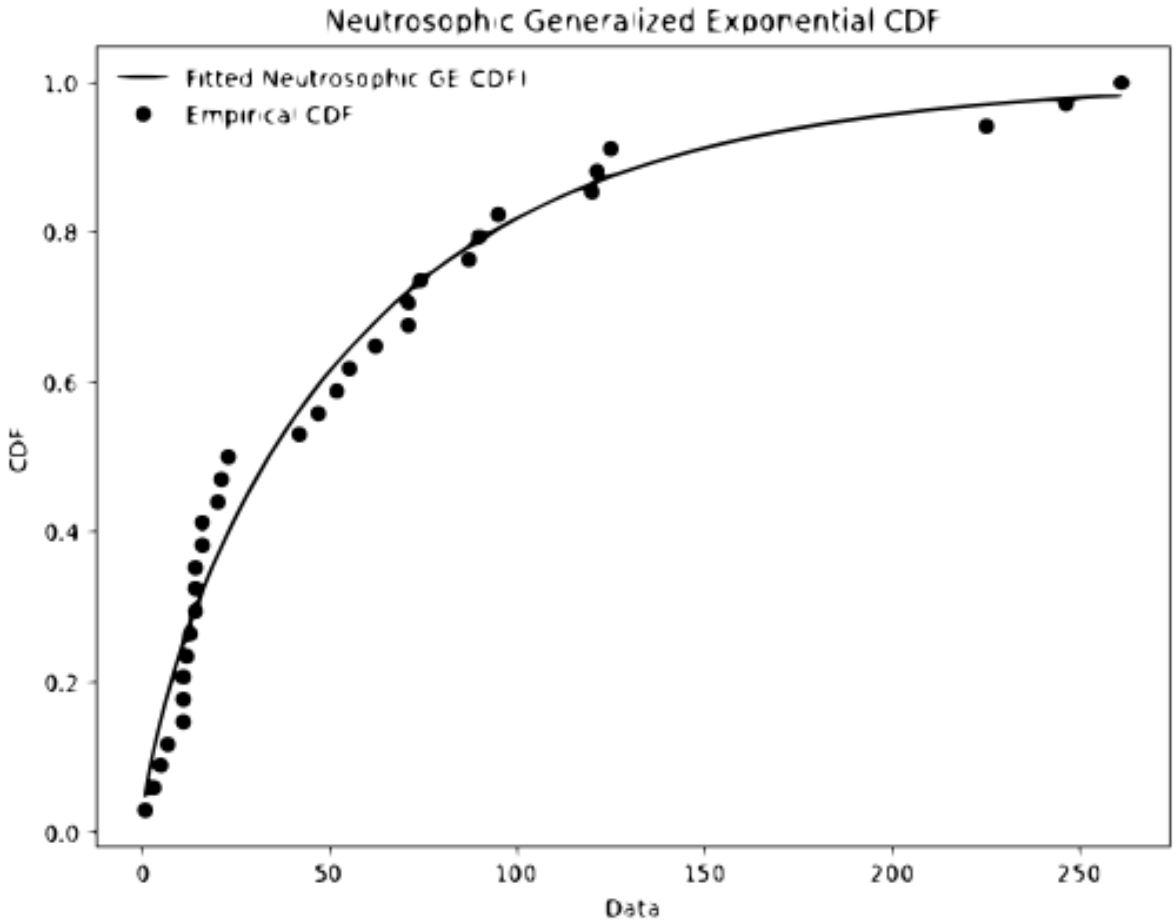

FIGURE 3. Curve fitting for given data based on Neutrosophic Generalized Exponential Distribution

Let us examine that the minimum specification for the product is L=25. Some observations during the experiment testing time were unclear, imprecise, or inconsistent.The producer risk associated with the aircraft cooling system cannot establish an appropriate sample size for product testing due to some unclear values or metrics. The air cooling system of the aircraft is characterized by the Neutrosophic Generalized Exponential Distribution involving a shape parameter $\delta_N \in \{2, 2.2\}$. Let AQL=0.01, LQL=0.035, $\alpha = 0.10$ and $\beta = 0.10$. From Table1 1, we get $r_N \in \{3, 4\}$,$k_{a_N} \in \{200.9, 221.5\}$,$k_{r_N} \in \{149.39, 167.39\}$.

- Step 1: Randomly select a sample $n_N \in [30, 30]$ from a lot.
- Step 2: We have $n_N = \{30, 30\}$ items on inspect and performs test till the failure of run time occurs. Note the failure time [121,125], [71, 74], [11, 13] and [52, 55].
- Step 3: Compute the Failure Censored Reliability Lifetest based on Neutrosophic quantity statistic v_N using equation15

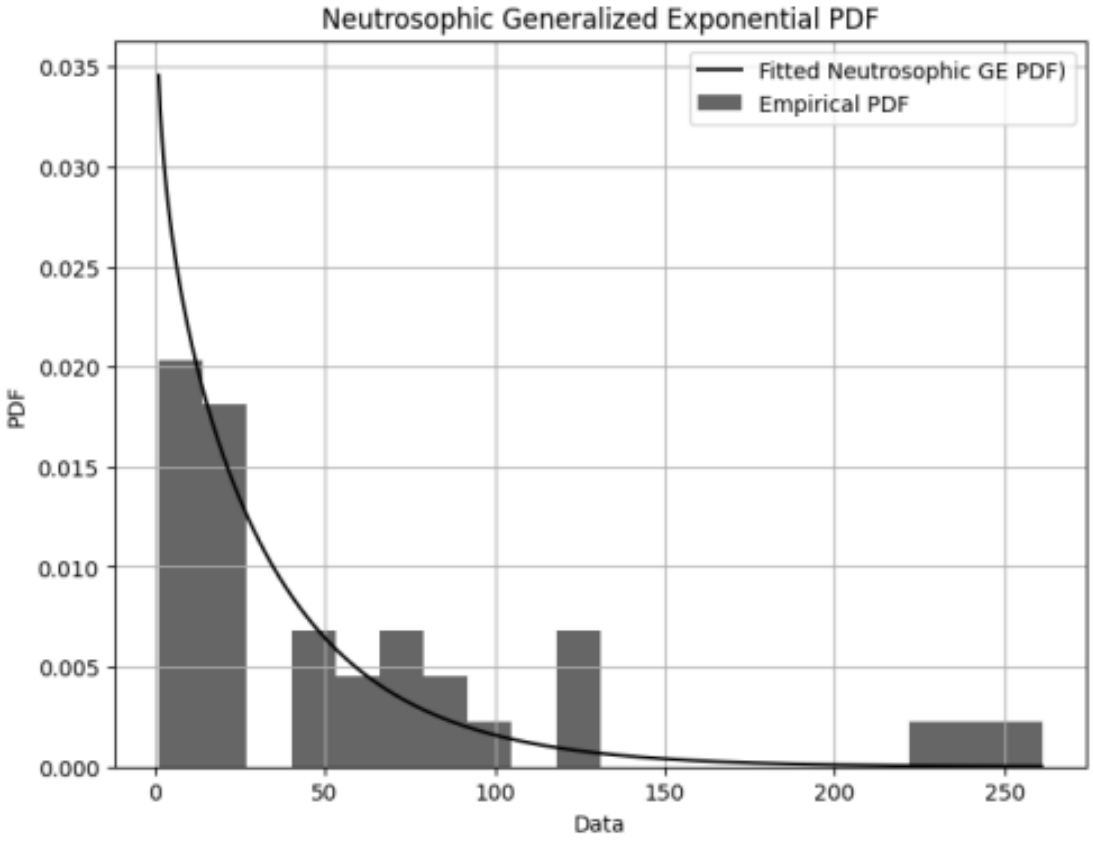

FIGURE 4. Normality using Histogram for given data follows Neutrosophic Generalized Exponential Distribution

- The Neutrosophic quantity v_N will be computed as:

$$\begin{aligned} &= (121, 125)^2 + (71, 74)^2 + (11, 13)^2 + (52, 55)^2 + \\ &\quad \left[(30 - 4)(52, 55)^2\right] \\ &= (22507, 24295) + (70304, 78650) \\ &\quad v_N \in (92811, 102945) \end{aligned}$$

- Step 4: We consider the values of $v_N \in (92811, 102945) < k_{a_N} L^{\delta_N} = [125562.5, 138437.5]$,$v_N \in (92811, 102945) < k_{a_N} L^{\delta_N} = [93368.75, 104618.75]$. As a result, the cooling system of the airplane product is accepted. The Neutrosophic statistical approach utilizes statistical values of cooling system in order to select the lot that should be submitted with the product.

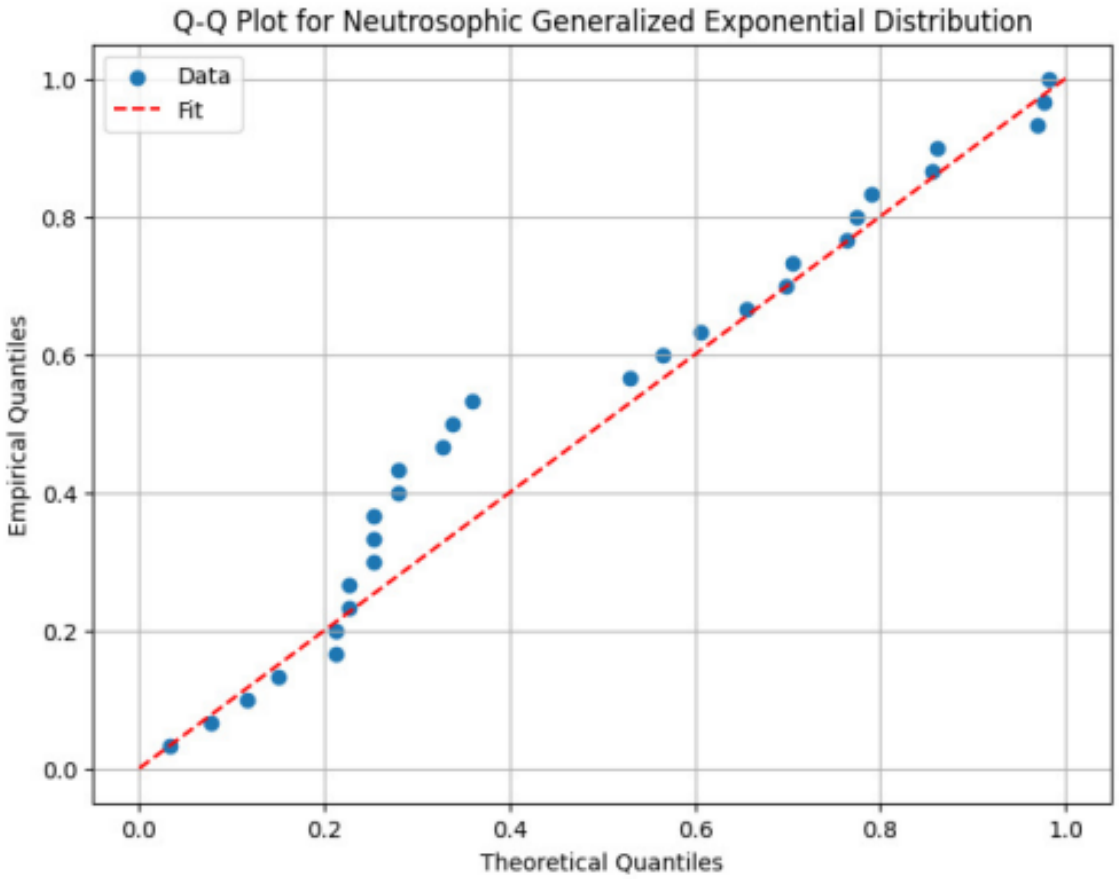

FIGURE 5. Q-Q plot for given data follows Neutrosophic Generalized Exponential Distribution

9. **Comparative Study:**

This proposed study compares the Neutrosophic interval method with classical statistics in order to determine its efficiency. Jun et al. (2010) use a Weibull distribution to estimate the values of plan parameters under classical statistics. Using classical statistics, we calculated the plan parameters of the proposed study. Table 5 presents plan parameters for selected AQL and LQL values when $\alpha = 0.05, \beta = 0.10$. It is important to use the same parameters in both plans when comparing them.

Table 5 indicates that the suggested plan features lesser interval values compared to the current plan according to traditional statistics. It is essential to highlight that the suggested plan is a range plan, while the current sampling plan is a fixed value. Consequently, the suggested sampling plan enables a more efficient examination of uncertain information. It is better suited for practical applications, and it is also more economical.

TABLE 5. Comparison of Proposed Sampling plan parameters when α=0.05,β=0.10

AQL	LQL	r_N	r
0.01	0.035	[4,6]	46
0.01	0.05	[3,5]	44
0.01	0.1	[2,3]	39
0.01	0.2	[1,3]	28
0.01	0.3	[1,2]	27
0.02	0.06	[6,8]	63
0.02	0.09	[5,6]	56
0.02	0.1	[4,5]	46
0.02	0.2	[2,4]	42
0.02	0.5	[1,2]	34
0.05	0.15	[7,9]	66
0.05	0.2	[6,7]	62
0.05	0.25	[5,6]	53
0.05	0.35	[3,5]	48
0.05	0.5	[2,3]	44

10. **Conclusion**

Our research aims to develop a failure-censored reliability testing plan using Neutrosophic Statistical Interval Method with Repetitive Group Sampling. The suggested sampling plan uses non-linear optimization grid search techniques to compute the tables. We provide a real data example to clarify our study's concept and present the findings clearly, comprehensibly and also this method provides increased flexibility in parameter selection and delivers more precise results.

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Ethical/Institutional Approval: The Ethical/Institutional Approval is not applicable for this Research.

Informed Consent from Research Participants: The informant consent from Research Participant is not applicable for this Research.

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Enhanced MCDM Methodology under Uncertainty: Applications to Energy Efficiency Evaluation of Mechanical and Electrical Systems in Public Buildings

Daoming Wang*

College of Modern Technology, Nanyang Arts and Crafts Vocational College, Zhenping, 474284, Henan, China

*Corresponding author, E-mail: gymwdm3258@163.com

Abstract: Evaluation of Mechanical and Electrical systems in public Buildings is a multi-criteria decision-making problem that has various criteria. These criteria can affect energy efficiency. So, we aim to propose an MCDM method for rate the criteria importance and proposed a set of systems to select the best one in energy firms. This section can increase energy efficiency. We applied the ARAS methodology to rank the alternatives. The ARAS methodology is an MCDM method. It applied into criteria independent. The ARAS method is integrated under the plithogenic set to deal with the vague and uncertainty information. The plithogenic set is an extension of neutrosophic set and solves the uncertainty in the opinions of experts. Case study with seven criteria and ten systems are proposed in this study to select the best one. The results show the Energy Consumption Performance criterion has the highest score and System Maintenance and Operational Efficiency criterion has the lowest score. We compared the proposed method with other MCDM methos such as TOPSIS, VIKOR, EDAS and MABAC methos. The results show the proposed method is robust compared with other MCDM methods.

Keywords: Multi-Criteria Decision-Making; Uncertainty Framework; ARAS Method; Energy Efficiency; Public Buildings.

1. Introduction

Protecting the facility and its infrastructure is a big part of facilities management, and security is undoubtedly a crucial component of that. However, mechanical and electrical services are also essential to a building's protection in terms of offering a secure

atmosphere, as well as to the building's suitability and capacity to house operations in a safe and efficient manner[1], [2].

A commercial property cannot operate without the safe and proper installation, testing, and monitoring of mechanical and electrical services, which are fundamental to any building's structure. Many small and medium-sized enterprises lack the funding necessary to hire in-house specialists in mechanical and electrical services[3], [4].

The mechanical systems connected with heating, ventilation, and air conditioning have progressed dramatically from the traditional steam-heating systems and general mechanical control methods. The range of systems and techniques employed today is far more intricate. The purpose of these more advanced systems is to give the building's occupants a secure, comfortable, and healthy atmosphere in addition to local control and energy saving, which the older systems were unable to do. Much work and technology has been developed to give both warmth and cooling. Evaluation of Mechanical and Electrical Systems in Public Buildings is a MCDM problem[5], [6].

Multi-criteria decision-making (MCDM) techniques are divided into two categories: multi-attribute and multi-objective. The former identifies the best options, while the latter identifies the best options that maximize the goal. To find the best answer to the decision-making problem, the MCDM approaches follow a set of phases[7], [8]. DM is the process of selecting the best option that, with the help of an expert, largely satisfies all the criteria; however, determining the weight of each criterion is vital. In certain situations, it is assumed that the criterion weights are equal, however this is not always the case. The criterion weight indicates the importance of the criteria, and it is therefore crucial to calculate the criterion weight[9], [10].

Researchers have recently created new plithogenic MCDM techniques. The plithogenic operators and the degree of contradiction are employed in these plithogenic decision-making models to determine the experts' collective opinion on the alternatives' criterion satisfaction rate[11], [12].

These days, a lot of people use computer-based technology to make their jobs easier. Similarly, Decision Support Systems (DSS) have been widely used in decision-making to facilitate management's decision-making process. Numerous DSS techniques have been used in a variety of spheres of life, including offices, commerce and economics, education, and so on[13], [14]. Thus, a decision assistance system was created to help decision makers to select the best alternative. The Additive Ratio Assessment (ARAS) approach was used in the development of this decision support system. Nowadays, a lot of different

fields employ the ARAS approach to help with decision-making. The optimum option is selected by employing the ARAS approach while constructing a decision support system[15], [16].

The rest of this study is organized as follows: Section 2 shows the proposed methodology steps. Section 3 shows the case study results. Section 4 shows comparative analysis. Section 5 shows the conclusions.

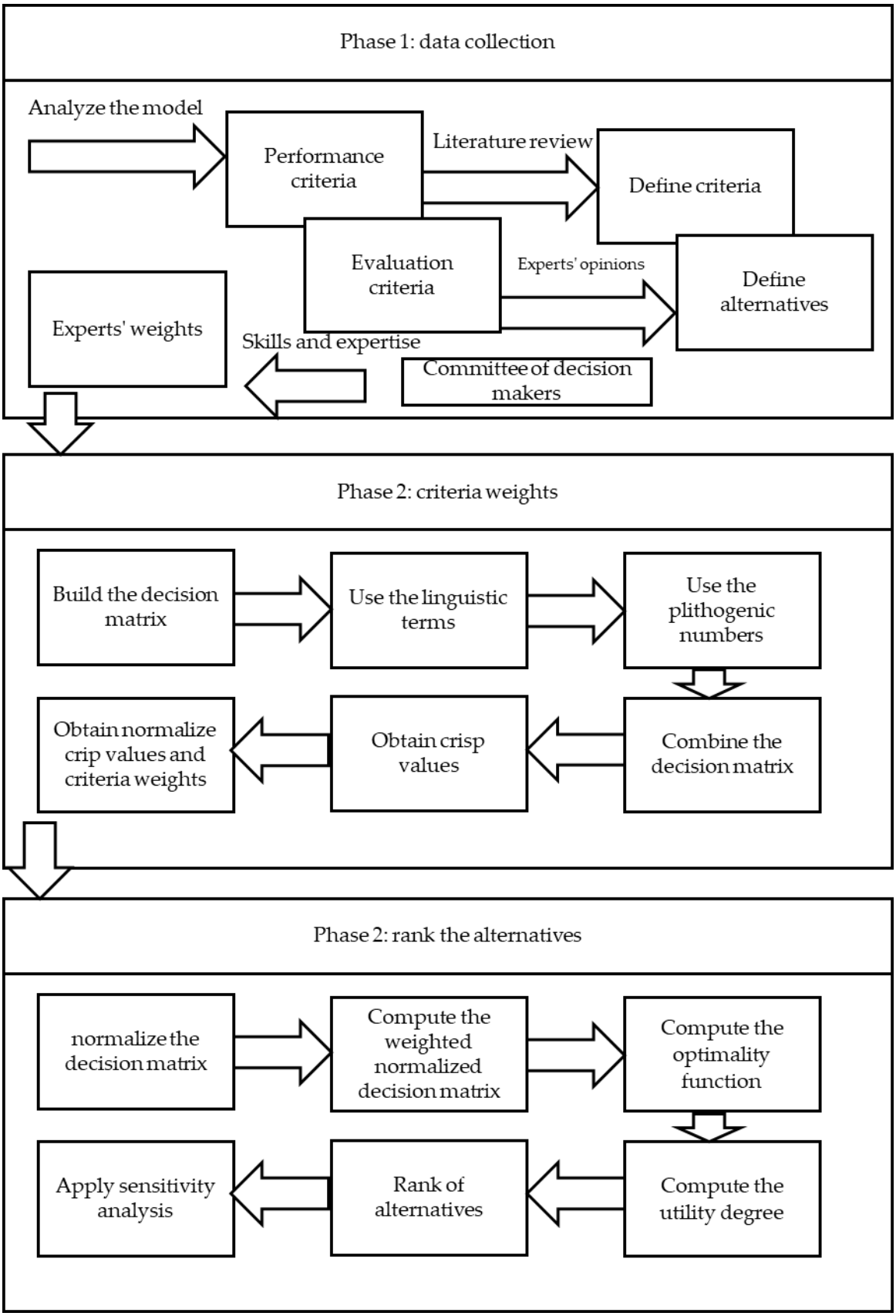
Phase 1: data collection
Analyze the model
Performance criteria
Literature review
Define criteria
Evaluation criteria
Experts' opinions
Define alternatives
Experts' weights
Skills and expertise
Committee of decision makers
Phase 2: criteria weights
Build the decision matrix
Use the linguistic terms
Use the plithogenic numbers
Obtain normalize crip values and criteria weights
Obtain crisp values
Combine the decision matrix
Phase 2: rank the alternatives
normalize the decision matrix
Compute the weighted normalized decision matrix
Compute the optimality function
Apply sensitivity analysis
Rank of alternatives
Compute the utility degree

Figure 1. The steps of the proposed methodology.

2. Methodology

The methodology of ARAS is used under the plithogenic set to find the ranking of alternatives[17], [18]. Figure 1 shows the steps of the proposed method. The steps involved in below:

Step 1. The initial decision matrix of order set of criteria and alternatives is built based on the opinions of experts and decision makers. The decision matrix contains the values of criteria and alternatives, these values are used from the linguistic terms such as medium, high, low. These terms have plithogenic numbers to evaluate the criteria and alternatives. Three decision makers and experts are used to creating the decision matrix. The combined decision matrix is obtained from the plithogenic intersection operators based on the representation fuzzy and neutrosophic sets of linguistic terms[19], [20].

Plithogenic fuzzy intersection

$$a \wedge Fb. \tag{1}$$

Plithogenic intuitionistic fuzzy intersection

$$a_1 a_2 \wedge IFS b_1 b_2 = a_1 \wedge F b_1 a_2 \wedge F b_2 \tag{2}$$

Plithogenic neutrosophic intersection

$$a_1 a_2 a_3 \wedge P b_1 b_2 b_3 = a_1 \wedge F b_{112} a_2 \wedge F b_2 + a_2 \vee a_2 F b_2 a_3 \wedge F b_3 \tag{3}$$

$$a \wedge Fb = ab, a \vee F = a + b - ab \tag{4}$$

Step 2. The weights of criteria are computed.

After aggregating the decision matrix. The crips values are computed using the score function. The normalized values of crips values are obtained to obtain the crisp values.

Step 3. After obtaining the criteria weights, we applied the steps of the Plithogenic ARAS method to rank the alternatives.

Step 3.1 normalize the decision matrix.

The decision matrix is normalized as:

$$y_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{5}$$

Step 3.2 Compute the weighted normalized decision matrix.

The normalized decision matrix is multiplied by the criteria weights such as:

$$d_{ij} = y_{ij} * w_j \quad (6)$$

Step 3.3. Compute the optimality function

The optimality function is computed and better has the larger value.

$$E_i = \sum_{j=1}^{n} d_{ij} \quad (7)$$

Step 3.4 Compute the utility degree

The utility degree is computed as follows:

$$U_i = \frac{E_i}{V_o} \quad (8)$$

Where V_o is a optimality value of E_i

3.5 Rank of alternatives.

3. Application to Decision-Making for Evaluation of Mechanical and Electrical Systems in Public Buildings

The proposed plithogenic ARAS MCDM method is illustrated with the decision on the Evaluation of Mechanical and Electrical Systems in Public Buildings. Energy efficiency is very important for evaluating and selecting the best alternatives. We collected seven criteria and ten alternatives from previous studies as shown in Table 1.

Table 1. The decision-making environment includes ten alternatives and seven criteria that are used in this study.

Criteria	Alternatives
System Maintenance and Operational Efficiency	System 1
Indoor Environmental Quality	System 2
Energy Consumption Performance	System 3
Energy-Saving Technologies	System 4
Energy Monitoring and Data Analytics	System 5
Compliance with Energy Standards and Certifications	System 6
Renewable Energy Utilization	System 7
	System 8
	System 9
	System 10

The initial plithogenic numbers of three experts to evaluate the decision matrix are as follows:

Expert 1

	C1	C2	C3	C4	C5	C6	C7
A1	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.6, 0.5, 0.3, 0)	(0.2, 0.5, 0.6, 0)	(0.6, 0.5, 0.3, 0)
A2	(0.9, 0.5, 0.0, 0.5)	(0.6, 0.5, 0.3, 0)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.5, 0.0, 0.4, 0)
A3	(0.8, 0.0, 0.1, 0)	(0.5, 0.0, 0.4, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.4, 0.0, 0.7, 0)
A4	(0.6, 0.5, 0.3, 0)	(0.4, 0.0, 0.7, 0)	(0.6, 0.5, 0.3, 0)	(0.2, 0.5, 0.6, 0)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.2, 0.5, 0.6, 0)
A5	(0.5, 0.0, 0.4, 0)	(0.2, 0.5, 0.6, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.6, 0.5, 0.3, 0)	(0.2, 0.5, 0.6, 0)	(0.9, 0.5, 0.0, 0.5)
A6	(0.4, 0.0, 0.7, 0)	(0.6, 0.5, 0.3, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.6, 0.5, 0.3, 0)
A7	(0.2, 0.5, 0.6, 0)	(0.5, 0.0, 0.4, 0)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.5, 0.0, 0.4, 0)
A8	(0.1, 0.0, 0.7, 0.5)	(0.4, 0.0, 0.7, 0)	(0.8, 0.0, 0.1, 0)	(0.2, 0.5, 0.6, 0)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.4, 0.0, 0.7, 0)
A9	(0.9, 0.5, 0.0, 0.5)	(0.2, 0.5, 0.6, 0)	(0.9, 0.5, 0.0, 0.5)	(0.9, 0.5, 0.0, 0.5)	(0.6, 0.5, 0.3, 0)	(0.2, 0.5, 0.6, 0)	(0.2, 0.5, 0.6, 0)
A10	(0.8, 0.0, 0.1, 0)	(0.6, 0.5, 0.3, 0)	(0.5, 0.0, 0.4, 0)	(0.4, 0.0, 0.7, 0)	(0.2, 0.5, 0.6, 0)	(0.1, 0.0, 0.7, 0.5)	(0.6, 0.5, 0.3, 0)

Expert 2

	C1	C2	C3	C4	C5	C6	C7
A1	(0.1, 0.0, 0.7, 0.5)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.1, 0.0, 0.7, 0.5)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)
A2	(0.2, 0.5, 0.6, 0)	(0.1, 0.0, 0.7, 0.5)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.1, 0.0, 0.7, 0.5)
A3	(0.4, 0.0, 0.7, 0)	(0.1, 0.0, 0.7, 0.5)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.8, 0.0, 0.1, 0)	(0.1, 0.0, 0.7, 0.5)	(0.2, 0.5, 0.6, 0)
A4	(0.5, 0.0, 0.4, 0)	(0.9, 0.5, 0.0, 0.5)	(0.4, 0.0, 0.7, 0)	(0.2, 0.5, 0.6, 0)	(0.1, 0.0, 0.7, 0.5)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)
A5	(0.6, 0.5, 0.3, 0)	(0.1, 0.0, 0.7, 0.5)	(0.6, 0.5, 0.3, 0)	(0.1, 0.0, 0.7, 0.5)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)
A6	(0.4, 0.0, 0.7, 0)	(0.2, 0.5, 0.6, 0)	(0.1, 0.0, 0.7, 0.5)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)
A7	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.2, 0.5, 0.6, 0)	(0.4, 0.0, 0.7, 0)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.2, 0.5, 0.6, 0)
A8	(0.1, 0.0, 0.7, 0.5)	(0.5, 0.0, 0.4, 0)	(0.4, 0.0, 0.7, 0.5)	(0.5, 0.0, 0.4, 0)	(0.6, 0.5, 0.3, 0)	(0.5, 0.0, 0.4, 0)	(0.4, 0.0, 0.7, 0)

A9	(0.95, 0.05, 0)	(0.65, 0.30, 0)	(0.50, 0.40, 0)	(0.65, 0.30, 0)	(0.65, 0.30, 0)	(0.65, 0.30, 0)	(0.50, 0.40, 0)
A10	(0.80, 0.10, 0)	(0.65, 0.30, 0)	(0.65, 0.30, 0)	(0.40, 0.70, 0)	(0.25, 0.60, 0)	(0.15, 0.70, 0)	(0.65, 0.30, 0)

Expert 3

	C1	C2	C3	C4	C5	C6	C7
A1	(0.25, 0.60, 0)	(0.40, 0.70, 0)	(0.50, 0.40, 0)	(0.65, 0.30, 0)	(0.80, 0.10, 0)	(0.25, 0.60, 0)	(0.40, 0.70, 0)
A2	(0.95, 0.05, 0)	(0.15, 0.70, 0.5)	(0.25, 0.60, 0)	(0.40, 0.70, 0)	(0.50, 0.40, 0)	(0.65, 0.30, 0)	(0.65, 0.30, 0)
A3	(0.80, 0.10, 0)	(0.15, 0.70, 0.5)	(0.50, 0.40, 0)	(0.65, 0.30, 0)	(0.80, 0.10, 0)	(0.80, 0.10, 0)	(0.80, 0.10, 0)
A4	(0.65, 0.30, 0)	(0.95, 0.00, 0.5)	(0.40, 0.70, 0)	(0.25, 0.60, 0)	(0.15, 0.70, 0.5)	(0.95, 0.00, 0.5)	(0.65, 0.30, 0)
A5	(0.50, 0.40, 0)	(0.80, 0.10, 0)	(0.65, 0.30, 0)	(0.15, 0.70, 0.5)	(0.25, 0.60, 0)	(0.15, 0.70, 0.5)	(0.95, 0.00, 0.5)
A6	(0.40, 0.70, 0)	(0.65, 0.30, 0)	(0.50, 0.40, 0)	(0.95, 0.00, 0.5)	(0.25, 0.60, 0)	(0.25, 0.60, 0)	(0.15, 0.70, 0.5)
A7	(0.25, 0.60, 0)	(0.50, 0.40, 0)	(0.65, 0.30, 0)	(0.80, 0.10, 0)	(0.40, 0.70, 0)	(0.40, 0.70, 0)	(0.25, 0.60, 0)
A8	(0.15, 0.70, 0.5)	(0.40, 0.70, 0)	(0.80, 0.10, 0)	(0.65, 0.30, 0)	(0.50, 0.40, 0)	(0.50, 0.40, 0)	(0.40, 0.70, 0)
A9	(0.95, 0.00, 0.5)	(0.25, 0.60, 0)	(0.95, 0.00, 0.5)	(0.95, 0.00, 0.5)	(0.65, 0.30, 0)	(0.65, 0.30, 0)	(0.50, 0.40, 0)
A10	(0.80, 0.10, 0)	(0.65, 0.30, 0)	(0.50, 0.40, 0)	(0.40, 0.70, 0)	(0.25, 0.60, 0)	(0.15, 0.70, 0.5)	(0.65, 0.30, 0)

We combine these plithogenic numbers into a single matrix using plithogenic operators. Then we apply the score function to obtain crisp values. Then we normalize the decision matrix using Eq. (5) as shown in Table 2.

Table 2. The normalized decision matrix

	C1	C2	C3	C4	C5	C6	C7
A1	0.042801	0.085048	0.100811	0.125242	0.124278	0.060455	0.096346
A2	0.142265	0.070188	0.052094	0.07836	0.113248	0.143289	0.091475
A3	0.129387	0.056879	0.100811	0.125242	0.143783	0.118042	0.103077
A4	0.118167	0.153431	0.093897	0.052841	0.036267	0.128465	0.091152

A5	0.110531	0.095456	0.116893	0.064669	0.091668	0.062861	0.149451
A6	0.078094	0.116334	0.07091	0.121166	0.088952	0.112133	0.101485
A7	0.052662	0.102786	0.081601	0.105772	0.096033	0.118179	0.072144
A8	0.022296	0.094112	0.127993	0.09801	0.108546	0.10835	0.079268
A9	0.15601	0.089835	0.145651	0.150337	0.138705	0.122631	0.088909
A10	0.147786	0.135931	0.109339	0.07836	0.058521	0.025596	0.126693

We compute the weighted normalized decision matrix as shown in table 3.

Table 3. The weighted normalized decision matrix.

	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**
A1	0.006405	0.011687	0.015251	0.01868	0.016737	0.007881	0.014206
A2	0.021291	0.009645	0.007881	0.011687	0.015251	0.01868	0.013487
A3	0.019364	0.007816	0.015251	0.01868	0.019364	0.015389	0.015198
A4	0.017685	0.021085	0.014206	0.007881	0.004884	0.016747	0.01344
A5	0.016542	0.013118	0.017685	0.009645	0.012345	0.008195	0.022035
A6	0.011687	0.015987	0.010728	0.018072	0.011979	0.014618	0.014963
A7	0.007881	0.014125	0.012345	0.015776	0.012933	0.015406	0.010637
A8	0.003337	0.012933	0.019364	0.014618	0.014618	0.014125	0.011687
A9	0.023348	0.012345	0.022035	0.022423	0.01868	0.015987	0.013109
A10	0.022117	0.01868	0.016542	0.011687	0.007881	0.003337	0.01868

Then we compute the optimality function using Eq. (7). Then we compute the utility degree using Eq. (8)

The utility degree is computed as follows:

$$U_i = \frac{E_i}{V_o} \tag{8}$$

Where V_o is a optimality value of E_i

3.5 Rank of alternatives.

Table 4. The rank of alternatives.

	Optimality function	**Utility degree**	**Rank of alternatives**
A1	0.090848	0.710155	3
A2	0.097924	0.765467	5
A3	0.111062	0.868166	9
A4	0.095927	0.74986	4

A_5	0.099565	0.778297	8
A_6	0.098035	0.766335	6
A_7	0.089104	0.696522	1
A_8	0.090683	0.708862	2
A_9	0.127927	1	10
A_{10}	0.098925	0.773288	7

4. Comparative Analysis

The ninth system was the highest score and the best alternative in Evaluation of Mechanical and Electrical Systems in Public Buildings procedure described above. The third system came in second, while the seventh system came in last. This section conducts a brief comparative analysis with current MCDM techniques, such as the TOPSIS method, VIKOR method, EDAS method, and MABAC method in a plithogenic set environment to confirm the superiority of the suggested model in Evaluation of Mechanical and Electrical Systems in Public Buildings and to further examine the variations in the scoring outcomes of different systems.

Figure 2 shows the rank of the proposed method compared with the other MCDM methods. We show the alternative 9 is the best in all MCDM methods and alternative 7 is the lowest in all MCDM methods. We show our model is strong compared with other MCDM methods.

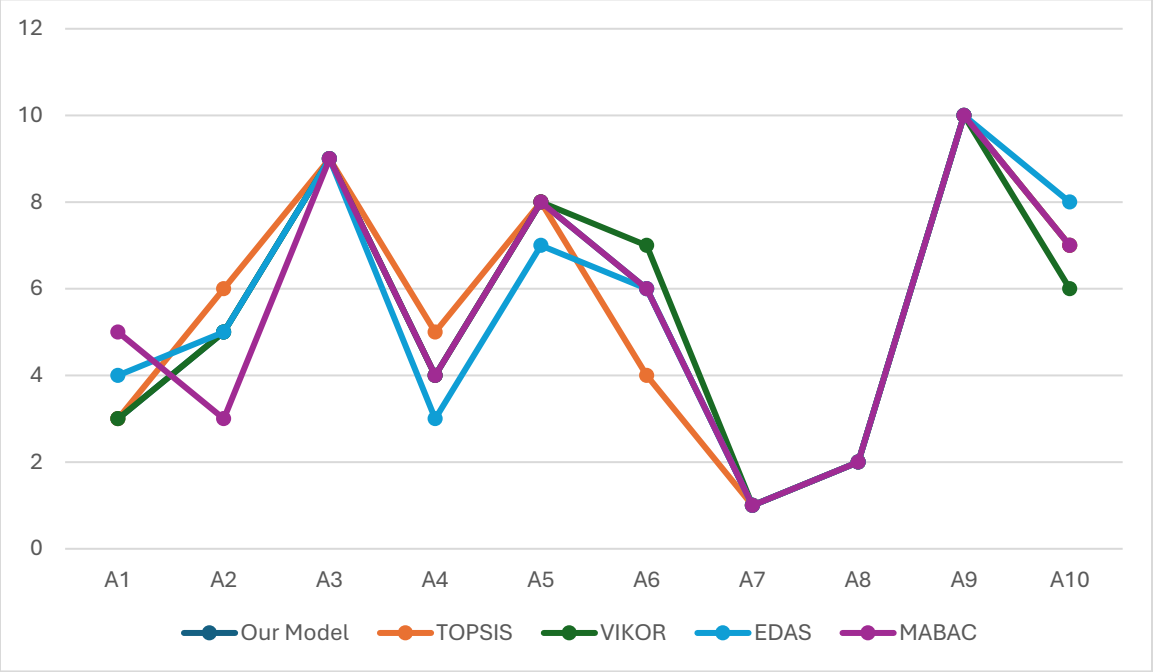

Figure 2. Comparative analysis.

We compute the correlation between our model and other MCDM methods to show the relationship between our model and other MCDM methods. We show the highest correlation between our model and other MCDM methods.

Table 4. The correlation between our model and other MCDM methods.

	Our Model	TOPSIS	VIKOR	EDAS	MABAC
Our Model	-	0.963636	0.987879	0.975758	0.951515
TOPSIS	0.963636	-	0.927273	0.927273	0.890909
VIKOR	0.987879	0.927273	-	0.951515	0.939394
EDAS	0.975758	0.927273	0.951515	-	0.951515
MABAC	0.951515	0.890909	0.939394	0.951515	-

5. Conclusions

This study proposed an MCDM method for evaluation of Mechanical and Electrical systems in public Buildings. This evaluation contains uncertainty and vague information. So, we proposed an ARAS method to rank the alternatives and compute the criteria weights. Three experts are invited to evaluate the criteria and alternatives. The ARAS method is integrated with the plithogenic sets to deal with uncertainty data. Then we combine plithogenic numbers into single matrix. Then we obtained crisp values. Then we obtained the criteria weights. The proposed method is applied into seven criteria and ten alternatives. The results show the 9 is the best and alternative 7 is the worst. We compared the proposed methodology with other MCDM methods. The results show the alternative 9 is the best in all MCDM methods. The results indicate the proposed methodology is strong compared with other MCDM methods.

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Analytical Solutions of Heat Transfer Model in Two-Dimensional Case of Neutrosophic Fredholm Integro-Differential Equations

Amer Darweesh[1,*], Kamel Al- Khaled[1], Marwan Alquran[1], Adel Almalki[2] and Sohad Al-Omari[1]

[1]Department of Mathematics and Stat., Jordan University of Science and Technology, Irbid 22110, Jordan; ahdarweesh@just.edu.jo; kamel@just.edu.jo: marwan04@just.edu.jo; szalomary15@sci.just.edu.jo.

[2]Department of Mathematics, Al-Gunfudah University College, Umm Al-Qura University, Mecca 21955, Saudi Arabia; aaamalki@uqu.edu.sa.

*Correspondence: ahdarweesh@just.edu.jo

Abstract. This article uses the fractional residual power series (FRPS) method to solve a linear neutrosophic fractional integro-differential equation in two dimensions. In what context does the term "fractional derivative" appeared, we presented the modified fractional power series method, a new technique that uses fractional power series expansion to approximate neutrosophic fractional integro-differential equations. A modified new method has been formulated, which is an improvement on the RPS, named as Modified Fractional Power Series Method (MFPSM), to solve the same problem under investigation. Novel results associated with the rate of convergent and error order of the (MFPSM) was examined, and some findings—along with detailed proof—were documented as theories. Several numerical examples are used to describe and test the validity and applicability of preset approaches. We investigate a semi-infinite rod using the solution of our model, where heat transfer is influenced by both the memory of past states and the current temperature distribution. The fractional derivative of order α is used to represent memory effects in heat transfer processes. To demonstrate the precision and efficacy of the two approaches, the results are shown in terms of tables and graphs. The modified fractional power series approach proved to be more effective, efficient, and straightforward for solving the neutrosophic two-dimensional integro-differential equations than the residual power series method, while also generating less error and computing time.

Keywords: Fractional residue power series method; numerical solutions; Heat Transfer; Neutrosophic Integral Equations; Fredholm integro-differential equation in two dimensions.

1. Introduction

In many fields, including science, engineering, and modeling, integral and integro-differential equations are used to represent issues. The intricacy of most of these equations makes it difficult to acquire analytical solutions; consequently, it is crucial to develop numerical methods to obtain approximate answers. Significant advancements have been made in the last fifty years in the development of analytical and numerical solutions for various types of integral equations; both linear and nonlinear examples have been taken into consideration. Few numerical techniques have been used to approximate the solutions of two-dimensional problems, despite the fact that there are numerous approaches for one-dimensional integral and integro-differential equations. Given the numerous uses of two-dimensional integral and integro-differential equations in physics, mechanics, modeling, engineering, and other applied sciences, methods for handling these equations also need further attention. Neutrosophic integro-differential equations are used in many different domains, such as engineering and medicine, proving their adaptability in simulating dynamic systems with inherent uncertainties. As a result, neutrosophic calculus enhances the theory of integral differential equations and provides a more thorough method for handling real-world complexity than interval computations.

Using an approximating subspace, a unique space of spline functions, [34] and [31] examined the numerical solution of two-dimensional Fredholm integral equations. [36] used artificial neutral networks for modeling and simulating complex real-world problems in two-dimensional Volterra-type fractional integro-differential equations. In [3, 17], it is demonstrated that the non-linear Fredholm integro-differential equation in two dimensions exists and has a unique solution.

The basic concepts of fractional calculus were first presented by the Marquis de Hopital and G.W. Leibniz in 1695, as is well known. Since then, a number of other people have defined fractional operators [29]. Due to their applicability in a variety of scientific fields, such as physics, chemistry, engineering, and others, fractional-order integro-differential equations have been the focus of more discussion in the literature. As a result, alternative numerical methods are being given much more consideration.

Abu Arqub was the first who proposed the residual power series method (RPSM) in 2013 [11]. He also introduced a new method to obtain a numerical solution for linear and non-linear fuzzy problems. Additionally, this method was used to obtain a series solution to the non-linear time fractional reaction diffusion equation [43]. Authors in [43] found that the method is effective and powerful in solving that equation. In [10], Alquran developed a new study of finding the analytic solution to the time fractional two-component evolutionary system of order 2 using RPSM. The time-fractional foam drainage equation was also introduced by [9].

The time fractional Zakharov-Kuznetsov equation was approximated by Alquran in [37] using RPSM. Also, they provided the RPSM convergence study and demonstrated the method's competitiveness, strength, dependability, and ease of use. The RPSM was used in [19] to obtain an approximation solution and an analytical solution for the fractional Susceptible-Infected-Recovered (SIR) epidemic model in the form of a convergent power series. The time fractional Korteweg de Vries was numerically solved by Senol and Ayse [38], and the RPSM is used to generate a modified version of the solution. The multi-pantograph delay differential equations system was approximated by the authors of [21] using RPSM. Similarly, using the RPSM, the authors of [22] discovered an analytic solution to the system of 1-dimensional Fredholm integral equations. Furthermore, Bayrak and Demir found numerical solutions to space-time fractional partial differential equations using the RPSM [12].

Numerous physical phenomena, including heat conduction in memory-containing materials and diffusion processes, are modeled by fractional integro-differential equations. In [27], Nawaz used the variational iteration method and the homotopy perturbation method to approximate solutions to the nonlinear boundary value problems for 4^{th}-order fractional integro-differential equations. While, for the linear Volterra integro-fractional differential equation, Ahmed and Salh in [2] employed the generalized Taylor matrix approach. Also, the homotopy perturbation method was used by Saeedi and Samimi [35] to approximate non-linear Fredholm integro-fractional differential equations. The Chebyshev-Legendre spectral approach for fractional Fredholm integro-differential equations was later proposed by the authors of [44]. Rashed [32] addressed a particular class of integro-differential equations. Additionally, Hu [20] proposed the interpolation collocation method to solve the Fredholm linear integro-differential equations analytically. Heris used the modified Laplace Adomian decomposition approach to find approximate solutions for integro-differential equations. However, no one has yet used RPSM to solve fractional integro-differential equations in two dimensions. Evaluating the function coefficients that show up in the definite integral term is the difficult part. Fractional integro-differential equation in two-dimensions has numerical solutions obtained in [15] using the Haar wavelet method.

Integro-differential equations are a special and intriguing area of mathematics. The theory of heat conduction is typically the source of problems involving time as an independent variable. For example, its solution yields the temperature at a distance x following a temperature distribution of t seconds. In practice, integro-differential equations representing dynamical systems frequently provide ambiguous or insufficient information, particularly when it comes to heat transfer equations. The initial conditions, boundary conditions, and other components of integro-differential equations can all exhibit this uncertainty. Therefore, in order to

get more accurate results than those based on real conditions, we solve integro-differential equations based on neutrosophic conditions. In [26], the authors give a scientific summary of neutrosophists.

An equation with an unknown neutrosophic function appearing under one or more integral signs is known as a neutrosophic integral equation, which has the following form

$$u(x,I) = f(x,I) + \lambda \int_a^b K(x,t,I)u(t,I)dt,$$

where $u(x,I)$ is the unknown neutrosophic function, I is an indeterminate number, while $f(x,I)$ and $K(x,t,I)$ are known functions, and λ, a, b are constants, for more details see [13, 40–42]. In [39], a suggested neutrosophic Laplace transform technique that makes it possible to solve integral equations involving non-linear neutrosophic numbers effectively is presented. A modified iterative approach in a neutrosophic setting has been used in [25] to determine the numerical solution of the second-kind Fredholm integral equation. Using certain theorems, it has been shown that the iterative method converges in a neutrosophic environment. For further studies of these results, these tools can be developed by linking them with other concepts that can be found in the following works [1, 4–8, 23].

In this article, we will examine two novel methods for finding exact or numerical solutions to the two-dimensional neutrosophic fractional integro-differential equations of the general form:

$$D_t^\alpha u(x,t,I) = f(x,t,I) + g(x,I) \int_0^a \int_0^b K(t,y,\tau,I)u(y,\tau,I)d\tau dy, \tag{1}$$

where $D_t^\alpha u(x,t,I)$ is the Caputo derivative of $u(x,t,I)$ with respect to t, and $0 < \alpha \leq 1$ denotes the fractional derivative of order α. We solve equation (1) subject to the neutrosophic initial condition $u(x,0,I) = F(x,I)$.

Examine a semi-infinite rod in which the memory of previous states and the current temperature distribution both affect heat transfer. Equation (1) provides a two-dimensional fractional Fredholm integro-differential equation that models this phenomenon. Memory effects in heat transfer processes are represented by the fractional derivative of order α, the kernel function characterizing the memory effect is $K(t,y,\tau,I)$, and the temperature distribution at position x and time t is $u(x,t,I)$. Here, the fractional-order rate of temperature change is captured by $D_t^\alpha u(x,t,I)$, which takes into consideration memory effects that are common in materials with thermal hysteresis. The cumulative effect of previous temperatures, weighted by the kernel K, is represented by the expression $g(x,I)\int_0^a \int_0^b K(t,y,\tau,I)u(y,\tau,I), d\tau, dy$. Applications of this model can be found in materials with unusual heat conduction characteristics, such as composite materials, biological tissues, or polymers, where thermal response is dependent on both the current and previous states (see [24, 30]).

2. Preliminaries and Basic Concepts

2.1. *Caputo's Fractional Derivative*

In this work, the fractional derivatives are considered in the sense of Caputo, which is defined in the following definition.

Definition 2.1 (Caputo derivative)**.** [16] Let $\alpha > 0$ such that $n-1 < \alpha < n$ for some $n \in \mathbb{N}$. The Caputo fractional derivative of $f(x)$ is defined by

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x f^{(n)}(t)(x-t)^{n-\alpha-1} dt. \tag{2}$$

It is not hard to check that Caputo derivative is linear operator. The following formulas are helpful in order to compute Caputo derivative of common functions (see [14, 16, 33]).

(1) $D_t^{\alpha}(t^p) = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & , \quad n-1 < \alpha < n,\ p > n-1,\ p \in \mathbb{R} \\ 0 \quad , & n-1 < \alpha < n,\ p \leq n-1,\ p \in \mathbb{N}_0. \end{cases}$

(2) The Caputo fractional derivative of the exponential function has the following form:

$$D_t^{\alpha} e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)}.$$

2.2. *Fractional Power series*

Definition 2.2. [18] A fractional power series (FBS) about c is a formal sum of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^{n\alpha} = a_0 + a_1(x-c)^{\alpha} + a_2(x-c)^{2\alpha} + \cdots,$$

where $\alpha > 0$ $x \geq c$, c is a constant, x is a variable, and $a_n's$ are the coefficients of the series.

As in the case of ordinary power series, the fractional power series has interval of convergence and radius of convergence, which we state in the following two theorems (see [18]).

Theorem 2.3. *We have two cases for the FPS* $\sum_{n=0}^{\infty} a_n x^{n\alpha}$, $x \geq 0$*:*

(1) If the FPS converges when $x = b > 0$*, then it converges whenever* $0 \leq x < b$*,*

(2) If the FPS diverges when $x = d > 0$*, then it diverges whenever* $x > d$*.*

Theorem 2.4. *The PS at* $c = 0$ *and* $-\infty < x < \infty$ *has radius of convergence* R *if and only if the FPS at* $c = 0$ *and* $x \geq 0$ *has radius of convergence* $R^{\frac{1}{\alpha}}$*.*

3. **Residual Power Series Method (RPSM)**

3.1. *Methodology of RPSM*

In this part, we solve 2-dimensional fractional Fredholm integro-differential equations using the residual power series approach. How to handle the evaluation of the coefficient functions that show up in the integral term is our contribution. To the best of our knowledge, we are the first to solve these kinds of problems using RPSM. Consider the following: Integro-differential equation of Fredholm

$$D_t^{\alpha} u(x,t,I) = f(x,t,I) + g(x,I) \int_0^a \int_0^b K(t,y,\tau,I) u(y,\tau,I) d\tau dy, \tag{3}$$

based on the condition

$$u(x,0,I) = C_0(x,I) \tag{4}$$

where $0 < \alpha \leq 1$ and $(x,t,I) \in [0,a] \times [0,b]$.
It is assumed by the residual power series approach [28] that the fractional power series form of the solution of (3) can be expanded

$$u(x,t) = \sum_{n=0}^{\infty} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

and defines the N^{th} approximation of $u(x,t)$ by

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{5}$$

Moreover, the k^{th} residual function for (3) is defined by

$$\mathrm{Res}_{u,k}(x,t) = D_t^{\alpha} u_k(x,t) - f(x,t) - g(x) \int_0^a \int_0^b K(t,y,\tau) u_N(y,\tau) d\tau dy. \tag{6}$$

Note that the fact that Caputo derivative of a constant is 0 implies that

$$D_t^{(k-1)\alpha} \mathrm{Res}_{u,k}(x,t)\Big|_{t=0} = 0, \ \texttt{for} \ k = 1,2,\ldots .$$

We must compute the function coefficients $C_1(x), C_2(x), ..., C_N(x)$ in order to get an approximate solution of N^{th}, where $C_0(x) = u(x,0)$ is already provided. Therefore, we build the system.

$$D_t^{(k-1)\alpha} \mathrm{Res}_{u,k}(x,t)\Big|_{t=0} = 0, \ \texttt{for} \ k = 1,2,...N. \tag{7}$$

We simplify (7) by computing

$$D_t^{(k-1)\alpha}\mathrm{Res}_{u,k}(x,t)\Big|_{t=0} = D_t^{(k-1)\alpha}\left(D_t^{\alpha}\left(\sum_{n=0}^{k} C_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right) - f(x,t) - g(x)\int_0^a\int_0^b K(t,y,\tau)\left(\sum_{n=0}^{k} C_n(y)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)d\tau dy\right)\Big|_{t=0}$$

$$= \sum_{n=0}^{k} C_n(x)\frac{D_t^{k\alpha}[t^{n\alpha}]}{\Gamma(n\alpha+1)} - D_t^{(k-1)\alpha}f(x,t) - g(x)\int_0^a\int_0^b D_t^{(k-1)\alpha}[K(t,y,\tau)]\left(\sum_{n=0}^{k} C_n(y)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)d\tau dy\Big|_{t=0}$$

$$= C_k(x) - D_t^{(k-1)\alpha}f(x,0) - g(x)\int_0^a\int_0^b D_t^{(k-1)\alpha}K(0,y,\tau)\left(\sum_{n=0}^{k} C_n(y)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)d\tau dy$$

$$= C_k(x) - D_t^{(k-1)\alpha}f(x,0) - g(x)\sum_{n=0}^{k}\int_0^a C_n(y)\left(\int_0^b D_t^{(k-1)\alpha}K(0,y,\tau)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}d\tau\right)dy.$$

If we set

$$h_{k,n}(y) = \int_0^b D_t^{(k-1)\alpha}K(0,y,\tau)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}d\tau, \quad \text{for } n = 0,1,2,...,k, \tag{8}$$

and

$$\eta_{k,n} = \int_0^a C_n(y)h_{k,n}(y)dy, \quad \text{for } n = 0,1,2,...,k. \tag{9}$$

Then, for $k = 1,2,3,...,N$, we have

$$D_t^{(k-1)\alpha}\mathrm{Res}_{u,k}(x,t)\Big|_{t=0} = C_k(x) - D_t^{(k-1)\alpha}f(x,0) - g(x)\sum_{n=0}^{k}\eta_{k,n} = 0,$$

or equivalently,

$$C_k(x) = D_t^{(k-1)\alpha}f(x,0) + g(x)\sum_{n=0}^{k}\eta_{k,n} = 0, \tag{10}$$

Our next step is to calculate $\eta_{k,n}$ for $n = 0,1,2,...,k$ and $k = 1,2,3,...,N$. The process is recursive. In Equation (10), enter $k = 1$ to obtain

$$C_1(x) = f(x,0) + g(x)\eta_{1,0} + g(x)\eta_{1,1}. \tag{11}$$

Observe that

$$\eta_{1,0} = \int_0^a\int_0^b K(0,y,\tau)C_0(y)d\tau dy.$$

Thus, $\eta_{1,1}$ is found in terms of $\eta_{1,0}$. Multiplying Equation (11) by $h_{1,1}(x)$ and integrating over $[0, a]$ will do this.

$$\eta_{1,1} - \int_0^a h_{1,1}(x) \left(f(x,0) + \eta_{1,0} g(x)\right) dx - \eta_{1,1} \int_0^a h_{1,1}(x) g(x) dx = 0.$$

It follows that

$$\eta_{1,1} = \frac{\int_0^a h_{1,1}(x) \left(f(x,0) + \eta_{1,0} g(x)\right) dx}{1 - \int_0^a h_{1,1}(x) g(x) dx},$$

and finally,

$$C_1(x) = f(x,0) + g(x) \left(\eta_{1,0} + \eta_{1,1}\right).$$

Now, plug $k = 2$ in Equation 10:

$$C_2(x) = D_t^{\alpha} f(x,0) + g(x)\eta_{2,0} + g(x)\eta_{2,1} + g(x)\eta_{2,2} \tag{12}$$

We find only $\eta_{2,2}$ because, once more, $\eta_{2,0}$ and $\eta_{2,1}$ are known. As in $\eta_{1,1}$, we solve for $\eta_{2,2}$ by multiplying Equation (12) by $h_{2,2}(x)$, integrating over $[0, a]$.

$$\eta_{2,2} = \frac{\int_0^a h_{2,2}(x) \left(D_t^{\alpha} f(x,0) + \eta_{2,0} g(x) + \eta_{2,1} g(x)\right) dx}{1 - \int_0^a h_{2,2}(x) g(x) dx}.$$

Inductively, we have

$$C_k(x) = D_t^{(k-1)\alpha} f(x,0) + g(x) \sum_{n=0}^{k} \eta_{k,n},$$

where

$$\eta_{k,n} = \int_0^a \int_0^b \frac{\tau^{n\alpha}}{\Gamma(n\alpha + 1)} D_t^{(k-1)\alpha} K(0, y, \tau) C_n(y) d\tau dy, \quad \text{for } n = 0, 1, 2, ..., k-1, \tag{13}$$

and

$$\eta_{k,k} = \frac{\int_0^a h_{k,k}(x) D_t^{(k-1)\alpha} f(x,0) dx + \int_0^a g(x) h_{k,k}(x) dx \sum_{n=0}^{k-1} \eta_{k,n}}{1 - \int_0^a h_{k,k}(x) g(x) dx}. \tag{14}$$

4. **Applications of FPSM to Linear Fractional Problem**

Here, we go over two cases to illustrate the process of the suggested approach. Following a review of the findings, we will offer some suggestions and remarks.

Example 4.1. Consider the two-dimensional linear fractional Fredholm integro-differential equation:

$$D_t^{0.5} u(x,t) = -\frac{4}{675} + \frac{3}{4}\sqrt{\pi}\, t\, x^2 + \int_0^1 \int_0^1 (y^2 - \tau^2) u(y,\tau)\, d\tau\, dy, \tag{15}$$

subject to the initial condition $u(x, 0) = 4$. The exact solution is given by: $u(x,t) = x^2\, t^{3/2} + 4$. and $f(x,t) = -\frac{4}{675} + \frac{3}{4}\sqrt{\pi}\, t\, x^2$ and $g(x) = 1$ provide the source term in this equation. According to the technique, we assume that the approximate solution takes the following form:

$$u(x,t) = \sum_{n=0}^{4} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)},$$

In this case, we employ $\alpha = 0.5$ and the first four fractional powers of the fractional power series expansion. To calculate $C_0(x)$, we simply utilize the initial condition, which yields:

$$C_0(x) = u(x, 0) = 4.$$

While, the coefficient $C_1(x)$ is given by:

$$\begin{aligned} C_1(x) &= f(x, 0) + g(x)\,(\eta_{1,0} + \eta_{1,1}) \\ &= -\frac{4}{675} + \eta_{1,0} + \eta_{1,1} = \frac{32}{5400 + 42525\sqrt{\pi}} - \frac{4}{675}. \end{aligned}$$

Therefore, the first residual power series approximate solution is:

$$u_1(x, t) = 4 - \frac{56\sqrt{t}}{75(8 + 63\sqrt{\pi})}.$$

Next, in order to compute $C_2(x)$, it is to be noted that the coefficient $C_2(x)$ is given by:

$$\begin{aligned} C_2(x) &= D_t^{\frac{1}{2}} f(x, 0) + g(x)\,(\eta_{2,0} + \eta_{2,1} + \eta_{2,2}) \\ &= \eta_{2,0} + \eta_{2,1} + \eta_{2,2} = 0. \end{aligned}$$

As a result, the second approximate residual power series solution is:

$$u_2(x, t) = 4 - \frac{56\sqrt{t}}{75(8 + 63\sqrt{\pi})}.$$

Finally, we compute $C_3(x)$. The coefficient $C_3(x)$ is given by:

$$C_3(x) = D_t^{(2)\frac{1}{2}} f(x, 0) + g(x) \sum_{n=0}^{3} \eta_{3,n} = \frac{3\sqrt{\pi}x^2}{4}.$$

Consequently, the following is the estimated third residual power series solution:

$$u_3(x, t) = 4 - \frac{56\sqrt{t}}{75(8 + 63\sqrt{\pi})} + t^{\frac{3}{2}}x^2.$$

We provide Table 1, which displays the absolute errors at chosen sites for particular values of x and t, to demonstrate the accuracy of the approximate solutions. Additionally, Figure (1) shows the comparison of the approximate and exact results.

Example 4.2. Consider the fractional Fredholm integro-differential equation (1) as

$$D_t^{\alpha} u(x, t) = f(x, t) + g(x) \int_0^1 \int_0^1 K(t, y, \tau) u(y, \tau)\, d\tau\, dy, \tag{16}$$

subject to the initial condition:

$$u(x, 0) = C_0(x), \tag{17}$$

where:

$$f(x, t) = -(2/3) + e^t \operatorname{Erf}(\sqrt{t}), \quad K(t, y, \tau) = y\tau, \quad g(x) = 1, \quad C_0(x) = 1 + x, \quad \text{and } \alpha = 0.5.$$

t	x	$u(x,t)$	$u_3(x,t)$	$\lvert u(x,t)-u_3(x,t)\rvert$
	0.1	4.00032	3.999125	1.19×10^{-3}
	0.2	4.00127	3.99956	1.71×10^{-3}
	0.3	4.00255	4.00168	1.04×10^{-3}
	0.4	4.00506	4.00475	8.70×10^{-4}
0.1	0.5	4.00791	4.00854	6.31×10^{-4}
	0.6	4.01139	4.01285	1.46×10^{-3}
	0.7	4.01549	4.01767	2.18×10^{-3}
	0.8	4.02020	4.02918	2.91×10^{-3}
	0.9	4.02554	4.02918	3.63×10^{-3}
	0.1	4.10207	4.10254	4.70×10^{-4}
	0.2	4.05578	4.05879	3.01×10^{-3}
	0.3	4.08795	4.08145	6.50×10^{-3}
	0.4	4.25454	4.25987	5.33×10^{-3}
0.3	0.5	4.38511	4.38125	3.86×10^{-3}
	0.6	4.41325	4.41456	1.31×10^{-3}
	0.7	4.65421	4.65897	4.76×10^{-3}
	0.8	4.81250	4.81453	2.03×10^{-3}
	0.9	4.85652	4.85132	5.20×10^{-3}
	0.1	4.32315	4.32520	2.05×10^{-3}
	0.2	4.15864	4.15352	5.12×10^{-3}
	0.3	4.30195	4.30315	1.20×10^{-3}
	0.4	4.36862	4.36121	7.41×10^{-3}
0.5	0.5	4.55385	4.55982	5.97×10^{-3}
	0.6	4.81148	4.81671	5.23×10^{-3}
	0.7	4.10541	4.10662	1.21×10^{-3}
	0.8	4.44472	4.44802	3.30×10^{-3}
	0.9	4.82362	4.82555	1.93×10^{-3}

TABLE 1. Absolute Error Analysis for the Approximate Solutions at Selected Points for Example 4.1

The exact solution is $u(x,t)=e^t+x$. It is assumed that the RPSM solution can be expressed as follows in order to determine the problem's approximate solution:

$$u(x,t)=\sum_{n=0}^{\infty}C_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{18}$$

where the N-th approximation is:

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{19}$$

Our next objective is to determine the coefficients and residual function: The residual function for that is:

$$\mathrm{Res}_{u,k}(x,t) = D_t^{\alpha} u_k(x,t) - f(x,t) - g(x) \int_0^a \int_0^b K(t,y,\tau) u_N(y,\tau)\, d\tau\, dy. \tag{20}$$

Recursively, the coefficients $C_k(x)$ are calculated using:

$$C_k(x) = D_t^{(k-1)\alpha} f(x,0) + g(x) \sum_{n=0}^{k} \eta_{k,n}, \tag{21}$$

$$\eta_{k,n} = \int_0^a C_n(y) \int_0^b \frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)} D_t^{(k-1)\alpha} K(0,y,\tau)\, d\tau\, dy. \tag{22}$$

For $k = 1$, we compute:

$$C_1(x) = f(x,0) + g(x)(\eta_{1,0} + \eta_{1,1}),$$

where:

$$\eta_{1,0} = \int_0^a \int_0^b K(0,y,\tau) C_0(y)\, d\tau\, dy.$$

We now determine the recursive coefficient computation. When the coefficients are greater ($k \geq 2$):

$$C_k(x) = D_t^{(k-1)\alpha} f(x,0) + g(x) \sum_{n=0}^{k} \eta_{k,n}, \tag{23}$$

$$\eta_{k,k} = \frac{\int_0^a h_{k,k}(x) D_t^{(k-1)\alpha} f(x,0) dx + \int_0^a g(x) h_{k,k}(x) \sum_{n=0}^{k-1} \eta_{k,n} dx}{1 - \int_0^a h_{k,k}(x) g(x) dx}. \tag{24}$$

For this example:

$$C_1(x) = -(1/4) - 1/(-4 + 10\sqrt{\pi}), \quad C_2(x) = 1, \quad C_3(x) = 0, \quad C_4(x) = 1, \ldots$$

Finally, the approximate solution is:

$$u_{\mathrm{app}}(x,t) = C_0(x) + \sum_{n=1}^{N} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{25}$$

where:

$$u_{\mathrm{app}}(x,t) = 1 + x + \frac{2\left(-(1/4) - 1/(-4+10\sqrt{\pi})\right)\sqrt{t}}{\sqrt{\pi}} + t + \frac{t^2}{2} + \frac{t^3}{6}.$$

In order to study the error analysis, we refer to table 2 that present the absolute error for various values of x and t. This illustration shows how to approximate solutions for fractional integro-differential equations using the Residual Power Series Method (RPSM).

t	x	Approximate Solution	Absolute Error
0.1	0.0	0.989961	0.104246
	0.1	1.089960	0.095596
	0.2	1.189960	0.088272
	0.3	1.289960	0.081990
	0.4	1.389960	0.076543
	0.5	1.489960	0.071774
0.3	0.0	1.149960	0.148090
	0.1	1.249960	0.137876
	0.2	1.349960	0.128980
	0.3	1.449960	0.121162
	0.4	1.549960	0.114238
	0.5	1.649960	0.108062
0.5	0.0	1.388230	0.157998
	0.1	1.488230	0.148963
	0.2	1.588230	0.140905
	0.3	1.688230	0.133675
	0.4	1.788230	0.127150
	0.5	1.888230	0.121232

TABLE 2. Absolute Error Analysis for the Approximate Solutions at Selected Points for Example 4.2

5. **Modified Fractional Power Series Method**

In order to address the same problem, the power series solution approach is modified to fractional order in this section's work.

5.1. *Methodology of Modified Fractional Power Series Method (MFPSM)*

In this section we present a new method using fractional series to solve fractional integro-differential equations of two dimensions. More precisely, we consider the following fractional integro-differential equation

$$D_t^{\alpha} u(x,t) = f(x,t) + \int_0^a \int_0^b K(y,\tau) u(y,\tau) dy d\tau, \tag{26}$$

subject to the initial condition

$$u(x,0) = C_0(x),$$

where $0 < \alpha \leq 1$ and $(x,t) \in [0,a] \times [0,b]$.

We assume the solution has the fractional series expansion

$$u(x,t)=\sum_{n=0}^{\infty}C_n(x)t^{n\alpha}. \tag{27}$$

Next, we approximate $u(x,t)$ by

$$u(x,t)\approx u_N(x,t)=\sum_{n=0}^{N}C_n(x)t^{n\alpha}.$$

Where $C_n(x)$ are unknown function coefficients to determined. Using the notation $\beta!=\Gamma(\beta+1)$, the fact that

$$D_t^{\alpha}\left(t^{n\alpha}\right)=\frac{(n\alpha)!}{((n-1)\alpha)!}t^{(n-1)\alpha},$$

and substituting $u_N(x,t)$ in (26) to get

$$\begin{aligned}D_t^{\alpha}(C_0(x)+C_1(x)t^{\alpha}+C_2(x)t^{2\alpha}+\cdots+C_N(x)t^{N\alpha})=f(x,t)\\+\int_0^a\int_0^b K(y,\tau)\left(C_0(y)+\cdots+C_N(y)t^{N\alpha}\right)d\tau dy.\end{aligned}$$

Define

$$A=\int_0^a\int_0^b K(y,\tau)\left(C_0(y)+\cdots+C_N(y)t^{N\alpha}\right)d\tau dy, \tag{28}$$

then we have

$$C_1(x)\alpha!+C_2(x)\frac{2\alpha!}{\alpha!}t^{\alpha}+\cdots+C_N(x)\frac{(N\alpha)!}{((N-1)\alpha)!}t^{(N-1)\alpha}=f(x,t)+A. \tag{29}$$

Now, evaluate Equation (29) at $t=0$ to obtain

$$C_1(x)\alpha!=f(x,0)+A.$$

Thus

$$C_1(x)=\frac{f(x,0)+A}{\alpha!}, \tag{30}$$

where A still unknown constant. Next, by differentiating Equation (29) α-fractional derivative and evaluating at $t=0$, one can have

$$C_2(x)(2\alpha)!=D_t^{\alpha}f(x,0),$$

thus

$$C_2(x)=\frac{D_t^{\alpha}f(x,0)}{(2\alpha)!}.$$

Inductively, differentiate Equation (29) n α-fractional derivatives and evaluate at $t=0$, to get

$$C_n(x)=\frac{D_t^{(n-1)\alpha}f(x,0)}{(n\alpha)!},\qquad n=2,3,4,\cdots,N. \tag{31}$$

Note that the coefficients $C_n(x)$ are known now, except for $n = 1$. Now, to evaluate $C_1(x)$ we need to compute the constant A. Go back to

$$A = \sum_{n=0}^{N} \int_0^a \int_0^b K(y,\tau) C_n(y) \tau^{n\alpha} d\tau dy,$$

and set

$$K_n = \int_0^a \int_0^b K(y,\tau) C_n(y) \tau^{n\alpha} d\tau dy.$$

Then,

$$\begin{aligned}
K_0 &= \int_0^a \int_0^b K(y,\tau) u(y,0) d\tau dy, \\
K_1 &= \int_0^a \int_0^b K(y,\tau) \frac{f(y,0) + A}{\alpha!} \tau^{\alpha} d\tau dy, \text{ and} \\
K_n &= \int_0^a \int_0^b K(y,\tau) \frac{D_t^{(n-1)\alpha} f(y,0) \tau^{n\alpha}}{(n\alpha)!} d\tau dy, \text{ for} n = 2,3,4,...,N.
\end{aligned}$$

Hence,

$$\begin{aligned}
A &= \sum_{n=0}^{N} K_n \\
&= (K_0 + K_2 + \cdots + K_N) + K_1 \\
&= (K_0 + K_2 + \cdots + K_N) + \int_0^a \int_0^b K(y,\tau) \frac{f(y,0)}{\alpha!} \tau^{\alpha} d\tau dy + A \int_0^a \int_0^b \frac{K(y,\tau)\tau^{\alpha}}{\alpha!} d\tau dy.
\end{aligned}$$

If we solve for A, we find that

$$A = \frac{K_0 + \sum_{n=2}^{N} K_n + \int_0^a \int_0^b K(y,\tau) \frac{f(y,0)}{\alpha!} \tau^{\alpha} d\tau dy}{1 - \int_0^a \int_0^b \frac{K(y,\tau)\tau^{\alpha}}{\alpha!} d\tau dy}. \tag{32}$$

Finally, we summaries the algorithm of the modified fractional power series method in the following steps. **Step 1.** Assume that the approximate solution of the form

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) t^{\alpha n}.$$

Step 2. Compute the coefficient functions

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}, \qquad n = 2,3,4,\cdots,N.$$

Step 3. Compute the constant

$$A = \frac{\sum_{n=0}^{N} A_n}{1 - \int_0^a \int_0^b \frac{K(y,\tau)\tau^{\alpha}}{\alpha!} d\tau dy},$$

where,

$$A_0 = \int_0^a \int_0^b K(y,\tau)u(y,0)d\tau dy,$$
$$A_n = \int_0^a \int_0^b K(y,\tau)\frac{D_t^{(n-1)\alpha} f(y,0)\tau^{n\alpha}}{(n\alpha)!}d\tau dy, \quad \text{for } n = 1,2,3,4,...,N.$$

Step 4. Finally, compute

$$C_1(x) = \frac{f(x,0) + A}{\alpha!}.$$

Therefore,

$$u(x,t) \approx \sum_{n=0}^{N} C_n(x)t^{\alpha n}$$

is now determined.

6. Convergence and Error Analysis of (MFPSM)

Novel results associated with the convergent of the (MFPSM) are studies in this section. The analysis show a fast convergence rate and small calculation error.

6.1. *Convergence of MFPSM*

In this section, we give some details analysis for the convergence of the modified approache.

Theorem 6.1. *Let the fractional integro-differential equation be given by*

$$D_t^{\alpha} u(x,t) = f(x,t) + \int_0^a \int_0^b K(x,t)u(x,t)\,dt\,dx,$$

with initial condition $u(x,0) = C_0(x)$*, where* $0 < \alpha \leq 1$ *and* $(x,t) \in [0,a] \times [0,b]$*. The series solution*

$$u(x,t) = \sum_{n=0}^{\infty} C_n(x)t^{n\alpha}$$

converges to the exact solution, and the truncated approximation

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x)t^{n\alpha}$$

converges uniformly to $u(x,t)$ *as* $N \to \infty$*, where the coefficients* $C_n(x)$ *satisfy:*

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}.$$

Proof. To demonstrate that the truncated series $u_N(x,t)$ converges to the exact solution $u(x,t)$.

$$E_N(x,t) = u(x,t) - u_N(x,t) = \sum_{n=N+1}^{\infty} C_n(x)t^{n\alpha}.$$

The coefficients $C_n(x)$ are determined as:

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}.$$

Assuming the fractional derivative $D_t^{(n-1)\alpha} f(x,0)$ is bounded for all n, there is a constant M_n according to which

$$|C_n(x)| \leq \frac{M_n}{(n\alpha)!}.$$

Therefore, the error term can be bounded by:

$$|E_N(x,t)| \leq \sum_{n=N+1}^{\infty} \frac{M_n |t|^{n\alpha}}{(n\alpha)!}.$$

We use the Weierstrass M-test to demonstrate uniform convergence. Let:

$$M_n = \sup_{x\in[0,a]} |D_t^{(n-1)\alpha} f(x,0)|.$$

Then:

$$\sum_{n=N+1}^{\infty} \frac{M_n |t|^{n\alpha}}{(n\alpha)!}$$

The factorial term $(n\alpha)!$ in the denominator rapidly grows, causing the series to converge. Thus, the series $\sum_{n=0}^{\infty} C_n(x)t^{n\alpha}$ converges uniformly on any compact subset of $[0,a] \times [0,b]$, implying

$$\lim_{N\to\infty} u_N(x,t) = u(x,t).$$

As a result, the MFPSM has achieved convergence. □

6.2. *Error Order of MFPSM*

This subsection discusses novel results on the rate error order of the MFPSM.

Theorem 6.2. *For the approximate solution given by:*

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x)t^{n\alpha},$$

the truncation error is of the order:

$$|E_N(x,t)| = |u(x,t) - u_N(x,t)| = \mathcal{O}\left(\frac{|t|^{(N+1)\alpha}}{((N+1)\alpha)!}\right).$$

Proof. Consider the error term:

$$E_N(x,t) = \sum_{n=N+1}^{\infty} C_n(x)t^{n\alpha}.$$

Recall that

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}.$$

Thus, the error can be bounded as:

$$|E_N(x,t)| \leq \sum_{n=N+1}^{\infty} \frac{|D_t^{(n-1)\alpha} f(x,0)||t|^{n\alpha}}{(n\alpha)!}.$$

Using Stirling's approximation for the factorial term $(n\alpha)!$ for large n.

$$(n\alpha)! \approx \sqrt{2\pi n\alpha}\left(\frac{n\alpha}{e}\right)^{n\alpha}.$$

As n increases, the terms $C_n(x)t^{n\alpha}$ decrease rapidly. For large N, the dominant term in the error sum is given by:

$$|E_N(x,t)| \approx \frac{|D_t^{N\alpha} f(x,0)||t|^{(N+1)\alpha}}{((N+1)\alpha)!}.$$

Consequently, the error is of the order:

$$|E_N(x,t)| = \mathcal{O}\left(\frac{|t|^{(N+1)\alpha}}{((N+1)\alpha)!}\right).$$

The factorial growth in the denominator causes a rapid decrease in error as N increases, confirming the method's high-order convergence. □

7. **Solving Equation (1) by MFPSM**

Here we will present some examples to study the effectiveness of the newly developed method (MFPSM) and its use in approximating solutions to the fractional integro-differential equations under study.

Example 7.1. Consider the two-dimensional linear fractional Fredholm integro-differential equation provided by:

$$D_t^{0.5} u(x,t) = \frac{\sqrt{\pi}}{2}e^x - \frac{2(e-1)t}{3} + \int_0^1 \int_0^1 u(y,\tau)t\, d\tau\, dy. \tag{33}$$

Subject to the initial condition $u(x,0) = 0$. The exact solution is given by $u(x,t) = \sqrt{t}e^x$. To solve the given equation, we use the Modified Fractional Power Series Method (MFPSM). First, we assume an approximate solution, where $\alpha = 0.5$, to be in the following form:

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x)t^{n/2}, \tag{34}$$

Next, we compute the coefficient functions $C_n(x)$ given by:

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}, \quad n = 2, 3, 4, \ldots, N. \tag{35}$$

Given that $D_t^{0.5} u(x,t) = \frac{\sqrt{\pi}}{2} e^x - \frac{2(e-1)t}{3}$, we first focus on the constant A using the integral terms. The constant A is given by:

$$A = \frac{\sum_{n=0}^{N} A_n}{1 - \int_0^a \int_0^b \frac{K(y,\tau)\tau^\alpha}{\alpha!}\, d\tau\, dy}, \tag{36}$$

where:

$$A_0 = \int_0^a \int_0^b K(y,\tau) u(y,0)\, d\tau\, dy,$$
$$A_n = \int_0^a \int_0^b K(y,\tau) \frac{D_t^{(n-1)\alpha} f(y,0)\tau^{n\alpha}}{(n\alpha)!}\, d\tau\, dy, \quad \text{for } n = 1, 2, 3, \ldots, N.$$

In the above integral equation, the kernel function is $K(y,\tau) = \tau$. Since $u(y,0) = 0$, we have:

$$A_0 = \int_0^1 \int_0^1 \tau \cdot u(y,0)\, d\tau\, dy = 0. \tag{37}$$

For $n \geq 1$:

$$A_n = \int_0^1 \int_0^1 \tau \cdot \frac{D_t^{(n-1)\alpha} f(y,0) \cdot \tau^{n\alpha}}{(n\alpha)!}\, d\tau\, dy. \tag{38}$$

Since $\alpha = 0.5$, we have $n\alpha = 0.5n$. Thus:

$$A_n = \int_0^1 \int_0^1 \frac{\tau^{1+0.5n}}{(n/2)!} \cdot D_t^{(n-1)\alpha} f(y,0)\, d\tau\, dy. \tag{39}$$

Given that the source term $f(x,t) = \frac{\sqrt{\pi}}{2} e^x - \frac{2(e-1)t}{3}$ yields zero higher-order derivatives at $t = 0$ for $n \geq 2$, we have:

$$D_t^{(n-1)\alpha} f(y,0) = 0 \quad \text{for } n \geq 2 \implies A_n = 0 \quad \text{for } n \geq 2. \tag{40}$$

To compute A_1, we have

$$A_1 = \int_0^1 \int_0^1 \frac{\tau \cdot D_t^0 f(y,0) \cdot \tau^{0.5}}{(0.5)!}\, d\tau\, dy = \int_0^1 \int_0^1 \frac{\tau^{1.5} \cdot \frac{\sqrt{\pi}}{2} e^y}{\Gamma(1.5)}\, d\tau\, dy. \tag{41}$$

Using $\Gamma(1.5) = \frac{\sqrt{\pi}}{2}$, we arrive at:

$$A_1 = \int_0^1 \int_0^1 e^y \cdot \tau^{1.5}\, d\tau\, dy. \tag{42}$$

Computing simple integrals, we obtain

$$A_1 = \int_0^1 e^y \cdot \frac{2}{5}\, dy = \frac{2}{5} \left[e^y\right]_0^1 = \frac{2}{5}(e-1). \tag{43}$$

That leads to the values of

$$A = \frac{\frac{2}{5}(e-1)}{1 - \frac{2}{5}} = \frac{\frac{2}{5}(e-1)}{\frac{3}{5}} = \frac{2}{3}(e-1). \tag{44}$$

and,

$$C_1(x) = \frac{f(x,0) + A}{\alpha!} = \frac{\frac{\sqrt{\pi}}{2}e^x + \frac{2}{3}(e-1)}{\frac{\sqrt{\pi}}{2}} = e^x. \tag{45}$$

For Higher-Order Terms $C_n(x)$, when $n \geq 2$ we have

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!} = 0 \quad \text{(since higher derivatives vanish)}. \tag{46}$$

Finally, the approximate solution, which matches the exact solution, and confirming the accuracy of the MFPSM is provided by

$$u_N(x,t) = \sum_{n=0}^{1} C_n(x) t^{n/2} = C_0(x) + C_1(x)t^{0.5} = 0 + e^x \cdot t^{0.5} = \sqrt{t}e^x. \tag{47}$$

Example 7.2. Consider the fractional Fredholm integro-differential equation:

$$D_t^{\alpha} u(x,t) = f(x,t) + g(x) \int_0^1 \int_0^1 K(t,y,\tau) u(y,\tau)\, d\tau\, dy, \tag{48}$$

subject to the initial condition:

$$u(x,0) = C_0(x), \tag{49}$$

where,

$$f(x,t) = -(2/3) + e^t \operatorname{Erf}(\sqrt{t}), \quad K(t,y,\tau) = y\tau, \quad g(x) = 1, \quad C_0(x) = 1 + x, \quad \text{and } \alpha = 0.5.$$

The exact solution is:

$$u(x,t) = e^t + x.$$

Define auxiliary function $f_1(x,t)$ by

$$f_1(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^{\alpha}}\, d\tau - \int_0^1 \int_0^1 K(x,t) u(x,t)\, dx\, dt. \tag{50}$$

Simplified, this becomes:

$$f(x,t) = -\frac{2}{3} + e^t \operatorname{Erf}(\sqrt{t}).$$

The coefficients $C_n(x)$ are calculated as:

$$C_n(x) = \left. \frac{\partial^{(n-1)\alpha} f(x,t)}{\Gamma(n\alpha+1)} \right|_{t=0}. \tag{51}$$

This generates the values

$$C_2 = 1,\ C_3 = 0,\ C_4 = \frac{1}{2},\ C_5 = 0,\ C_6 = \frac{1}{6},\ C_7 = 0,\ C_8 = \frac{1}{24}, \ldots$$

While, the constant A is computed as:

$$A = \frac{\sum_{n=2}^{M} K_n + K_0 + \int_0^1 \int_0^1 K(x,t) \frac{f_1(x,0)}{\Gamma(\alpha+1)} t^{\alpha}\, dt\, dx}{1 - \int_0^1 \int_0^1 K(x,t) \frac{1}{\Gamma(\alpha+1)} t^{\alpha}\, dt\, dx}. \tag{52}$$

Here:

$$K_n = \int_0^1 \int_0^1 K(x,t)C_n(x)t^{n\alpha}\,dt\,dx,$$
$$K_0 = \int_0^1 \int_0^1 K(x,t)C_0(x)\,dt\,dx.$$

Upon passing simple calculations, we arrive at the value $A \approx 0.666667$. The approximate solution is given by:

$$u_{\text{app}}(x,t) = C_0(x) + C_1(x)t^{\alpha} + \sum_{n=2}^{M} C_n(x)t^{n\alpha}. \tag{53}$$

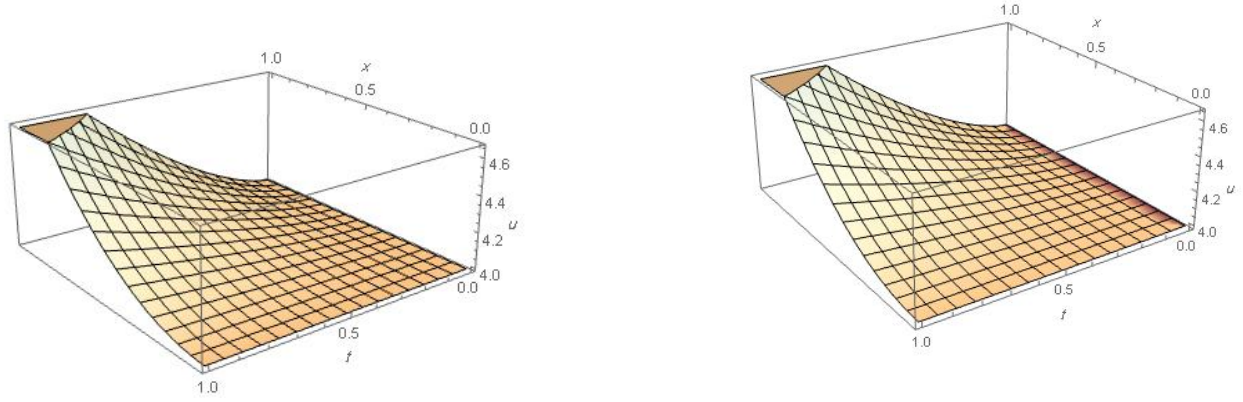
Substituting the coefficients, the solution becomes:

$$u_{\text{app}}(x,t) = 1 + x - 1.52113 \times 10^{-9}\sqrt{t} + t + 0.5t^2 + 0.166667t^3 + \dots$$

In regrads to the error analysis, Table (3) present the absolute error for various values of x and t. The efficiency of the Modified Fractional Power Series Method (MFPSM) in approximating solutions to fractional integro-differential equations is illustrated by this example.

t	x	Absolute Error
0.1	0.0	8.5223×10^{-8}
	0.1	8.5223×10^{-8}
	0.2	8.5223×10^{-8}
	0.3	8.5223×10^{-8}
	0.4	8.5223×10^{-8}
	0.5	8.5223×10^{-8}
0.3	0.0	2.1308×10^{-5}
	0.1	2.1308×10^{-5}
	0.2	2.1308×10^{-5}
	0.3	2.1308×10^{-5}
	0.4	2.1308×10^{-5}
	0.5	2.1308×10^{-5}
0.5	0.0	2.8377×10^{-4}
	0.1	2.8377×10^{-4}
	0.2	2.8377×10^{-4}
	0.3	2.8377×10^{-4}
	0.4	2.8377×10^{-4}
	0.5	2.8377×10^{-4}

TABLE 3. Absolute Error for the Approximate Solutions at Selected Points for Example 7.2



(A) Exact solution. (B) The third RPS approximation.

FIGURE 1. The behavior of the third RPS approximation together with exact solution for Example 4.1.

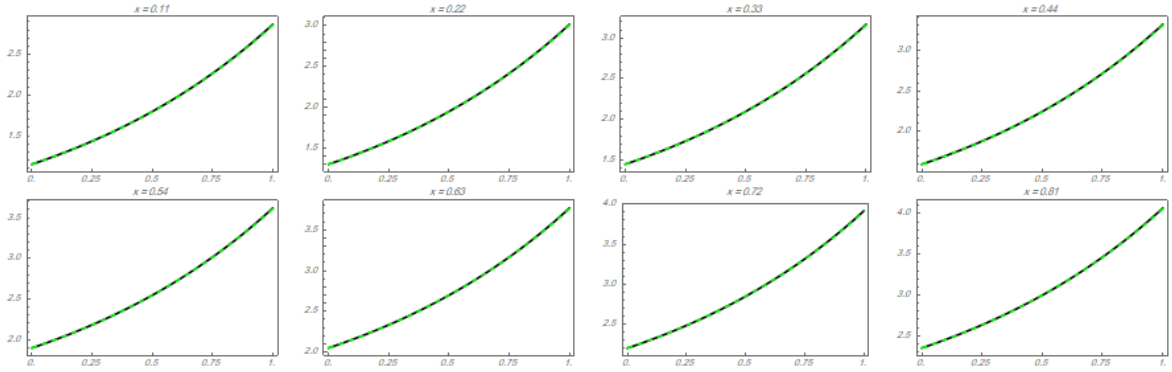


FIGURE 2. Approximate solution for different values of x in Example 7.2.

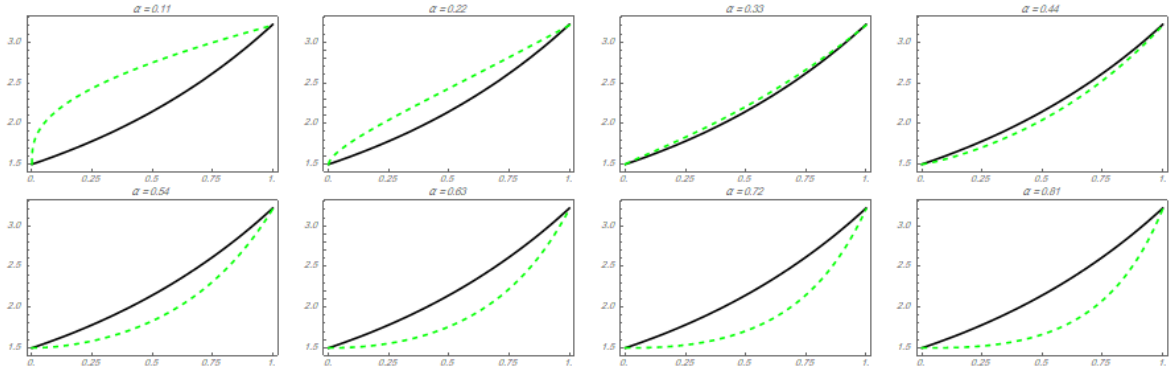


FIGURE 3. Approximate solution for different values of α in Example 7.2.

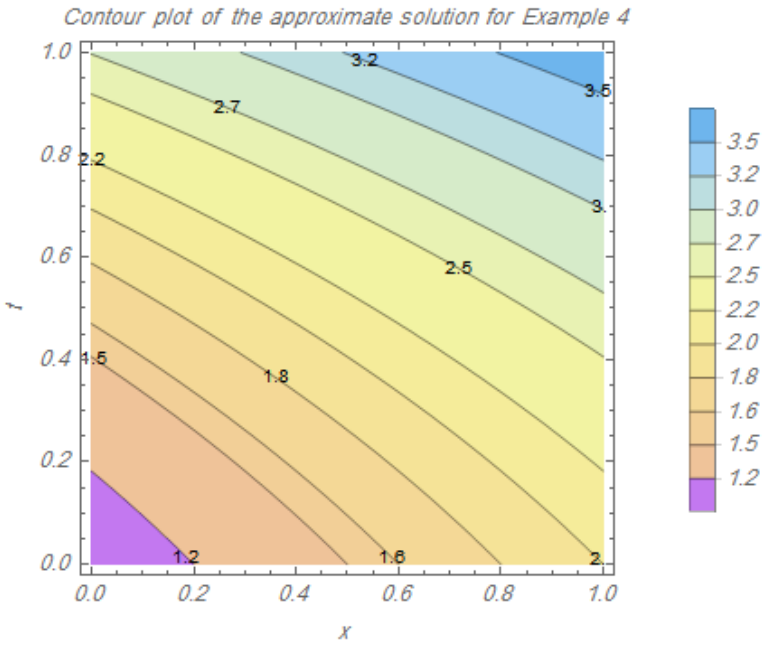


FIGURE 4. The approximate solution in Example 7.2 .

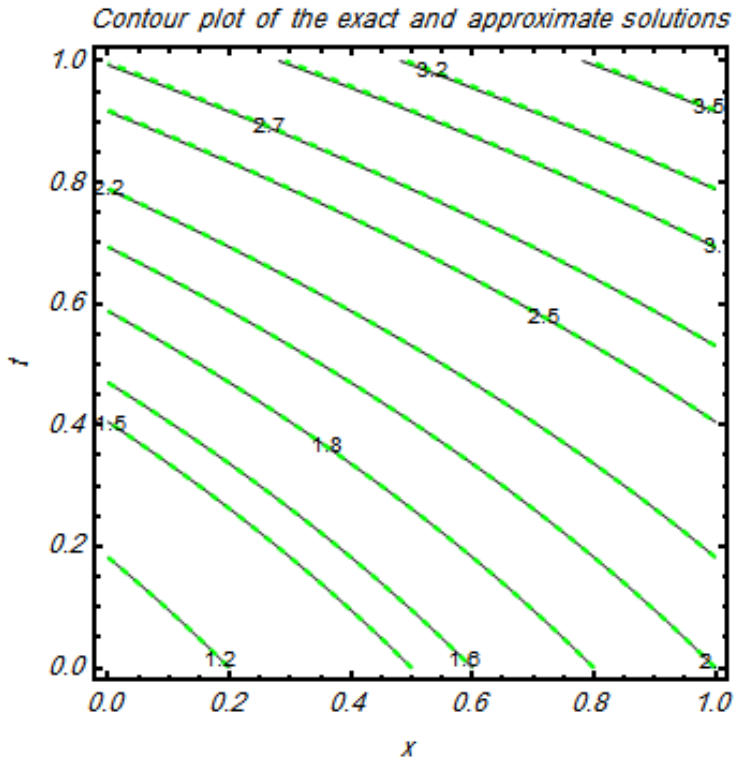


FIGURE 5. Values for both exact (Black) and approximate (Green) solutions in Example 7.2.

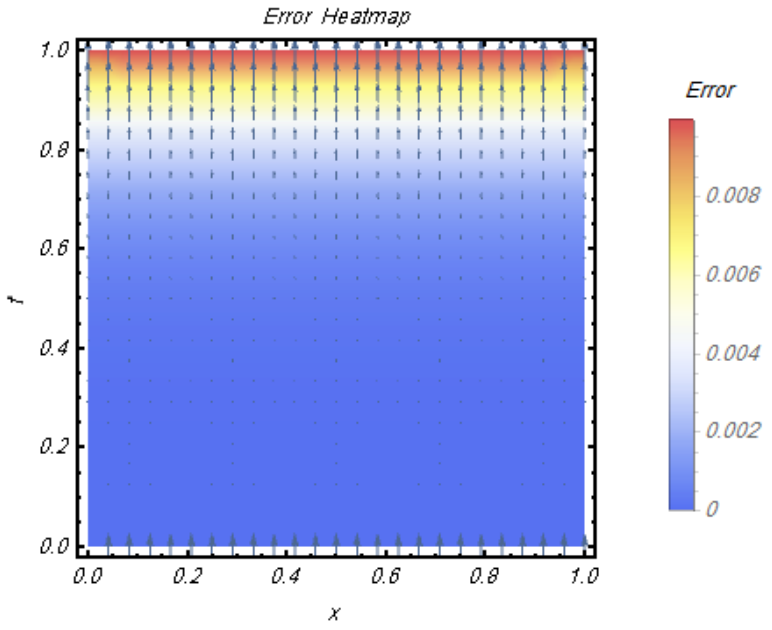


FIGURE 6. The error between the exact and approximate solution, where the blue color indicates a small error in example 7.2.

A quick look at Figure 2, when the value of $\alpha = 0.5$ (where the solution is exact) is present and we plot the approximate solution for several different values of the variable x, we found that the approximate solution matched the exact solution. In contrast, looking at Figure 3, where we draw the approximate solution for several different α, we notice from the figure that the closer the value of α to 0.5, the more the approximate solution is identical to the exact solution, and vice versa, that the further the value of α is from 0.5, the farther the approximate solution is from the exact solution. Figure 4 represents a contour graph of the approximate solution when $\alpha = 0.5$ with values of $u(x,t)$ labeled along the contour lines. In Figure 5 we plot both the exact solution (black) and the approximate one (green) together with the values of $u(x,t)$ labeled along the contour lines, showing that the approximate solution agrees with the exact. Figure 6 show the absolute error for the obtained approximate solution when $\alpha = 0.5$, and both x, t are between 0 and 1, we observe that the dark blue color indicates the area where the solution is more accurate.

For comparison purposes between RPSM and MFPSM, we introduce the following example.

Example 7.3. The same fractional Fredholm integro-differential equation (1) is solved using the Modified Residual Power Series Method (MRPSM) and the Residual Power Series Method (RPSM), with

$$f(x,t) = -(2/3) + e^t \operatorname{Erf}(\sqrt{t}), \quad K(t,y,\tau) = y\tau, \quad g(x) = 1, \quad u(x,0) = 1 + x, \quad \alpha = 0.5.$$

The exact solution is:

$$u(x,t) = e^t + x.$$

In the table 4, we provide an overview of the absolute errors for the Residual Power Series Method (RPSM) and the Modified Residual Power Series Method (MRPSM) over a range of x and t values.

t	x	Error (MRPSM)	Error (RPSM)
0.1	0.0	8.5223×10^{-8}	0.104246
	0.1	8.5223×10^{-8}	0.095596
	0.2	8.5223×10^{-8}	0.088272
	0.3	8.5223×10^{-8}	0.081990
	0.4	8.5223×10^{-8}	0.076543
	0.5	8.5223×10^{-8}	0.071774
0.3	0.0	2.1308×10^{-5}	0.148090
	0.1	2.1308×10^{-5}	0.137876
	0.2	2.1308×10^{-5}	0.128980
	0.3	2.1308×10^{-5}	0.121162
	0.4	2.1308×10^{-5}	0.114238
	0.5	2.1308×10^{-4}	0.108062
0.5	0.0	2.8377×10^{-4}	0.157998
	0.1	2.8377×10^{-4}	0.148963
	0.2	2.8377×10^{-4}	0.140905
	0.3	2.8377×10^{-4}	0.133675
	0.4	2.8377×10^{-4}	0.127150
	0.5	2.8377×10^{-4}	0.121232

TABLE 4. Absolute Error Analysis using both MRPSM and RPSM for for Example 7.3

The following are our accuracy findings: For all x and t values, the MRPSM provides significantly less absolute errors than RPSM. Additionally, the errors using MRPSM are orders of magnitude reduced, particularly at smaller values of t. In terms of error consistency, we find that MRPSM errors are consistent across different x values at the same t, indicating the robustness of the solution approach. However, RPSM exhibits more errors and is more volatile. In terms of appropriateness, MRPSM performs better in situations requiring a high level of precision, such as scientific computations or problems where accuracy is essential. Nonetheless,

the RPSM provides a less complex and computationally intensive method, which is advantageous for exploratory studies or circumstances in which approximations suffice. Lastly, we find that MRPSM requires more computing work to handle higher-order modifications, but that the precision of MRPSM outweighs the expense in high-stakes scenarios. However, RPSM is useful for fast estimations since it strikes a mix between convenience and reasonable accuracy.

Conclusion

In this paper, we applied the residual power series method to the two dimensional neutrosophic fractional integro differential equations with smooth kernel. All the previous works did not involve the Fredholm type using the residual power series method, the problem is to find the coefficients $C_1, C_2, \ldots, C_n$. We introduced a technique to find these coefficients. We found analytic solution and numerical solution for some of two-dimensional fractional- Fredholm integro differential equations to show the accuracy of this method. To obtain higher accuracy, we suggested a new residual power series method and we applied this method to some examples. The examples show that this modified Residual power series method is efficient and accurate more than the ordinary Residual power series method. Most the time we obtain an exact solution.

Conflicts of Interest: The authors declare no conflict of interest.

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Enhanced Technique for Operation Evaluation of Distribution Network Automation Systems Based on the Uncertainty Environment

Dingqun Zhang*

School of Intelligent Manufacturing, Nanyang Vocational College of Science and Technology, Dengzhou, 474150, Henan, China

*Corresponding author, E-mail: zdq720410@163.com; j0104072@nyist.edu.cn

Abstract:

Evaluation of the distribution network automation systems is a complex task due to it containing various criteria. So, this evaluation is an multi-criteria decision making (MCDM) problem. We used the VIKOR method to rank the alternatives. The VIKOR method is integrated with the plithogenic set to deal with uncertainty information in the evaluation distribution network automation system. Three experts are invited to evaluate the criteria and alternatives. An application with eight criteria and 12 alternatives is collected to select the highest importance criterion and highest score alternatives. The results show the System Reliability and Stability criterion has the highest importance and the User Interface and Control Systems criterion has the lowest importance. We performed sensitivity analysis with ten cases to show the different rank of alternatives. The results show the proposed method is stable in different cases.

Keywords: Distribution Network Automation Systems; Multi-Criteria Decision Making; Uncertainty Environment; VIKOR Method.

1. Introduction

One of the most effective strategies to increase the dependability of electrical distribution networks is network automation. It shortens the time that consumers are interrupted and the number of impacted customers. Network automation relies heavily on fault indicators (FPIs) and remotely operated and supervised switches (such as reclosers and

sectionalizing switches (RCSs). Distributed generators (DGs) operating on islands can increase distribution networks' dependability even more. An island will be constructed if generation can supply loads without breaching the operational limits during the island operation. Therefore, when designing for network automation, the best island partitioning should be considered. This is made possible by automatic switching devices, such as reclosers and RCSs[1], [2].

The network automation planning challenge becomes even more difficult as a result. Numerous strategies have been put forth that maximize the quantity, kind, and placement of automation devices (ADs) in distribution networks while taking DG island operation into account. They consider reliability indices, the cost to consumers of both short-term and long-term interruptions, the cost of automation devices, and the cost of DGs to determine the optimal network automation strategy and DG placement[3], [4]. Evaluation of the network automation systems is a MCDM problem.

Experts feel at ease using linguistics such as "Very Important," "Important," "Medium," and "Average" to convey the necessary information. Due to their ambiguity, these concepts must be quantified to be used for analysis. To employ these linguistic notions in analysis, Zadeh created the idea of fuzzy sets, in which an element's degree of membership determines how gradually it changes from a member to a non-member[5], [6]. These sets were further expanded to include intuitionistic fuzzy sets (IFS), interval valued intuitionistic fuzzy sets (IVIFS), and interval valued fuzzy sets (IVFS). Research tackling MCDM issues under ambiguity has made extensive use of fuzzy sets[7], [8].

However, none of these sets were able to account for the element's membership indeterminacy component. Any of the fuzzy sets listed above are unable to handle the following scenarios: 0.4 for the statement to be true, 0.3 for the statement to be false, and 0.5 for the statement to be uncertain[9], [10]. To get around the drawbacks of fuzzy sets where indeterminacy is openly described, Smarandache created the neutrosophic set (NS). In a universal set X, where X contains real standard or nonstandard subsets, a neutrophilic set is defined as set A that is separately characterized by a truth membership function, indeterminacy membership function, and falsity membership function[11], [12].

When applied with consideration for the degree of disagreement between each attribute value and the dominating attribute values of the data, the plithogenic set, which was developed as an extension of the neutrosophic set, can handle data uncertainty and produce results that are significantly better than the neutrosophic set. Plithogenic sets were successfully applied by researchers to MCDM issues[13], [14].

In MCDM, a compromise ranking technique called VIKOR is employed. By considering conflicting criteria and offering compromise solutions that maximize collective value and reduce individual regret, it aids in ranking and choosing alternatives. VIKOR, which focuses on identifying the option that is closest to the optimum solution, is especially helpful when decision makers are unable to articulate their preferences[15], [16].

The main purpose of this paper is

(1) To aim to convert the plithogenic number into crisp values.
(2) To establish an MCDM method to compute the criteria weights and rank the distribution network systems.
(3) To develop a VIKOR method to rank the alternatives.
(4) To illustrate with an example and the effectiveness of the proposed method.
(5) To perform the sensitivity analysis.

The rest of this study is organized as follows: Section 2 shows the Preliminaries operations. Section 3 shows the proposed methodology steps. Section 4 shows the illustrative example of this study. Section 5 shows the sensitivity analysis. Section 6 shows the conclusions.

2. Preliminaries

This section shows some definitions of plithogenic sets and their operations.

2.1 Plithogenic Set

As an expansion of the concepts of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets, Smarandache introduced the idea of plithogenic sets. One or more qualities may be present in a plithogenic set P, and each attribute may have many values. According to certain specified criteria, the value v of each attribute corresponds to the (fuzzy, intuitionistic fuzzy, or neutrosophic) degree of appurtenance d(x, v) of the element x to the set P. A (fuzzy, intuitionistic fuzzy, or neutrosophic) contradiction (dissimilarity) degree between every characteristic value and the dominant characteristic value is established in order to improve the accuracy of the plithogenic aggregation operations. The most significant characteristic value is determined and given a dominating value by experts. If a dominant attribute value is absent or has many dominant attribute values, experts might choose to suppress it or create another association function.

2.2 Plithogenic Set Theoretic Operations

Plithogenic operations can be defined as:

$$d_{AB}(x,v_T)=\begin{pmatrix}(T,I,F)\\ \big((1-cd_i)[d_A(x,v_T)]t_{conorm}\ d_B(x,v_T)\big)\\ +cd_i[d_A(x,v_T)]t_{conorm}\ d_B(x,v_T)\ ,\\ \frac{1}{2}[d_A(x,v_I)]t_{conorm}\ d_B(x,v_I)+[d_A(x,v_I)]t_{conorm}\ d_B(x,v_I),\\ (1-cd_i)[d_A(x,v_F)]t_{norm}\ d_B(x,v_F)\\ +cd_i[d_A(x,v_F)]t_{norm}\ d_B(x,v_F)\end{pmatrix} \tag{1}$$

$$d_{AB}(x,v_T)=\begin{pmatrix}(T,I,F)\\ \Big((1-cd_i)[d_A(x,v_T)]\bigvee_f\ \ d_B(x,v_T)\Big)\\ +cd_i[d_A(x,v_T)]\bigwedge_f d_B(x,v_T)\ ,\\ \frac{1}{2}[d_A(x,v_I)]\bigwedge_f d_B(x,v_I)+[d_A(x,v_I)]\bigvee_f d_B(x,v_I),\\ (1-cd_i)[d_A(x,v_F)]\bigwedge_f d_B(x,v_F)\\ +cd_i[d_A(x,v_F)]\bigvee_f d_B(x,v_F)\end{pmatrix} \tag{2}$$

3. The Proposed Model

The proposed model uses plithogenic numbers and VIKOR method to evaluate the criteria and rank the alternatives. Set of experts identify the framework to evaluate the criteria and list of alternatives and these criteria and affect on the decision-making process. They can decide the contradiction degrees of criteria, obtain combined alternatives score using the plithogenic sets and VIKOR method. Figure 1 shows the steps of the proposed methodology. The process is summarized below:

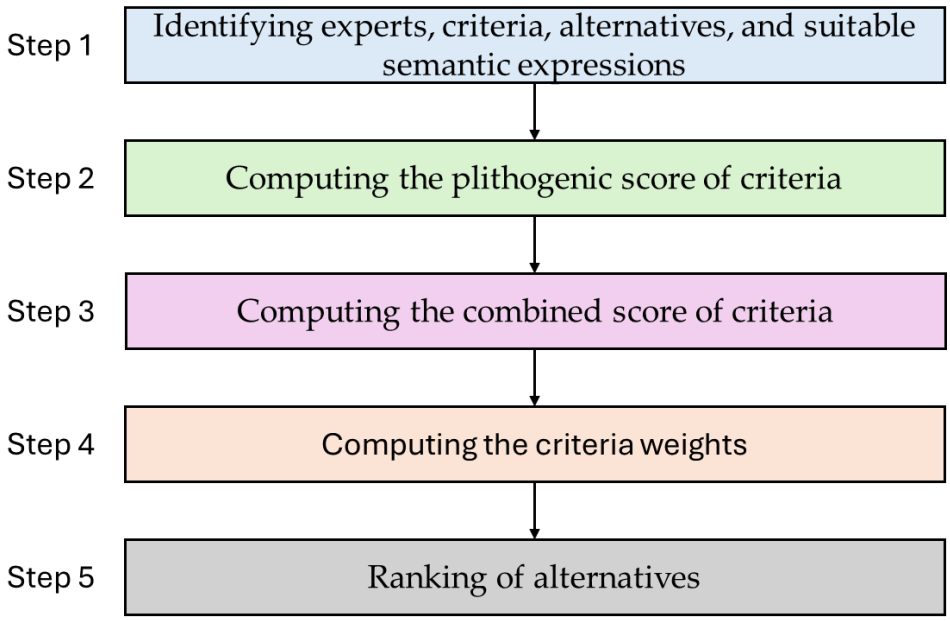

Figure 1. Flowchart of proposed model.

Step 1. Identifying experts, criteria, alternatives, and suitable semantic expressions.

We form a committee of k experts $E = \{E_1, \dots, E_k\}$ to define list n criteria $C = \{C_1, \dots, C_j\}$ and list m alternatives $A = \{A_1, \dots, A_i\}$. If $E_t = (a_t, b_t, c_t)$ is a plithogenic number expressing the importance of experts and crisp value can be obtained as:

$$S(E) = \frac{(2+a_t-b_t-c_t)}{3} \tag{3}$$

After normalizing these crisp values for experts, we can obtain weights of experts.

Step 2. Computing the plithogenic score of criteria

Plithogenic score of criteria can be computed by the product of plithogenic descriptions. The plithogenic score can be computed using the plithogenic operator.

Step 3. Computing the combined score of criteria

The criteria score obtained for all decision makers and experts using the plithogenic operations intersection. Then we apply the score function to obtain crisp values in the combined matrix. Then we normalize the matrix to obtain the criteria weights.

Step 4. Ranking of alternatives.

We apply the steps of the VIKOR method under the plithogenic numbers to rank the alternatives.

Compute the maximum and minimum score of each column in the combined decision matrix for beneficial and non-beneficial criteria.

$$\begin{Bmatrix} y_j^* = \max y_{ij} \\ y_j^- = \min y_{ij} \end{Bmatrix} \tag{4}$$

$$\begin{Bmatrix} y_j^* = \min y_{ij} \\ y_j^- = \max y_{ij} \end{Bmatrix} \tag{5}$$

Where $i = 1, \dots, m; j = 1, \dots, n$

The S and R indexes

$$S_i = \sum_{j=1}^{n} w_j \frac{\left(y_j^* - f_{ij}\right)}{\left(y_j^* - y_j^-\right)} \tag{6}$$

$$R_i = \max\left[w_j \frac{\left(y_j^* - f_{ij}\right)}{\left(y_j^* - y_j^-\right)}\right] \tag{7}$$

The VIKOR index

$$U_i = v \times \left[\frac{(S_i - \max S_i)}{(\min S_i - \max S_i)}\right] + (1 - v \times)\left[\frac{(R_i - \max R_i)}{(\min R_i - \max R_i)}\right] \tag{8}$$

The final ranking of alternatives

The alternatives are ranked by the lowest value in VIKOR index

4. Illustrative Example

Consider an organization system and need to select the optimal Operation Evaluation of Distribution Network Automation Systems. Here are three experts from the network automation system who are invited through semi structured interviews to identify and list the various criteria and alternatives.

Step 1. In this step we collect seven criteria and 12 systems to be evaluated and select the highest score system to be applied in the operations firm. Figure 2 shows the list of criteria that are used in this study.

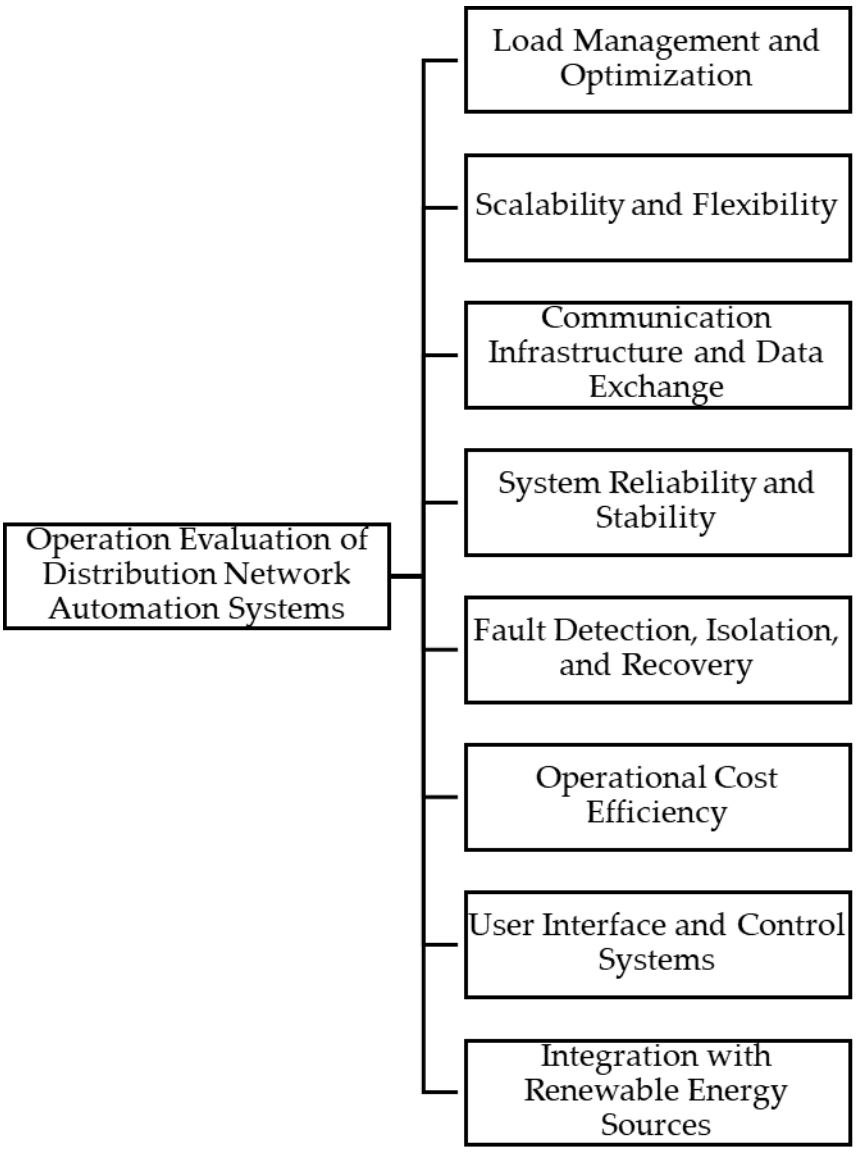
Operation Evaluation of Distribution Network Automation Systems
Load Management and Optimization
Scalability and Flexibility
Communication Infrastructure and Data Exchange
System Reliability and Stability
Fault Detection, Isolation, and Recovery
Operational Cost Efficiency
User Interface and Control Systems
Integration with Renewable Energy Sources

Figure 2. The operations of distributions network automation system criteria.

Step 2. This step lets three experts and decision makers evaluate the criteria. Then we replaced their opinions by using the plithogenic numbers as shown in Tables 1-3.

Table 1. The plithogenic numbers by expert 1

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_2	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_4	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_5	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_6	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_7	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_8	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A_9	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)

A_{10}	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_{12}	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)

Table 2. The plithogenic numbers by expert 2

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_2	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_3	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_4	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_5	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_6	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_7	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_8	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)
A_9	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_{10}	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)
A_{12}	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)

Table 3. The plithogenic numbers by expert 3

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_2	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_4	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_5	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_6	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_7	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_8	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_9	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_{10}	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_{12}	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)

Step 3. This step combines the different opinions of experts by using the plithogenic operator. Then we obtain crisp values. Then we normalize these values to obtain the criteria weights. Then we rank the criteria as:

$$C4 > C5 > C1 > C8 > C3 > C2 > C6 > C7$$

We show criterion 4 has the highest importance and criterion 7 has the lowest importance.

Step 4. This step applies the steps of the VIKOR methodology to rank the alternatives.

Then we obtain the normalized decision matrix shown in Table 4. Then we obtain the matrix of weighted normalized as in Table 5. Then we obtained the values of the S and R index.

Table 4. The normalized matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.309892	1	0.534896	0.714349	0.400332	0	0	0.524087
A_2	0.553889	0.632791	0.145436	0.171169	0	1	0.772905	1

A_3	0.449574	0.645141	0.455726	0.300376	0.877332	0.073825	0.520457	0.7211
A_4	0.380254	0.68643	0.200871	0.044951	0.452466	0.823258	0.340121	0.508677
A_5	0.129519	0.167356	0.056307	0.022254	0.950549	0.331694	0.072447	0.136463
A_6	0.147277	0.095609	0.400622	1	0.252747	0.618731	0.180559	0
A_7	0.153308	0.167356	0	0.602192	0.652568	0.418047	0.40959	0.005216
A_8	0.492759	0.377937	1	0.26384	0.877332	0.242266	0.380254	0.793789
A_9	0.407282	0.284159	0.049585	0	0.124329	0.154106	0.757232	0.494263
A_{10}	0.329846	0.885237	0.166763	0.562555	0.309865	0.390531	0.378392	0.322617
A_{11}	1	1	1	0.562555	0.877332	0.317513	0.404601	0.651304
A_{12}	0	0	0.145436	0.891275	1	0.431805	1	0.218777

Table 5. The weighted normalized matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.039996	0.120884	0.064683	0.097212	0.053843	0	0	0.063395
A_2	0.071488	0.076494	0.017587	0.023294	0	0.120058	0.090832	0.120963
A_3	0.058025	0.077987	0.05511	0.040877	0.117997	0.008863	0.061165	0.087227
A_4	0.049078	0.082978	0.024291	0.006117	0.060855	0.098839	0.039971	0.061531
A_5	0.016716	0.020231	0.006809	0.003028	0.127845	0.039823	0.008514	0.016507
A_6	0.019008	0.011558	0.048446	0.136085	0.033993	0.074284	0.021219	0
A_7	0.019787	0.020231	0	0.08195	0.087768	0.05019	0.048135	0.000631
A_8	0.063598	0.045686	0.120927	0.035905	0.117997	0.029086	0.044688	0.096019
A_9	0.052566	0.03435	0.005996	0	0.016722	0.018502	0.08899	0.059788
A_{10}	0.042572	0.107011	0.020166	0.076556	0.041675	0.046886	0.044469	0.039025
A_{11}	0.129066	0.120884	0.120927	0.076556	0.117997	0.03812	0.047549	0.078784
A_{12}	0	0	0.017587	0.12129	0.134496	0.051842	0.117521	0.026464

Then we obtained the VIKOR index as in Table 6. We used v=0.5. Then we ranked the alternatives as in Table 6.

Table 6. The rank of alternatives.

	S	R	*VIKOR index*	*Rank*
A_1	0.440014	0.120884	0.452847	6
A_2	0.520717	0.120963	0.369741	9
A_3	0.50725	0.117997	0.414165	7
A_4	0.42366	0.098839	0.697641	3
A_5	0.239473	0.127845	0.585276	5
A_6	0.344594	0.136085	0.392823	8
A_7	0.308691	0.087768	0.929428	2
A_8	0.553907	0.120927	0.336278	10

A_9	0.276914	0.08899	0.949172	1
A_{10}	0.41836	0.107011	0.618483	4
A_{11}	0.729882	0.129066	0.07264	12
A_{12}	0.469199	0.134496	0.282233	11

5. Sensitivity Analysis

The sensitivity analysis is performed to see the effectiveness of the proposed models under uncertainty environment. In the VIKOR method, we put the v=0.5 in the VIKOR index. In this section, we change the v value from 0 to 1. Then we rank the alternatives. Figure 3 shows the rank of alternatives under different v values. We show the rank of alternatives is stable under different v values. So, the proposed model is effective.

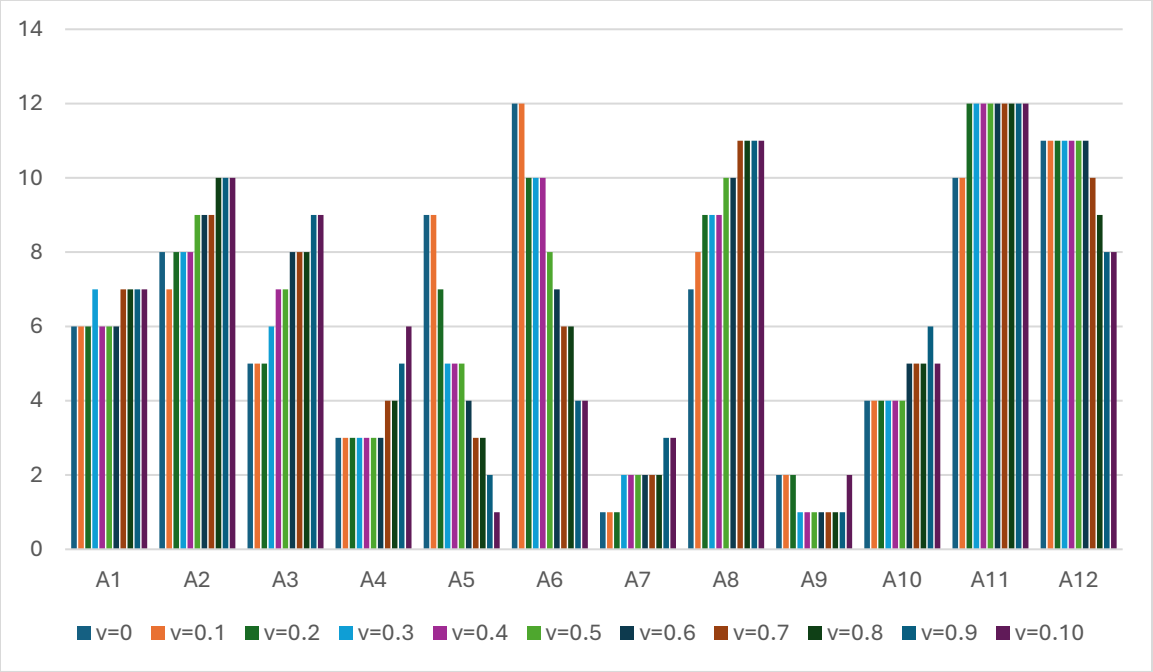

Figure 3. The rank of alternatives under different v values.

6. Conclusions

The goal of this study is to propose an MCDM methodology under the plithogenic sets and VIKOR methodology. We used plithogenic numbers to evaluate the criteria and alternatives. Three experts and decision makers are invited to evaluate the criteria and alternatives to build the decision matrix. We combined these numbers into a single matrix using the plithogenic operator. We obtain crisp values instead of plithogenic numbers. Eight criteria and 12 alternatives are collected in this study. We compute the criteria

weights to rank the criteria to show the highest and lowest importance. Then we applied the VIKOR method to rank the alternatives. We show alternative 11 is the best and alternative 9 is the worst.

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On Consistent and Weak Transitive Neutrosophic Fuzzy Matrices

K. Radhika[1], M.Kavitha[2] , K. Thirumalai[3] , R. Geethanjaliyadav[4] , N.Buvaneswari[5], V. Kamalakannan[6]

[1]Department of Mathematics, Rajah Serfoji Government College (Autonomous) (Affiliated to Bharathidasan University, Tiruchirappalli), Thanjavur - 613005, Tamilnadu, India.
radhi9791222789@gmail.com

[2]Department of Mathematics, Panimalar Engineering College, Chennai, Tamilnadu, India
mkavitha.saran@gmail.com

[3]Department of Mathematics, Saveetha Engineering College, Chennai - 602105, Tamilnadu, India.
thirumalaikmaths@gmail.com

[4]Department of Mathematics, Rajalakshmi institute of Technology, Poonamalle, Chennai, Tamilnadu, India.
rvgeetha20@gmail.com

[5]Department of Mathematics, Siga College of Management and Computer Science, Villupuram Affiliated to Annamalai University, Chidambaram, Tamilnadu, India.
n.buvanvardhana@gmail.com

[6]Department of Mathematics, Panimalar Engineering College, Chennai, Tamilnadu, India
vkamalakannan@panimalar.ac.in

Abstract: This paper delves into the properties of two specific types of Neutrosophic Fuzzy Matrices (NFM) namely consistent and weakly transitive NFM. A significant focus is placed on nilpotent and transitive NFM, highlighting their critical role in the analysis. It is demonstrated that these two types of matrices are controllable, and a formula is derived for determining the canonical form of a weakly transitive NFM. To support and clarify the findings, counterexamples are provided throughout the discussion. We introduce an operation on NFM, referred to as the Gödel implication operator. Utilizing this operator, we establish several significant results for NFM, with a particular emphasis on properties related to pre-orders. Our analysis focuses primarily on reflexive and transitive NFM, enabling us to derive meaningful insights. Additionally, we demonstrate a method for constructing an idempotent NFM from any given matrices.

Keywords: Weakly transitive, Gödel implication operator, Consistent, Idempotent NFM.

1. Introduction

The concept of fuzzy sets, presented by Zadeh [1], has significantly influenced research in modeling uncertainty and imprecision. This foundational work laid the groundwork for various extensions, including intuitionistic fuzzy sets by Atanasov [15, 33], which incorporate membership, non-membership, and hesitation degrees, offering a more generalized framework for representing

imprecise information. These advancements have spurred the development of fuzzy and intuitionistic fuzzy matrices, essential tools for solving complex mathematical problems and decision-making challenges. The study of fuzzy matrices began with Thomason [2], who examined their convergence properties, and Kim and Roush [3, 41], who introduced generalized fuzzy matrices. Research by Hashimoto [8, 35,38-39] and Chenggong [7] further advanced the field by exploring canonical forms and transitivity properties of fuzzy matrices. Additionally, Emam and Fendh [6, 35] investigated max-min and min-max compositions, providing new insights into fuzzy relations and their applications. In the context of intuitionistic fuzzy matrices, Khan and Pal [22, 44] explored generalized inverses, while Muthuraji et al. [24, 24, 27] focused on decomposition techniques and methods to reduce intuitionistic fuzzy matrices to standard fuzzy matrices. These contributions have expanded the utility of intuitionistic fuzzy matrices in areas such as preference modeling [14, 32], aggregation operations [28], and transitivity analysis [29].

Smarandache [52] introduced neutrosophic sets, which extend fuzzy and intuitionistic fuzzy sets by integrating a third component for indeterminacy. This innovation has inspired researchers to explore neutrosophic fuzzy matrices (NFM) as a framework for analyzing uncertainty in data. Despite significant progress, challenges remain. For instance, while properties like transitivity [9, 27] and idempotence [10, 45] have been extensively studied, their extension to newer constructs such as neutrosophic fuzzy matrices requires further exploration. Anandhkumar et al [4–5] have studied pseudo Similarity of Neutrosophic Fuzzy matrices and On various Inverse of Neutrosophic Fuzzy Matrices. Anandhkumar et al [50] have presented Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices. Moreover, relationships between various transitivity-related concepts and their practical implications warrant deeper investigation [29, 47]. Sudha et al [54-56] have studied A new technique for using Symmetric pentagonal Intuitionistic fuzzy numbers to solve Assignment problem, An Approach for solving fuzzy sequencing problem with Triangular fuzzy Neutrosophic numbers using Ranking function and A Method for finding Critical path problem with Triangular Neutrosophic Numbers by utilizing numerous methods for Ranking.

Anandhkumar et al [42-43] have presented Secondary k-column symmetric Neutrosophic Fuzzy Matrices and Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. This paper builds on the foundational work of Zadeh [1], Atanasov [15, 33], and others, extending the analysis of intuitionistic fuzzy and neutrosophic fuzzy matrices. We focus on reflexive and transitive properties and propose methods for constructing idempotent matrices, addressing existing gaps and advancing the understanding of these mathematical constructs. This article delves into the properties of two specific types of Neutrosophic Fuzzy Matrices (NFM) namely consistent and weakly transitive NFM. A significant focus is placed on nilpotent and transitive NFM, highlighting their critical role in the analysis. It is demonstrated that these two types of matrices are controllable, and a formula is derived for determining the canonical form of a weakly transitive NFM. To support

and clarify the findings, counterexamples are provided throughout the discussion. We introduce an operation on NFM, referred to as the Gödel implication operator. Utilizing this operator, we establish several significant results for NFM, with a particular emphasis on properties related to pre-orders. Our analysis focuses primarily on reflexive and transitive NFM, enabling us to derive meaningful insights. Additionally, we demonstrate a method for constructing an idempotent NFM from any given matrices.

1.1 Abbreviations

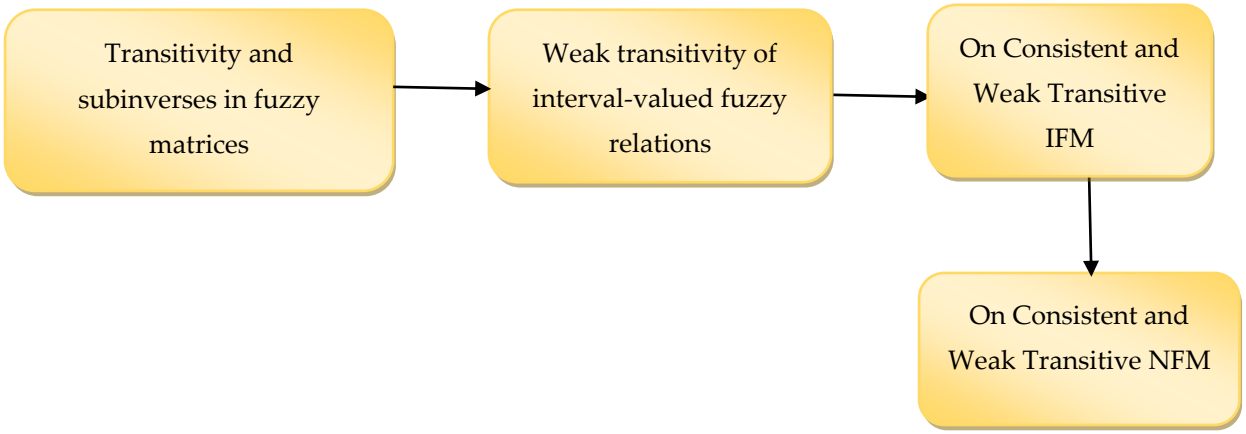
IFM = Intuitionistic Fuzzy Matrices
NFM = Neutrosophic Fuzzy Matrices
NFPM = Neutrosophic Fuzzy Permutation Matrices
TNFM = Transitive Neutrosophic Fuzzy Permutation Matrices

1.2 Research gap

Mishref, Emam and Xu YJ, Wang H M, Yu DJ [45,33,53] introduced the concepts of Transitivity and subinverses in fuzzy matrices, Weak transitivity of interval-valued fuzzy relations, On Consistent and Weak Transitive IFM. Building on their work, we have applied these principles to NFM. In this study, we examine several results and extend these concepts to NFMs.

Table:1 An overview of the advancements in extending NFM.

References	**Extension of NFM.**	**Year**
E.G. Emam [45]	Transitivity and subinverses in fuzzy matrices	1992
Xu YJ et al [32]	Weak transitivity of interval-valued fuzzy relations	2014
E. G. Emam [53]	On Consistent and Weak Transitive IFM	2022
Proposed	On Consistent and Weak Transitive NFM	2024


Transitivity and subinverses in fuzzy matrices
Weak transitivity of interval-valued fuzzy relations
On Consistent and Weak Transitive IFM
On Consistent and Weak Transitive NFM

Flowchart for the research gap

1.3 Structure of the Article

This article is structured to provide a logical and detailed presentation of our research on Neutrosophic Fuzzy Matrices (NFM). It begins with a Literature Review [2], where we analyze existing studies, identifying key advancements and limitations while establishing the relevance of NFM in addressing contemporary challenges. Following this, the section on Novelty [4] and Contribution of Our Work [5] highlights the unique aspects of our approach, showcasing how our research fills critical gaps in the existing body of knowledge. In Preliminaries [5] , we introduce the foundational concepts, definitions, and notations that form the basis for the proposed methodologies. This ensures that readers are equipped with the necessary tools to engage with the subsequent sections. The Theorems and Results section [6] presents the core findings of the study, including the primary theorems, supported by rigorous proofs and illustrative examples.

The article further delves into the Reflexivity and Transitivity [7] of Neutrosophic Fuzzy Matrices, exploring these essential properties in depth and discussing their implications for theoretical and practical applications. Finally, the Conclusion and Future [8] Work section summarizes the key outcomes of our research and provides insights into potential avenues for future exploration in this domain. This structured approach ensures clarity and coherence, guiding the reader through the progression of our work.

1.4 Comparison of Neutrosophic Fuzzy Matrices with Existing Models

To comprehensively understand the significance and advantages of Neutrosophic Fuzzy Matrices (NFM), it is essential to compare them with existing models such as fuzzy matrices and intuitionistic fuzzy matrices.

(i) **Fuzzy Matrices**

Fuzzy matrices are widely used for modeling uncertainty, where each element of the matrix is represented by a value in the interval [0,1][0,1][0,1], reflecting the degree of membership of an element to a set. However, fuzzy matrices are limited in addressing situations with incomplete or conflicting information. They operate solely within the framework of membership values and lack the ability to quantify non-membership or indeterminate states.

(ii) **Intuitionistic Fuzzy Matrices (IFM)**

Intuitionistic fuzzy matrices extend fuzzy matrices by incorporating both membership μ and non-membership ν values, ensuring that $\mu+\nu \leq 1$ While this model improves the representation of uncertainty, it still assumes that the sum of the membership and non-membership values accounts for all possibilities, neglecting the explicit representation of indeterminacy. This limitation makes intuitionistic fuzzy matrices less effective in scenarios involving a high degree of vagueness or incomplete knowledge.

(iii) **Neutrosophic Fuzzy Matrices (NFM)**

Neutrosophic fuzzy matrices overcome the limitations of both fuzzy and intuitionistic fuzzy matrices by introducing three independent components: membership (T), non-membership (F), and indeterminacy (I), where $T,F,I \in [0,1]$ and $T+F+I \leq 3$. This three-component structure provides a more flexible and comprehensive framework for representing uncertainty, conflict, and indeterminacy simultaneously. Consequently, NFM is particularly suitable for complex systems where multiple sources of uncertainty coexist.

Comparative Insights

While fuzzy matrices and intuitionistic fuzzy matrices have been effective in handling uncertainty to varying degrees, NFM offers superior expressiveness and adaptability. By explicitly accommodating indeterminacy, NFM allows for more nuanced modeling and analysis of uncertain systems, making it a powerful tool for solving problems in areas like decision-making, systems analysis, and artificial intelligence.

2. Literature Review

Fuzzy set theory, presented by Zadeh [1], has laid a robust foundation for handling uncertainty and imprecision, influencing a wide array of fields, including matrix theory. The study of fuzzy matrices began with Thomason [2], who explored their convergence properties, and was further developed by Kim and Roush [3, 41], who generalized the concept of FM. Punithavalli et al [12] have presented Kernel and k-Kernel Symmetric Intuitionistic Fuzzy Matrices**.** Anandhkumar et al [13] have discussed Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices. These works emphasized the role of FMs in modeling dynamic systems and their structural properties. Hashimoto [8, 38–40] contributed significantly to understanding the canonical forms and decomposition of fuzzy matrices, providing key insights into their structural analysis. Chenggong [7] focused on strongly transitive matrices, developing canonical forms that enhance their applicability in system analysis. Emam and Fendh [6, 35] investigated the max-min and min-max compositions of bifuzzy matrices, revealing novel relationships between fuzzy relations and operations. Anandhkumar et al [19] have studied Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. Punithavalli et al [20] have focussed on Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices.

In the domain of IFMs, Atanasov [15, 33] extended the fuzzy set theory by incorporating membership, non-membership, and hesitation degrees. Pal et al. [18, 46] and Khan and Pal [22, 44] explored intuitionistic fuzzy matrices, focusing on generalized inverses and algebraic structures. Muthuraji et al. [23,24,27] advanced this research by examining decomposition methods, representation techniques, and reduction strategies, enabling the transformation of intuitionistic fuzzy matrices into fuzzy matrices while preserving critical algebraic properties. The properties of transitivity and idempotence in fuzzy and intuitionistic fuzzy matrices have also been widely

studied. Hashimoto [39] and Jian-Xin [10, 11] analyzed. Anandhkumar et al [25] have presented Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices. Radhika et al [26] have characterized On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices.

3. Novelty

This research introduces several innovative contributions to the study of neutrosophic fuzzy matrices, addressing key gaps in the literature. One significant advancement is the extension of the Gödel implication operator to NFM, enabling a deeper exploration of pre-order relations, particularly in reflexive and transitive matrices. Radhika et al [30] have studied Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making. Prathab et al [31] presented Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses. Anandhkumar et al [49] have studied Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems. While earlier studies [4, 25] focused on basic operations in fuzzy matrices, this work broadens their scope by applying them to NFM. Another notable contribution is the derivation of idempotent neutrosophic fuzzy matrices (NFM) from any given matrix, a topic that remains largely unexplored in existing research [9, 27]. Anandhkumar et al [36-37] have studied Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic fuzzy matrices and Generalized Symmetric Neutrosophic Fuzzy Matrices.

Furthermore, this study introduces canonical forms for weakly transitive NFM, building upon prior work on strongly transitive matrices [6, 38] and extending their applicability to cases involving weaker transitivity conditions. Additionally, by integrating neutrosophic set theory [52] into fuzzy matrix analysis, the research bridges the gap between classical fuzzy matrices and their neutrosophic counterparts, allowing for the modeling of indeterminacy alongside membership and non-membership values. These contributions collectively enhance the theoretical understanding and practical utility of neutrosophic fuzzy matrices, offering new tools and methodologies for applications in decision-making, systems analysis, and beyond.

4. Contribution of Our Work

(i) **Analysis of Consistent and Weakly Transitive NFM**:
This study examines two specific types of neutrosophic fuzzy matrices (NFM): consistent and weakly transitive NFM. We demonstrate their controllability and derive a formula for the canonical form of weakly transitive NFM, providing a novel analytical perspective.

(ii) **Significance of Nilpotent and Transitive NFM**:
A significant focus is placed on the properties of nilpotent and transitive NFM, emphasizing

their critical role in matrix analysis and contributing to the broader understanding of their applications in uncertainty modeling.

(iii) **Gödel Implication Operator on NFM**:
We introduce the Gödel implication operator as a novel operation on NFM, extending its applicability and enabling the derivation of significant results. This operator plays a key role in analyzing pre-orders and their implications in reflexive and transitive NFM.

(iv) **Construction of Idempotent NFM**:
A method for constructing idempotent NFM from any given matrix is presented, addressing an underexplored area in existing literature. This contribution provides a new approach for generating matrices with desirable algebraic properties.

(v) **Counterexamples for Clarity**:
To enhance the understanding and practical applicability of the results, counterexamples are provided throughout the discussion, clarifying the theoretical findings.

5. Preliminaries

In this part, we introduce operations for NFMs. For two NFMs P and Q, we define the subsequent operations $U \vee V$, $U \wedge V$.

$$U \vee V = \left[u_{ij} \vee v_{ij}\right] = \left[\max < u_{ij}^{T}, v_{ij}^{T} >, \max < u_{ij}^{I}, u_{ij}^{I} >, \min < u_{ij}^{F}, u_{ij}^{F} >\right]$$

$$U \wedge V = \left[u_{ij} \wedge v_{ij}\right] = \left[\min < u_{ij}^{T}, v_{ij}^{T} >, \min < u_{ij}^{I}, v_{ij}^{I} >, \max < u_{ij}^{F}, v_{ij}^{F} >\right]$$

Definition: 5.1 A NFSs P on the universe of discourse Y is well-defined as

$U = \left\{< y, u^{T}(y), u^{I}(y), u^{F}(y) >, y \in Y\right\}$, everywhere $u^{T}, u^{I}, u^{F} : Y \to]^{-}0,1^{+}[$ also

$0 \le u^{T} + u^{I} + u^{F} \le 3.$

Definition:5.2 A neutrosophic Fuzzy Matrices U is less then or equal to V (or U and V are comparable)

That is $U \le V$ if $\left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right) \le \left(v_{ij}^{T}, v_{ij}^{I}, v_{ij}^{F}\right)$ means $u_{ij}^{T} \le v_{ij}^{T}, u_{ij}^{I} \le v_{ij}^{I}, u_{ij}^{F} \ge v_{ij}^{F}$.

Example: 5.1 Let us consider 3x3 NFM

$$U = \begin{bmatrix} < 0.5, 0.5, 0.3 > & < 0.8, 0.2, 0.3 > & < 0.8, 0.2, 0.1 > \\ < 0.8, 0.1, 0.3 > & < 0.8, 0.2, 0.3 > & < 0.8, 0.2, 0.3 > \\ < 0.8, 0.1, 0.1 > & < 0.8, 0.2, 0.3 > & < 0.8, 0.2, 0.2 > \end{bmatrix}$$

$$V=\begin{bmatrix} <0.9,0.6,0.1> & <0.7,0.5,0.2> & <0.9,0.3,0.1> \\ <0.8,0.7,0.2> & <1,0.7,0.2> & <0.9,0.8,0.2> \\ <0.8,0.4,0.1> & <1,0.6,0.1> & <0.9,0.3,0.1> \end{bmatrix}$$

$$(0.5,0.5,0.3)\leq(0.9,0.6,0.1)$$

$$0.5\leq 0.9, 0.5\leq 0.6, 0.3\geq 0.1$$

Therefore, all the above elements satisfy the same manner

Definition 5.3 . A NFM is considered null if all its elements are (0,0,0). This type of matrix is denoted by$N_{(0,0,0)}$. On the other hand, an NFM is defined as zero if all its elements are (0,0,1) and it is represented by O.

Definition 5.4 A square NFM is referred to as a Neutrosophic Fuzzy Permutation Matrix (NFPM) if each row and each column contains exactly one element with a value of (1,1,0) while all other entries are (0,0,1).

Example: 5.2 Let us consider 3x3 NFPM

$$P=\begin{bmatrix} (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

Definition 5.5 For identity NFM of order nx n is represented by I_n and is well-defined by

$$\left(\delta_{ij}^{\ T},\delta_{ij}^{\ I},\delta_{ij}^{\ F}\right)=\begin{cases}(1,1,0)\ if\ \ i=j\\(0,0,1)\ if\ \ i\neq j\end{cases}.$$

Example: 5.3 Let us consider 3x3 INFM

$$P=\begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

We now present the following operations for NFMs $U=\left(u^T,u^I,u^F\right)$ and $V=\left(v^T,v^I,v^F\right)$ us defined the binary operation

(i) $U*V=\left[u_{ij}*v_{ij}\right].$

(ii) $U\Theta V=\left[u_{ij}\Theta v_{ij}\right].$

(iii) $U\times V=\left[\bigcup_{k=1}^{n}\left(u_{ik}\wedge v_{kj}\right)\right].$

(iv) $U^{k+1} = U^k \times U, (k = 1, 2, 3, ...)$

(v) $U^T = \left[u_{ji}{}^{\alpha}, u_{ji}{}^{\beta}, u_{ji}{}^{\gamma} \right]$ (the transpose of U)

(vi) $\Delta U = U \Theta U^T$

(vii) $\nabla U = U \wedge U^T$

(viii) $U^2 = U$ (U is idempotent)

(ix) $U \wedge I_n = 0$ (U is irreflexive)

(x) $U^T = U$ (U is SNFM)

(xi) $U \wedge U^T \leq I_n$ (P is antisymmetric)

(xii) $u \Theta v = \begin{cases} (0,0,1) & if\ u^T \leq v^T,\ u^I \leq v^I \text{and} u^F \geq v^F \\ (0,0,u^F) & if\ u^T \leq v^T, u^I \leq v^I \text{and} u^F < v^F \\ (u^T, u^I, u^F) & if\ u^T > v^T, u^I > v^I\ and\ (u^F \geq v^F\ or\ u^F < v^F) \end{cases}$

$u \leq v$ iff $u^T \leq v^T, u^I \leq v^I, u^F \geq v^F$

Definition 5.6 **Let** $i, j, k \leq n$ and let $U = \left[u_{ij} = \left(u_{ij}{}^T, u_{ij}{}^I, u_{ij}{}^F \right) \right]$ be an NFM. Then U is called

Transitive iff $U^2 = UU \leq U$ *i.e.*, $u_{ik}{}^T \wedge u_{kj}{}^T \leq u_{ij}{}^T, u_{ik}{}^I \wedge u_{kj}{}^I \leq u_{ij}{}^I$ and $u_{ik}{}^F \vee u_{kj}{}^F \geq u_{ij}{}^F$ for every $i, j, k \leq n$.

Nilpotent iff $U^n = UU...U$ (n-times) = 0

consistent iff $u_{ij} \geq u_{ji}$ and $u_{jk} \geq u_{kj}$ implies $u_{ik} \geq u_{ki}$.

Weak transitive if and only if $u_{ij} > u_{ji}$ and $u_{jk} > u_{kj}$ implies $u_{ik} > u_{ki}$

Example: 5.4 Let assume that 3x3 Weak transitive NFM

$$\text{NFM} U = \begin{bmatrix} <0.8, 0.1, 0.1> & <0.1, 0.2, 0.3> & <0.4, 0.5, 0.1> \\ <0.2, 0.3, 0.4> & <0.8, 0.2, 0.3> & <0.8, 0.2, 0.3> \\ <0.5, 0.6, 0.2> & <0.8, 0.2, 0.3> & <0.8, 0.2, 0.1> \end{bmatrix}$$

Definition:5.7 (Idempotent NFM) An idempotent neutrosophic fuzzy matrix U satisfies the condition: $U^2 = U$ where U^2 denotes the matrix product of U with itself. This means that when you multiply the matrix by itself, it results in the same matrix.

Example: 5.5 Let assume that 3x3 Idempotent NFM

$$U=\begin{bmatrix} <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \\ <0.8,0.1,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.3> \\ <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \end{bmatrix}$$

$$U^2=\begin{bmatrix} <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \\ <0.8,0.1,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.3> \\ <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \end{bmatrix}$$

$\Rightarrow U^2=U$

Definition 5.8 . A NFM is considered null if all its elements are (0,0,0). This type of matrix is denoted by $N_{(0,0,0)}$. On the other hand, an NFM is defined as zero if all its elements are (0,0,1) and it is represented by O.

Definition 5.9 The transpose of a NFM is an operation that flips the matrix over its diagonal. In other words, the rows of the original matrix become the columns in the transposed matrix, and vice versa.

i.e., $U^T=\left[u_{ji}{}^{\alpha},u_{ji}{}^{\beta},u_{ji}{}^{\gamma}\right]$

Example: 5.6 Let assume that 3x3 NFM

$$U=\begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0.8,0.2,0.1> & <0,0,1> \end{bmatrix}$$

$$U^T=\begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0.8,0.2,0.1> \\ <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$$

Definition 5.10 An n × n NFM U is said to be controllable if there exists an Neutrosophic Fuzzy permutation matrix P such that $F=PRP^T$ satiesfies $f_{ij}\geq f_{ji}$ for every $i>j$. The matrix F here is called the canonical form of U.

6. Theorems and Results

In this section, we explore and analyze the properties of **consistency** and **weak transitivity** in **NFMs**. These properties are employed to develop a **canonical form** of an NFM that is either consistent or weakly transitive.Throughout this section, let $U=\left[u_{ij}=\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)\right]$ represent an n×n NFM, We begin by presenting the following proposition to establish foundational concepts:

Proposition 6.1: If U is consistent (weak transitive) iff ΔU is consistent (weak transitive).

Proof: We provide a proof for the case of consistency, and the proof for weak transitivity can be established in a similar fashion. Let U be a consistent NFM and let $V = \Delta U$. Assume that $v_{ij} \geq v_{ji}$ and $v_{jk} \geq v_{kj}$. i.e., Suppose that $\left(u_{ij} \Theta u_{ji}\right) \geq \left(u_{ji} \Theta u_{ij}\right)$ and $\left(u_{ik} \Theta u_{ki}\right) \geq \left(u_{kj} \Theta u_{jk}\right)$.

Now we have several cases for the inequality $\left(u_{ij} \Theta u_{ji}\right) \geq \left(u_{ji} \Theta u_{ij}\right)$

Case(i) If $u_{ij}^{T} < u_{ji}^{T}, u_{ij}^{I} < u_{ji}^{I}, u_{ij}^{F} > u_{ji}^{F}$ then

i.e., if $u_{ij} < u_{ji}$ then $\left(u_{ij} \Theta u_{ji}\right) = (0,0,1)$ and $\left(u_{ji} \Theta u_{ij}\right) = \left(u_{ji}^{T}, u_{ji}^{I}, u_{ji}^{F}\right)$

which is a contradiction since we have that $\left(u_{ij} \Theta u_{ji}\right) \geq \left(u_{ji} \Theta u_{ij}\right)$.

Case(ii) If $u_{ij}^{T} < u_{ji}^{T}, u_{ij}^{I} < u_{ji}^{I}, u_{ij}^{F} < u_{ji}^{F}$, then

$\left(u_{ij} \Theta u_{ji}\right) = \left(0,0,u_{ij}^{F}\right)$ but $\left(u_{ji} \Theta u_{ij}\right) = \left(u_{ji}^{T}, u_{ji}^{I}, u_{ji}^{F}\right)$ which is a contradiction since $u_{ij}^{F} < u_{ji}^{F}$.

Case(iii) If $u_{ij}^{T} < u_{ji}^{T}, u_{ij}^{I} < u_{ji}^{I}, u_{ij}^{F} = u_{ji}^{F}$,

then $\left(u_{ij} \Theta u_{ji}\right) = (0,0,1)$ and $\left(u_{ji} \Theta u_{ij}\right) = \left(u_{ji}^{T}, u_{ji}^{I}, u_{ji}^{F}\right)$ and this is also a contradiction.

Case(iv) If $u_{ij}^{T} > u_{ji}^{T}, u_{ij}^{I} > u_{ji}^{I}, u_{ij}^{F} > u_{ji}^{F}$,then $\left(u_{ij} \Theta u_{ji}\right) \geq \left(u_{ji} \Theta u_{ij}\right)$ and this is a contradiction since $u_{ij}^{F} > u_{ji}^{F}$.

Case(v) If $u_{ij}^{T} = u_{ji}^{T}, u_{ij}^{I} = u_{ji}^{I}, u_{ij}^{F} > u_{ji}^{F}$, then $\left(u_{ij} \Theta u_{ji}\right) = (0,0,1)$ and

$\left(u_{ji} \Theta u_{ij}\right) = \left(0,0,u_{ji}^{F}\right)$. Again,this is a contradiction since and $u_{ij}^{F} > u_{ji}^{F}$.

Case(vi) If $u_{ij}^{T} \geq u_{ji}^{T}, u_{ij}^{I} \geq u_{ji}^{I}, u_{ij}^{F} \leq u_{ji}^{F}$ i.e., if, then

$\left(u_{ij} \Theta u_{ji}\right) = \left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right)$ and $\left(u_{ji} \Theta u_{ij}\right) = (0,0,1)$.

From the above cases, we conclude that if $\left(u_{ij} \Theta u_{ji}\right) \geq \left(u_{ji} \Theta u_{ij}\right)$, then $u_{ij} \geq u_{ji}$. Similarly, ifwe have $\left(u_{jk} \Theta u_{kj}\right) \geq \left(u_{kj} \Theta u_{jk}\right)$, then $u_{jk} \geq u_{kj}$. But since we have that R is consistent, we get $u_{ik} \geq u_{ki}$ which implies $\left(u_{ik} \Theta u_{ki}\right) \geq \left(u_{ki} \Theta u_{ik}\right)$. That is $v_{ik} \geq v_{ki}$ and $V = \Delta U$ is thus consistent. The reverse implication is straightforward.

Hence the Theorem.

It is observed that U iff U satisfies weak transitive and $u_{ij} = u_{ji}$ and $u_{jk} = u_{kj}$ implies $u_{ik} = u_{ki}$.

Proposition 6.2: Let U and V be two m×n NFM such that U≺ V. Then for any l×m NFM W we have WU≺WV.

Proof: Similarly, if $v_{tj}^{F} = 1$, then $u_{tj}^{F} = 1$ and so $x_{ij}^{F} = 1$.Also, if $w_{it} = 1$, then $x_{ij}^{F} = 1$.Thus $f_{ij}^{F} = 1$ implies $x_{ij}^{F} = 1$. Therefore, $f_{ij} = 0$ implies $x_{ij} = 0$ and hence WU≺WV.

It is observed that the relation ≺ is reflexive and transitive NFM. The following example illustrates this proposition.

Example 6.1: Let us assume that the NFM

$$U = \begin{bmatrix} <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.7> \\ <0.7,0.3,0.3> & <0.5,0.3,0.4> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0.4,0.3,0.6> \end{bmatrix}$$

$$V = \begin{bmatrix} <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.7> \\ <0.8,0.3,0.2> & <0.6,0.3,0.4> & <0,0,1> \\ <0.2,0.3,0.8> & <0,0,1> & <0.9,0.3,0.1> \end{bmatrix}$$

and $W = \begin{bmatrix} <0.5,0.3,0.4> & <0,0,1> & <0.3,0.3,0.7> \\ <0.5,0.3,0.5> & <0,0,1> & <0,0,1> \\ <0.7,0.3,0.3> & <0,0,1> & <0,0,1> \end{bmatrix}$

Then it is clear that U≺V.

Also, $WU = \begin{bmatrix} <0,0,1> & <0.3,0.3,0.4> & <0.3,0.3,0.7> \\ <0,0,1> & <0.3,0.3,0.5> & <0.2,0.3,0.5> \\ <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.3> \end{bmatrix}$

and $WV = \begin{bmatrix} <0.2,0.3,0.8> & <0.3,0.3,0.4> & <0.3,0.3,0.7> \\ <0,0,1> & <0.3,0.3,0.5> & <0.2,0.3,0.7> \\ <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.7> \end{bmatrix}$

and it is clear also that CU≺CV

Proposition6.3: If U is weak transitive if and only if $(\Delta U)^2 \prec \Delta U$.

Proof: Conversely, suppose that $S^2 \prec S$ and let $u_{ij}^{T} > u_{ji}^{T}, u_{jk}^{T} > u_{jk}^{T}$, $u_{ij}^{I} > u_{ji}^{I}, u_{jk}^{I} > u_{jk}^{I}$

And let also $u_{ij}^{F} < u_{ji}^{F}, u_{jk}^{F} < u_{jk}^{F}$ .Then $u_{ij}^{T} \wedge u_{jk}^{T} > u_{ji}^{T} \wedge u_{kj}^{T}$, $u_{ij}^{I} \wedge u_{jk}^{I} > u_{ji}^{I} \wedge u_{kj}^{I}$ and $u_{ij}^{F} \vee u_{jk}^{F} < u_{ji}^{F} \wedge u_{kj}^{F}$ .Now let $u_{ik}^{T} \leq u_{ki}^{T}, u_{ik}^{I} > u_{ki}^{I}$ and $u_{ik}^{F} \geq u_{ki}^{F}$. i.e., let $s_{ik} = 0$. Since $S^2 \prec S$ we get $s_{ik}^{(2)} = 0$. That is $s_{it} \wedge s_{tk} = 0$ for every $t \geq n$ and so $s_{ij} \wedge s_{jk} = 0.$ If $s_{ij} = 0$, then $u_{ij}^{T} \leq u_{ji}^{T}, u_{ij}^{I} \leq u_{ji}^{I}$ and $u_{ij}^{F} \geq u_{ji}^{F}$. Also, if $s_{jk} = 0,$ then $u_{jk}^{T} \leq u_{kj}^{T}, u_{jk}^{I} \leq u_{kj}^{I}$ and $u_{jk}^{F} \geq u_{kj}^{F}$.

Therefore, $u_{ij}^{T} \wedge u_{jk}^{T} \leq u_{ji}^{T} \wedge u_{kj}^{T}, u_{ij}^{I} \wedge u_{jk}^{I} \leq u_{ji}^{I} \wedge u_{kj}^{I}$ and $u_{ij}^{F} \vee u_{jk}^{F} < u_{ji}^{F} \vee u_{kj}^{F}$ which is a contradiction. Hence $u_{ik}^{T} > u_{ki}^{T}, u_{ik}^{I} > u_{ki}^{I}$ and $u_{ik}^{F} < u_{ki}^{F}$. Thus, U is weak transitive.

Remark6.1: Consistency is evidently a stronger property than weak transitivity. Specifically, if U is consistent, then it is weakly transitive; however, the converse does not necessarily hold. Furthermore, it is observed that transitivity implies weak transitivity.

Proposition6.4: If U is weak transitive, then ΔU is nilpotent.

Proof: By Proposition 6.3, we obtain $(\Delta U)^2 \prec \Delta U.$ and also by Proposition3.2,we can see that $(\Delta U)^3 \prec (\Delta U)^2$ and soon. But by the transitivity of the relation $\prec$, we get $(\Delta U)^k \prec (\Delta U)^{k-1}$ for every $k \leq n.$ Now, let $S = \Delta U.$

Then $s_{ii} = 0$ and hereafter $s_{ii}^{(k)} = 0$ for all $k, i \in I_n.$

Therefore by, we conclude $S = \Delta U$ is nilpotent.

Proposition 6.5: If U is controllable if and only if ΔU is nilpotent. By Proposition since we obtain ΔU is nilpotent for a weak transitive NFM U, we conclude by Definition that U is controllable and so, there exists a PNFM P such that F = PRP^T satisfies $f_{ij} \geq f_{ji}$ for every $i > j$ which is th ecanonical form of the weak transitive NFM U.

Let $E_{(i,j)}$) represents the matrix derived from the identity NFM I_n by exchanging the ith row by jth row for i, j ≤ n. We can now present the following proposition.

Proposition6.6: If U is consistent (weak transitive) iff $E_{(i,j)}$ U $E_{(i,j)}$ is consistent (weak transitive).

Proof: Let U be consistent and let V = $E_{(i,j)}$U $E_{(i,j)}$.

The elements of E(i,j) can be expressed as follows:

$$e_{kl} = \begin{cases} a_{kl} & if\ \ k \neq i \neq j\ \ or\ \ l \neq i \neq j, \\ 1 & if\ \ k = i\ \ and\ l = j\ \ or\ if\ \ k = i\ and\ i = l, \\ 0 & otherwise \end{cases}$$

The elements of the NFM are derived from the elements of U as follows

$$v_{kl} = \begin{cases} u_{rs} & if\ \ r \neq s \neq i \neq j, \\ u_{rs} & if\ \ r = i\ \ and\ v = j\ \ or\ if\ \ u = j\ and\ v = i, \\ u_{ri} & if\ \ s = j, \\ u_{rj} & if\ \ s = i, \\ u_{is} & if\ \ r = j, \\ u_{is} & if\ \ r = i, \\ u_{ii} & if\ \ r = s = j, \\ u_{jj} & if\ \ r = s = i, \end{cases}$$

Next, we demonstrate that V is consistent, relying on the consistency of U. This requires evaluating several cases. To perform these computations, $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ .We will show that $v_{hl} \geq v_{lh}$ in all the following cases where $h \neq k \neq l$ everywhere.

Case(i) If $j \neq i \neq h \neq k \neq l,$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hk} \geq u_{kh}$ and $u_{kl} \geq u_{lk}$ .Thus $u_{hl} \geq u_{lh}$ and hence $v_{hl} \geq v_{lh}$.

Case(ii) If $h = i$ and $k = j$ or if $h = j$ and $k = i$, then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hk} \geq u_{kh}$ and $u_{il} \geq u_{li}$ .Thus $u_{kh} \geq u_{hk}$ and $v_{hl} \geq v_{lh}$. Hence $u_{kl} \geq u_{lk}$ by the consistency of U. That is $u_{jl} \geq u_{lj}$ and therefore $v_{hl} \geq v_{lh}$.

Case(iii) If $k = i$ and $l = j$ or if $k = j$ and $l = i$, then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hj} \geq u_{jh}$ and $u_{lk} \geq u_{kl}$ .Thus $u_{hj} \geq u_{jh}$ and $u_{ji} \geq u_{ij}$ .Hence $u_{hi} \geq u_{ih}$ and $v_{hl} \geq v_{lh}$.

Case(iv) If $k = i$ and $l \neq j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hj} \geq u_{jh}$ and $u_{jl} \geq u_{lj}$ .Thus $u_{hl} \geq u_{lh}$ and so $v_{hl} \geq v_{lh}$.

Case(v) If $k = j$ and $i \neq l$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hi} \geq u_{ih}$ and $u_{il} \geq u_{li}$ .Thus $u_{hl} \geq u_{lh}$ and so $v_{hl} \geq v_{lh}$.

Case(vi) If $j \neq l$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{jk} \geq u_{kj}$ and $u_{kl} \geq u_{lk}$ .Thus $u_{jl} \geq u_{lj}$ and so $v_{hl} \geq v_{lh}$.

Case(vii) If $h = j$ and $i \neq l$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{ik} \geq u_{ki}$ and $v_{kl} \geq v_{lk}$ and $u_{ik} \geq u_{ki}$ .Thus $u_{il} \geq u_{lj}$ and so $v_{hl} \geq v_{lh}$.

Case(viii) If $l = i$ and $l \neq j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hk} \geq u_{kh}$ and $u_{kj} \geq u_{jk}$ .Thus $u_{hj} \geq u_{jh}$ and so $v_{hl} \geq v_{lh}$..

Case(ix) If $l \neq i$ and $l = j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hk} \geq u_{kh}$ and $u_{ki} \geq u_{ik}$ .Thus $u_{hi} \geq u_{ih}$ and so $v_{hl} \geq v_{lh}$.

Case(x) If $h = i$ and $l = j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{jk} \geq u_{kj}$ and $u_{ki} \geq u_{ik}$ .Thus $u_{lh} \geq u_{hl}$ and so $v_{hl} \geq v_{lh}$.

Case(xi) If $h = j$ and $l = i$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{ik} \geq u_{ki}$ and $u_{kj} \geq u_{jk}$ .Thus $u_{ij} \geq u_{ji}$ that is $u_{lh} \geq u_{hl}$ and so $v_{hl} \geq v_{lh}$. **Conversely,** suppose $V = E_{(i,j)}UE_{(i,j)}$ is consistent. Then $E_{(i,j)}VE_{(i,j)}$ is consistent. But $E_{(i,j)}VE_{(i,j)} = E_{(i,j)}E_{(i,j)}UE_{(i,j)}UE_{(i,j)} = IUI = U$

Thus U is consistent. This concludes the proof. Similarly, the case for weak transitivity can be demonstrated in an analogous manner.

Example 6.2: Let us assume that the NFM

$$U = \begin{bmatrix} <0.6,0.3,0.4> & <0.7,0.3,0.2> & <0.5,0.3,0.5> & <0.4,0.3,0.5> \\ <0.6,0.3,0.3> & <0.9,0.3,0.1> & <0.6,0.3,0.3> & <0.5,0.3,0.5> \\ <0.4,0.3,0.6> & <0.7,0.3,0.2> & <0.8,0.3,0.2> & <0.9,0.3,0.0> \\ <0.3,0.3,0.6> & <0.4,0.3,0.6> & <0.6,0.3,0.4> & <0.7,0.3,0.1> \end{bmatrix}$$

By the definition of weak transitivity, it is evident that U is weakly transitive. For instance $u_{13} > u_{31}$

and $u_{32} > u_{23}$ implies $u_{12} > u_{21}$. Also, $u_{32} > u_{23}$ and $u_{24} > u_{42}$ implies $u_{34} > u_{43}$ and so on for all $i \neq j \neq k \leq 4$.

Now $\Delta U = U \Theta U^T = \begin{bmatrix} <0,0,1> & <0.7,0.3,0.2> & <0.5,0.3,0.5> & <0.4,0.3,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0.5,0.3,0.5> \\ <0,0,1> & <0.7,0.3,0.2> & <0,0,1> & <0.9,0.3,0.0> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$

Whichisweaktransitiveandmoreover

$(\Delta U)^2 = \begin{bmatrix} <0,0,1> & <0.5,0.3,0.5> & <0,0,1> & <0.4,0.3,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0.5,0.3,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix} \prec \Delta U$

and $(\Delta U)^3 = 0$ (thezeromatrix) which means that ΔU is nilpotent. Let

$P = \begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> & <1,1,0> \\ <0,0,1> & <1,1,0> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <1,1,0> & <0,0,1> \\ <1,1,0> & <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$

Be a permutation NFM.Then

$F = PUP^T = \begin{bmatrix} <0.7,0.3,0.1> & <0.4,0.3,0.6> & <0.6,0.3,0.4> & <0.3,0.3,0.6> \\ <0.5,0.3,0.5> & <0.9,0.3,0.1> & <0.6,0.3,0.3> & <0.6,0.3,0.3> \\ <0.9,0.3,0.0> & <0.7,0.3,0.2> & <0.8,0.3,0.2> & <0.4,0.3,0.6> \\ <0.4,0.3,0.5> & <0.7,0.3,0.2> & <0.5,0.3,0.5> & <0.6,0.3,0.4> \end{bmatrix}$

With $f_{ij} \geq f_{ji}$ for every $i > j$ and U is thus controllable. Observe that F is weakly transitive.

This example illustrates the results derived in Propositions 6.3, 6.4, 6.5, and 6.6. Additionally, it demonstrates that the weakly transitive NFM U is controllable.

7.Reflexivity and Transitivity o f NFM

Definition:7.1 For $U = (u^T, u^I, u^F)$ and $V = (v^T, v^I, v^F) \in (NFM)$, we define as:

$$u \to v = \begin{cases} (1,1,0) & if\ u^T \leq v^T,\ u^I \leq v^I, \\ (v^T, 0, 0) & if\ u^T > v^T, u^F \geq v^F, \\ (v^I, 0, 0) & if\ u^I > v^I \text{ and } u^F \geq v^F, \\ (u^T, u^I, u^F) & if\ u^T > v^T, u^I > v^I \text{ and } u^F < v^F \end{cases}$$

However, theoperator is the Godel implication operator which is well known in many branches of fuzzy mathematics.

The min $\rightarrow$ composition of two NFM $U=\left[u_{ij}=\left(u^{T}{}_{ij},u^{I}{}_{ij},u^{F}{}_{ij}\right)\right]$ and $W=\left[w_{ij}=\left(w^{T}{}_{ij},w^{I}{}_{ij},w^{F}{}_{ij}\right)\right]$ is denoted by $X=U\rightarrow W$ and is defined as $x_{ij}=\bigcap_{k=1}^{n}\left(u_{ik}\rightarrow w_{kj}\right).$

In this section, we explore various properties of the operations defined earlier. Additionally, we briefly examine some properties of NFMs representing Neutrosophic fuzzy pre-orders using these operations $\rightarrow$. Our focus will be on the reflexive and transitive IFM. Now, let us highlight some key properties of these operations $\rightarrow$

Lemma 7.1 For $u=\left(u^{T},u^{I},u^{F}\right),v=\left(v^{T},v^{I},v^{F}\right),w=\left(w^{T},w^{I},w^{F}\right)\in\left(NFM\right)$, we have $\left(u\rightarrow v\right)\leftarrow w=u\rightarrow\left(v\leftarrow w\right)=v\leftarrow\left(u\wedge w\right).$

Proof. Based on the definition of the operation, we identify the following four cases, each of which contains four subcases. The cases are as follows

Csse(i) If $u^{T}\leq v^{T}$, $u^{I}\leq v^{I}$

Csse (ii) If $u^{T}>v^{T},u^{F}\geq v^{F}$,

Csse (iii) If $u^{I}>v^{I}$ and $u^{F}\geq v^{F}$

Csse (iv) If $u^{T}>v^{T},u^{I}>v^{I}$ *and* $u^{F}<v^{F}$

The subcases of each case are:

Csse(i) If $w^{T}\leq v^{T}$, $w^{I}\leq v^{I}$

Csse (ii) If $w^{T}>v^{T},w^{F}\geq v^{F}$,

Csse (iii) If $w^{I}>v^{I}$ and $w^{F}\geq v^{F}$

Csse (iv) If $w^{T}>v^{T},w^{I}>v^{I}$ *and* $w^{F}<v^{F}$

We prove one case,namely, Case(ii) and (iii) and the proofs of the other two cases are similar. To do that, suppose $w^{T}>v^{T},w^{F}\geq v^{F}$, $w^{I}>v^{I}$ and $w^{F}\geq v^{F}$

(i)If $w^T \le v^T$, $w^I \le v^I$, then

$$\left[\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(v^T,0,0\right)\leftarrow\left(w^T,w^I,w^F\right)=\left(1,1,0\right)$$

and $\left(u^T,u^I,u^F\right)\to\left[\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(u^T,u^I,u^F\right)\to\left(1,1,0\right)=\left(1,1,0\right)$

Since $u^T\wedge u^F\le u^I$,

we get $\left(v^T,v^I,v^F\right)\leftarrow\left[\left(u^T,u^I,u^F\right)\wedge\left(w^T,w^I,w^F\right)\right]=\left(1,1,0\right)$

(ii) If $w^T>v^T, w^I>v^I$ and $w^F<v^F$,then

$$\left[\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)\right]\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)$$

and $\left(u^T,u^I,u^F\right)\to\left[\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)=\left(v^T,0,0\right)$

Since $u^T\wedge u^F>u^I$ and $v^T\vee v^F\ge v^I$ we get

$\left(v^T,v^I,v^F\right)\leftarrow\left[\left(u^T,u^I,u^F\right)\wedge\left(w^T,w^I,w^F\right)\right]=\left(v^T,0,0\right)$

(iii)If $w^T>v^T$ and $w^F\ge v^F$, then

$$\left[\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)\right]\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)$$

and $\left(u^T,u^I,u^F\right)\to\left[\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(u^T,u^I,u^F\right)\to\left(v^T,0,0\right)=\left(v^T,0,0\right)$

Last, $u^T\wedge u^F>u^I$ and $v^T\vee v^F\ge v^I$ implies

$\left(v^T,v^I,v^F\right)\leftarrow\left[\left(u^T,u^I,u^F\right)\wedge\left(w^T,w^I,w^F\right)\right]=\left(v^T,0,0\right)$.

Hence from all the above cases we conclude

$\left(u\to v\right)\leftarrow w=u\to\left(v\leftarrow w\right)=v\leftarrow\left(u\wedge w\right)$ for every $u,v,w\in(NFM)$.

Proposition 7.1 Let $U=\left[u_{ij}\right]_{m\times p}, V=\left[v_{ij}\right]_{p\times g}$ and $W=\left[w_{ij}\right]_{g\times n}$ be three NFM. Then

$\left(U\to V\right)\leftarrow V=U\to\left(V\leftarrow V\right)$.

Proof. Let $X=\left(U\to V\right)\leftarrow V$ and $Y=U\to\left(V\leftarrow V\right)$

Then by Lemma 7.1,

$$x_{ij}=\bigcap_{l=1}^{g}\left[\bigcap_{k=1}^{p}\left(u_{ik}\to v_{kl}\right)\leftarrow w_{lj}\right]=\bigcap_{l=1}^{g}\bigcap_{k=1}^{p}\left(u_{ik}\to v_{kl}\leftarrow w_{lj}\right)$$

Also, $y_{ij} = \bigcap_{k=1}^{p}\left[u_{ik} \to \bigcap_{k=1}^{p}\left(v_{kl} \leftarrow w_{lj}\right)\right] = \bigcap_{k=1}^{p}\bigcap_{l=1}^{g}\left(u_{ik} \to v_{kl} \leftarrow w_{lj}\right)$

$x_{ij} = y_{ij}.$

Proposition 7.2 If an NFM U is reflexive and transitive, then U is idempotent.

Proof. Since U is transitive, it is sufficient to prove that $U^2 \geq U$

Now, let U be of order nxn and let $S = U^2$.

Then $s_{ij} = \left(s_{ij}^{\ T}, s_{ij}^{\ I}, s_{ij}^{\ F}\right) = \left[\bigcup_{k=1}^{n}\left(u_{ik}^{\ T} \wedge u_{kj}^{\ T}\right), \bigcup_{k=1}^{n}\left(u_{ik}^{\ I} \wedge u_{kj}^{\ I}\right), \bigcap_{k=1}^{n}\left(u_{ik}^{\ F} \vee u_{kj}^{\ F}\right)\right]$

$\geq \left(u_{ij}^{\ T} \wedge u_{ij}^{\ T}, u_{ij}^{\ I} \wedge u_{ij}^{\ I}, u_{ij}^{\ F} \vee u_{ij}^{\ F}\right)$

$= \left(u_{ij}^{\ T} \wedge 1, u_{ij}^{\ I} \wedge 1, u_{ij}^{\ F} \vee 0\right)$

$= \left(u_{ij}^{\ T}, u_{ij}^{\ I}, u_{ij}^{\ F}\right)$

$= u_{ij}$ (since U is reflexive).

i.e., $U^2 \geq U$ and U is so idempotent.

Theorem 7.1 Let U and V be two mxn NFM such that $U \leq V.$ Then $U \leftarrow V^t$ and $V^t \to U$ are transitive.

Proof. Let $W = U \leftarrow V^t.$ To show that W is transitive, we must prove that

$w_{ij} \geq w_{il} \wedge w_{li}$ for all $l \leq n.\ w_{il} \wedge w_{li} = x > 0.$ then

$w_{il} = \bigcap_{k=1}^{n}\left(u_{ik} \leftarrow v_{ik}\right) \geq x$ and $w_{lj} = \bigcap_{k=1}^{n}\left(u_{lk} \leftarrow v_{jk}\right) \geq x$

If $w_{ij} = \left(w_{ij}^{\ T}, w_{ij}^{\ I}, w_{ij}^{\ F}\right)$ be such that $w_{ij}^{\ T} < x^T$, $w_{ij}^{\ I} < x^I$ and $w_{ij}^{\ F} > x^F$

Where $x = \left(x^T, x^I, x^F\right)$

That is, if $\left(u_{ih}^{\ T}, u_{ih}^{\ I}, u_{ih}^{\ F}\right) \leftarrow \left(v_{jh}^{\ T}, v_{jh}^{\ I}, v_{jh}^{\ F}\right) < \left(x^T, x^I, x^F\right)$,

Then $\left(u_{ih}^{\ T}, u_{ih}^{\ I}, u_{ih}^{\ F}\right) \leftarrow \left(v_{jh}^{\ T}, v_{jh}^{\ I}, v_{jh}^{\ F}\right) \neq (1,1,0)$

Since $0 \leq x^F$, So that $\left(u_{ih}^{\ T}, u_{ih}^{\ I}, u_{ih}^{\ F}\right) \leftarrow \left(v_{jh}^{\ T}, v_{jh}^{\ I}, v_{jh}^{\ F}\right) = \left(u_{ih}^{\ T}, u_{ih}^{\ I}, u_{ih}^{\ F}\right) < \left(x^T, x^I, x^F\right).$

Thus, $u_{ih}^{\ T} < x^T, u_{ih}^{\ I} < x^I$ and $u_{ih}^{\ F} > x^F$ and in this case, $u_{ih}^{\ T} < v_{jh}^{\ T}, u_{ih}^{\ I} < v_{jh}^{\ I}$ and $u_{ih}^{\ F} > v_{jh}^{\ F}$

By the definitionof $\leftarrow$

Since $w_{il} \geq x,$ we have $\left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right)$

and so $\left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) \neq \left(u_{ih}{}^{T}, 0, 0\right), \left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right).$

Therefore, $\left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) = (1,1,0).$

But $u_{ih}{}^{T} < v_{jh}{}^{T}, u_{ih}{}^{I} < v_{jh}{}^{I}$ and $u_{ih}{}^{F} > v_{jh}{}^{F}.$

Then $u_{ih}{}^{T} \geq v_{jh}{}^{T}, u_{ih}{}^{I} \geq v_{jh}{}^{I}$ and $u_{ih}{}^{F} \leq v_{jh}{}^{F}$

Also, $w_{lj} \geq x,$ we have that $\left(u_{lh}{}^{T}, u_{lh}{}^{I}, u_{lh}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right)$

and so that $u_{lh}{}^{T} \geq v_{jh}{}^{T}, u_{lh}{}^{I} \geq v_{jh}{}^{I}$ and $u_{lh}{}^{F} \leq v_{jh}{}^{F}$

Since $U \leq V,$ we have $x^{T} > u_{ih}{}^{T} \geq v_{lh}{}^{T} \geq u_{lh}{}^{T} \geq v_{jh}{}^{T}, x^{I} > u_{ih}{}^{I} \geq v_{lh}{}^{I} \geq u_{lh}{}^{I} \geq v_{jh}{}^{I}$

However, this contradicts that $u_{ih}{}^{T} < v_{jh}{}^{T}, u_{ih}{}^{I} < v_{jh}{}^{I}.$ So that $x_{ij}{}^{T} \geq x^{I}.$

Aiso, since $U \leq V,$ We have $x^{F} < u_{ih}{}^{F} \leq v_{lh}{}^{T} \leq u_{lh}{}^{T} \leq v_{jh}{}^{T}.$

However, this contradicts that $x^{F} \geq u_{ih}{}^{F}$ .So that $w \leq x^{F}$

Therefore, $w_{ij} = \left(w^{T}, w^{I}, w^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right) = x$ and W is transitive.

The transitivity of the matrix $V^{t} \rightarrow V$ can be obtained in a similar manner.

Corollary 7.1 For any mxn NFM U, the NFM $V \rightarrow V^{t}$ and $V^{t} \rightarrow V$ are idempotent.

Proof. It is easy to see that U $V \rightarrow V^{t}$ and $V^{t} \rightarrow V$ are reflexive. Then by Theorem 7.1 and Proposition 7.2 they are idempotent.

Theorem3.2 Let U be any nxn NFM. Then U is reflexive and transitive if and only if $V \leftarrow V^{t} = V$ (also if and only if $V^{t} \rightarrow V = V.$)

Proof. Suppose U is reflexive and transitive and let $W = V \leftarrow V^{t}$

$$i.e., w_{ij} = \bigcap_{k=1}^{n} \left[\left(u_{ik}{}^{T}, u_{ik}{}^{I}, u_{ik}{}^{F}\right) \leftarrow \left(u_{jk}{}^{T}, u_{jk}{}^{I}, u_{jk}{}^{F}\right) \right].$$

Suppose $w_{ij} = \left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right) = \left(x^{T}, x^{I}, x^{F}\right) > (0,0,1).$

Then $\left(u_{ij}{}^{T}, u_{ij}{}^{I}, u_{ij}{}^{F}\right) \leftarrow \left(u_{jj}{}^{T}, u_{jj}{}^{I}, u_{jj}{}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right).$

That is $\left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right) \geq (0,0,1) \geq \left(x^{T}, x^{I}, x^{F}\right)$ (by the reflexivity of U)which yielding

$u_{ij} = \left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right) = w_{ij}$ and so $U \geq U \leftarrow U^{t}$. On the other hand we will show that $U \geq U \leftarrow U^{t}$. using the transitivity of U. .

Suppose that $u_{ij} = \left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right) = \left(x^{T}, x^{I}, x^{F}\right) > (0,0,1)$

and $w_{ij} = \left(w_{ij}^{T}, w_{ij}^{I}, w_{ij}^{F}\right) = \left(u_{il}^{T}, u_{il}^{I}, u_{il}^{F}\right) \leftarrow \left(u_{jl}^{T}, u_{jl}^{I}, u_{jl}^{F}\right)$

For a given $l \leq n.$ According to the definition of the operation, the following three cases arise:
$W \geq U.$

Case(i) $\left(u_{il}^{T}, u_{il}^{I}, u_{il}^{F}\right) \leftarrow \left(u_{jl}^{T}, u_{jl}^{I}, u_{jl}^{F}\right) = (0,0,1)$ then it is clear that $w_{ij} \geq x = u_{ij}$. Thus

Case(ii) If $\left(u_{il}^{T}, u_{il}^{I}, u_{il}^{F}\right) \leftarrow \left(u_{jl}^{T}, u_{jl}^{I}, u_{jl}^{F}\right) = \left(u_{jl}^{T}, u_{jl}^{I}, u_{jl}^{F}\right) < \left(x^{T}, x^{I}, x^{F}\right),$

Then $u_{il}^{T} < x^{T}, u_{il}^{I} < x_{ij}^{I}$ and $u_{il}^{F} > x^{F}$

$i.e., u_{il}^{T} < x^{T} \wedge u_{jl}^{T}, u_{il}^{I} < x^{I} \wedge u_{jl}^{I}$ and $u_{il}^{F} > x^{F} \vee u_{jl}^{F}.$

Now since we have that U is transitive, we get $\left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right) \wedge \left(u_{jl}^{T}, u_{jl}^{I}, u_{jl}^{F}\right) \leq \left(u_{il}^{T}, u_{il}^{I}, u_{il}^{F}\right).$

Thus $\left(x^{T}, x^{I}, x^{F}\right) \wedge \left(u_{jl}^{T}, u_{jl}^{I}, u_{jl}^{F}\right) \leq \left(u_{il}^{T}, u_{il}^{I}, u_{il}^{F}\right).$

Therefore, $u_{il}^{T} < x^{T} \wedge u_{jl}^{T} \leq u_{il}^{T}, u_{il}^{I} < x^{I} \wedge u_{jl}^{I} \leq u_{il}^{I}$ and $u_{il}^{F} > x^{F} \vee u_{jl}^{F} \geq u_{il}^{F}.$ However, these are contradictions and so $w_{jl}^{T} \geq x^{T}, w_{jl}^{I} \geq x^{I}$ and $w_{jl}^{F} \leq x^{F}$. Thus $w_{ij} \geq x =$ u_{ij} and so $W \geq U.$

Case(iii) If $w_{ij} = \left(u_{il}^{T}, 0, 0\right)$, Then $u_{il}^{T} < u_{jl}^{T}$. Since we have that $0 < x^{F}$

it is enough to show that $u_{il}^{T} \geq x^{T}.$ Suppose that $u_{il}^{T} < x^{I}.$

Then $u_{il}^{T} < x^{T} \wedge u_{jl}^{T} = u_{ij}^{T} \wedge u_{jl}^{T} \leq u_{il}^{T}$ (Again by the transitivity of U) and also we have a contradiction and so $u_{il}^{T} \geq x^{T}$ and $w_{ij} \geq x.$ Thus $w_{ij} \geq x = u_{ij}$ $i.e., W \geq U.$

Conversely, if $U \leftarrow U^{t} = U$ then by the proof of Corollary 310, U is reflexive and transitive so it is idempotent.

Similarly, we can prove that $U^{t} \rightarrow U = U$.

Theorem 3.2 shows interesting properties of preorders. Thus U is a matrix representing a preorder iff $U \leftarrow U^t = U$ (or $U^t \rightarrow U = U$). However, since $U \leftarrow U^t$ is obtained by using U if we multiply $U \leftarrow U^t$ by U, any information is no tadded to U. That is, the product $\left(U \leftarrow U^t\right)U$ is equal to U.

Example: 7.1 Let us consider the NFM

$$U = \begin{bmatrix} <0.5,0.3,0.4> & <0.7,0.3,0.3> \\ <0.6,0.3,0.3> & <0.8,0.3,0.2> \\ <0.9,0.3,0.0> & <0.4,0.3,0.6> \end{bmatrix}$$

Then $U \leftarrow U^t = \begin{bmatrix} <0.5,0.3,0.4> & <0.7,0.3,0.3> \\ <0.6,0.3,0.3> & <0.8,0.3,0.2> \\ <0.9,0.3,0.0> & <0.4,0.3,0.6> \end{bmatrix}$

$$\leftarrow \begin{bmatrix} <0.5,0.3,0.4> & <0.6,0.3,0.3> & <0.9,0.3,0.0> \\ <0.7,0.3,0.3> & <0.8,0.3,0.2> & <0.4,0.3,0.6> \end{bmatrix}$$

$$= \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

It is clear that $U \leftarrow U^t$ is reflexive and

$$\left(U \leftarrow U^t\right)^2 = \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix} = U \leftarrow U^t$$

That is is reflexive and transitive and so idempotent. Moreover, if we let $V = U \leftarrow U^t$ .Then

$$V \leftarrow V^t = \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

$$\leftarrow \begin{bmatrix} <1,1,0> & <1,1,0> & <0.4,0.3,0.6> \\ <0.5,0.3,0.4> & <1,1,0> & <0.4,0.3,0.6> \\ <0.5,0.3,0.4> & <0.6,0.3,0.3> & <1,1,0> \end{bmatrix}$$

$$= \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix} = V$$

$$\left(V \leftarrow V^t\right)V = \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

$$\begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

$$= \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix} = V .$$

8. Conclusion and Future Work

This paper has thoroughly investigated the properties of consistent and weakly transitive Neutrosophic Fuzzy Matrices (NFM), emphasizing their controllability and the derivation of a canonical form for weakly transitive NFM. The study has highlighted the significance of nilpotent and transitive NFM in matrix analysis, underscoring their importance in uncertainty modeling. By introducing the Gödel implication operator as a novel operation on NFM, we extended the theoretical framework to explore properties related to pre-orders, particularly in reflexive and transitive matrices. Additionally, a method for constructing idempotent NFM from any given matrix was presented, contributing a practical tool for further applications. The use of counterexamples throughout the discussion has provided clarity and validated the theoretical findings, ensuring their robustness and practical relevance.

Building on these findings, future research could focus on extending the analysis to higher-order properties, such as symmetry and anti-symmetry, to enhance the theoretical understanding of NFM. Applications of consistent and weakly transitive NFM in real-world decision-making problems, such as multi-criteria decision analysis and preference modeling, offer promising avenues for exploration. Developing efficient algorithms for constructing idempotent and transitive NFM could improve their computational usability. Additionally, integrating NFM with other fuzzy systems, such as intuitionistic and bipolar fuzzy systems, could address more complex problems requiring multi-dimensional uncertainty modeling. Finally, empirical studies validating these theoretical contributions in practical domains, including network analysis, machine learning, and knowledge representation, could further demonstrate their applicability and impact.

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An Integrated Plithogenic MCDM Approach College English Blended Teaching Quality Evaluation

Yuan Huang[1]*, Jie Yu[2]

[1]School of Foreign Languages, Henan University of Science and Technology, Luoyang, 471000, Henan, China

[2]Changde College, Changde, 415000, Hunan, China (Dominic_82768@163.com)

*Corresponding author, E-mail: hyhaust0517@163.com

Abstract: The Outcome-Based Education method has garnered increased attention due to its focus on curriculum creation based on learning outcomes and its concern for the abilities that society requires. This study proposed a multi-criteria decision making (MCDM) methodology for college English blended teaching quality evaluation. The MCDM method used to deal with different criteria. We proposed an uncertainty framework under the Plithogenic sets to deal with uncertainty and vague data. The Plithogenic sets integrated with the WASPAS method. We used the WASPAS method to rank the alternatives. Eight criteria and eleven colleges are used in this study. The results show the Learning Outcome Achievement criterion has the highest importance and Student Satisfaction has the lowest importance. In the rank of alternatives, the results show the alternative 6 is the best and alternative 10 is the worst. The sensitivity analysis was conducted in this study to show different rank of alternatives. The results show our proposed model obtained stable rank of alternatives.

Keywords: WASPAS Method; Multi-Criteria Decision-making; English Blended Teaching; Evaluation.

1. Introduction

A blended teaching paradigm, which is typically defined as the combination of traditional classroom methods with online activities, started to emerge with the

convergence of education and information technology. The focus of China's education reform in recent years has shifted to blended learning. To establish a diverse teaching and learning environment, colleges and universities should fully utilize information technology, according to the Guidelines on College English Teaching. Top-notch undergraduate courses, including blended, online, and virtual simulation experiment courses, must be designed to help students form the habits of active, independent, and customized learning[1], [2].

In the English teaching process, blended learning methods have grown significantly over the last few years. The growth of large-scale online instruction, particularly in 2020 when the epidemic was prevalent, speeds up the integration of information technology and English teaching reform. This not only guarantees meaningful communication in the classroom and online synchronously, but also fosters a cohesive learning environment and gives students the flexibility to change their time and location, among other advantages[3].

The methodical design of the instructional objectives in blended learning classrooms needs to be reevaluated, even though the English blended learning model can improve students' ability development to some extent. This is because, like traditional classroom instruction, blended learning is always designed strictly according to the prescribed teaching process and teaching contents, which may result in unsuitability for learners' developmental needs and impractical results[4], [5]. Evaluation of the College English Blended Teaching Quality is a MCDM methodology due to it include different criteria. Different MCDM methods are applied on the decision-making problem[6], [7].

WASPAS is an MCDM method. It is used to rank alternatives. To increase accuracy, the approach combines the Weighted Sum Model (WSM) and Weighted Product Model (WPM) approaches. The essay also warns against using qualitative data and emphasizes that the WSM and WPM approaches are only appropriate for quantitative data. Initial procedures are shared by WSM and WPM, the foundational steps of WASPAS. Since MCDM problems are specified by sets of m choices and n criteria, the first step is to construct the Decision Matrix[8], [9].

A set whose elements are defined by one or more qualities, each of which may have many values, is called a plithogenic set P. According to certain specified criteria, the value v of each attribute corresponds to the degree of appurtenance d(x,v) of the element x to the set P[10], [11].

A degree of disagreement (dissimilarity) between every attribute value and the dominant (most significant) attribute value is defined to improve the accuracy of the plithogenic aggregation operators. The degrees of contradiction can also be fuzzy, intuitionistic fuzzy, neutrosophic, or any other fuzzy extension [12], [13].

1.1 The main contributions of this study

The main contribution of this study is organized as follows:

- ✓ We proposed a MCDM methodology for Evaluation of the College English Blended Teaching Quality.
- ✓ The MCDM method is integrated under the plithogenic sets to deal with vague and uncertainty information.
- ✓ The WASPAS method is used under plithogenic to rank the alternatives.
- ✓ The sensitivity analysis is applied to show the different rank of alternatives.

1.2 The rest of this study

The rest of this study is organized as follows: Section 2 shows the method steps. Section 3 shows the proposed method steps. Section 4 shows the results of this study. Section 5 shows the discussion and sensitivity analysis. Section 6 shows the conclusions of this work.

2. Method

This section shows some operations of the plithogenic sets. Plithogeny presents to enhance the new entities from groups of contradictory or non-contradictory various old entities. Plithogenic sets can be defined as (P, A, V, d, c) is a set that includes of number of components defined by a set of criteria. Plithogenic sets can deal with uncertainty information in the decision-making process[14], [15].

Because of its two primary characteristics—contradiction degree and appurtenance degree—the plithogenic set takes uncertainty into account to enhance more accurate findings. Each attribute value is distinguished from the dominant (most desired) characteristic value by the contradiction (dissimilarity) degree function c(v,D), where the appurtenance degree function of the element x with regard to the set of specified criteria is denoted as d(x,v). Intersection, union, complement, inclusion, and equality are examples of lithogenic set actions[16], [17].

Let two Plithogenic numbers and their operations as $x = (x_1, x_2, x_3)\ and\ y = (y_1, y_2, y_3)$.

Plithogenic intersection

$$\left((x_{i1},x_{i2},x_{i3}),1\le i\le n\right)\wedge P\left((y_{i1},y_{i2},y_{i3}),1\le i\le n\right)=\left(\begin{matrix}(x_{i1}\wedge y_{i1}),\\ \left(\frac{1}{2}(x_{i2}\wedge y_{i2})+\frac{1}{2}(x_{i2}\vee y_{i2})\right),\\ (x_{i3}\vee y_{i3})\end{matrix}\right),1\le i\le n \quad (1)$$

Plithogenic union

$$\left((x_{i1},x_{i2},x_{i3}),1\le i\le n\right)\vee P\left((y_{i1},y_{i2},y_{i3}),1\le i\le n\right)=\left(\begin{matrix}(x_{i1}\vee y_{i1}),\\ \left(\frac{1}{2}(x_{i2}\wedge y_{i2})+\frac{1}{2}(x_{i2}\vee y_{i2})\right),\\ (x_{i3}\wedge y_{i3})\end{matrix}\right),1\le i\le n \quad (2)$$

Where

$$x_{i1}\wedge P y_{i1}=\left[\left(1-c(v_D,v_1)\right)\right].t_{norm}(v_D,v_1)+c(v_D,v_1).t_{norm}(v_D,v_1) \quad (3)$$

$$x_{i1}\vee P y_{i1}=\left[\left(1-c(v_D,v_1)\right)\right].t_{conorm}(v_D,v_1)+c(v_D,v_1).t_{norm}(v_D,v_1) \quad (4)$$

Where,

$$t_{norm}=\wedge_F y=ayt_{conorm}x\vee_F y=x+y-xy \quad (5)$$

Plithogenic complement

$$\left((x_{i1},x_{i2},x_{i3}),1\le i\le n\right)=(x_{i3},x_{i2},x_{i1}),1\le i\le n \quad (6)$$

The Weighted Aggregates Sum Product Assessment (WASPAS)

This method can rank alternatives[18], [19]. The steps of this method are shown as:

Step 1. Normalize the decision matrix

$$n_{ij}=\frac{x_{ij}}{\max x_{ij}};i=1,\dots,m;j=1,\dots,n \quad (7)$$

Step 2. Compute the additive relative importance.

$$U_i^{(1)}=\sum_{j=1}^{n}n_{ij}w_j \quad (8)$$

Step 3. Compute the multiplicate relative importance.

$$U_i^{(2)} = \prod_{j=1}^{n} (n_{ij})^{w_j} \tag{9}$$

Step 4. Compute the joint generalization criterion

$$U_i = \frac{1}{2}\left(U_i^{(1)} + U_i^{(2)}\right) = \frac{1}{2}\sum_{j=1}^{n} n_{ij} w_j + \prod_{j=1}^{n} (n_{ij})^{w_j} \tag{10}$$

Step 5. Rank the alternatives.

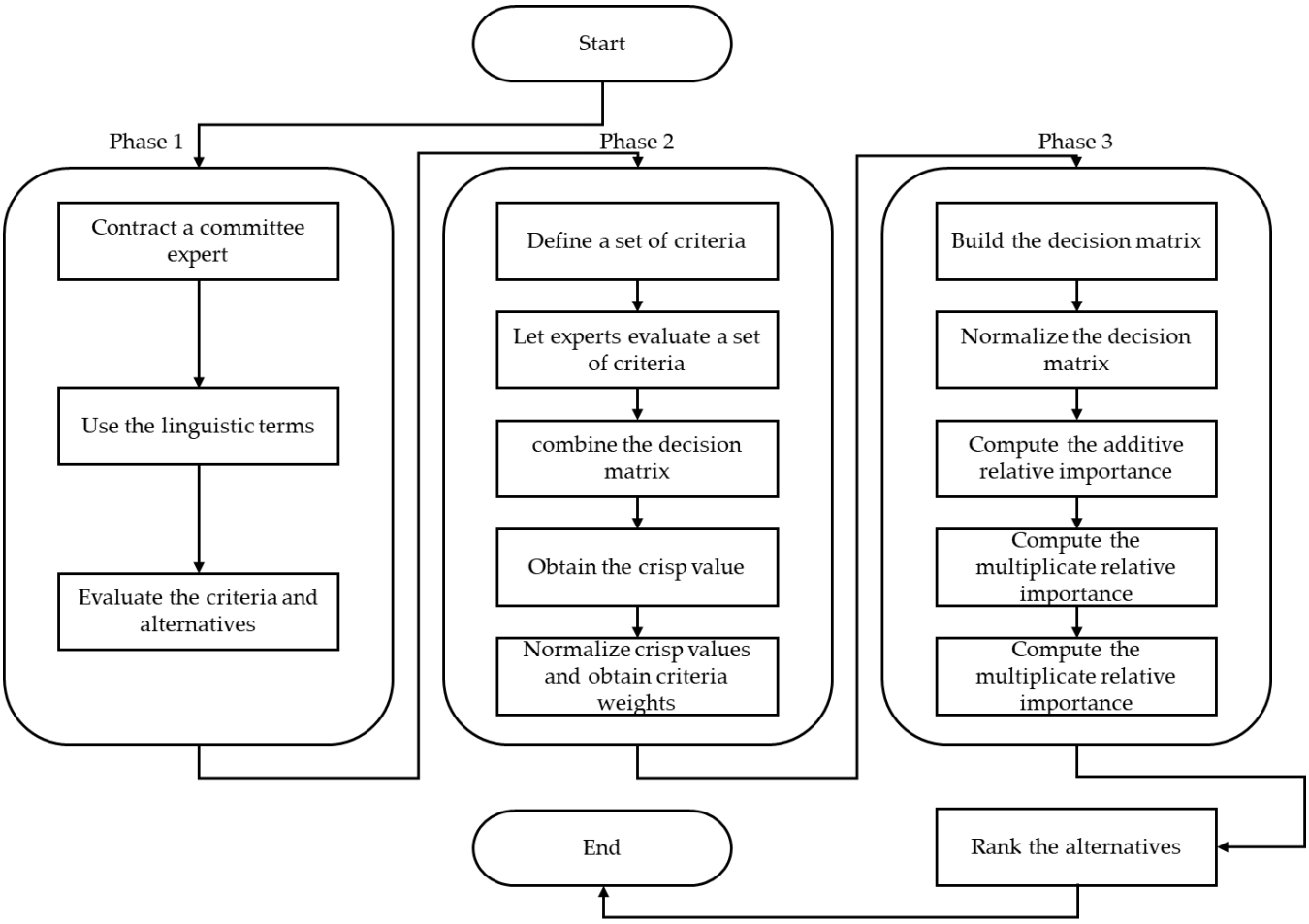

Figure 1. The steps of the plithogenic WASPAS.

3. The Suggested Methodology

In this study we propose plithogenic MCDM methodology for ranking the alternatives and compute the criteria weights. In this section we show the proposed plithogenic methodology based on the WASPAS methodology. Figure 1 shows the steps of the proposed methodology.

Phase 1. Contract a committee expert with expiries in the education filed. $EX = \{d_1, d_2, \dots, d_k\}$, where k refers to the number of experts. Define the problem dimensions including set of criteria and alternatives such as $C = \{c_1, \dots, c_n\}$; $A = \{A_1, \dots, A_m\}$.

Phase 2. Compute the criteria weights of this problem.

Step 1. Define a set of criteria. A set of criteria are gathered from previous studies to be evaluated.

Steo 2. Let experts evaluate a set of criteria.

Step 3. We applied the steps of the plithogenic operations to combine the decision matrix into one matrix.

Step 4. Obtain the crisp value as:

$$S(x) = \frac{2+T-I-F}{3} \tag{11}$$

Step 5. Normalize crisp values. To obtain the criteria weights.

Step 6. Rate the criteria weights to show the highest criterion and lowest criterion.

Phase 3. Rank the alternatives.

Step 7. Build the decision matrix between criteria and alternatives.

Step 8. Normalize the decision matrix using Eq. (7).

Step 9. Compute the additive relative importance using Eq. (8).

Step 10. Compute the multiplicate relative importance using Eq. (9)

Step 11. Compute the joint generalization criterion using Eq. (10).

Step 5. Rate the alternatives.

4. Case Study and Results

This section shows the results of the proposed methodology.

Case Study Dimension

This study proposed a Plithogenic MCDM for evaluation of college English blended teaching quality. We define eight criteria and eleven alternatives. The criteria of this study and alternatives are shown in Figure 2.

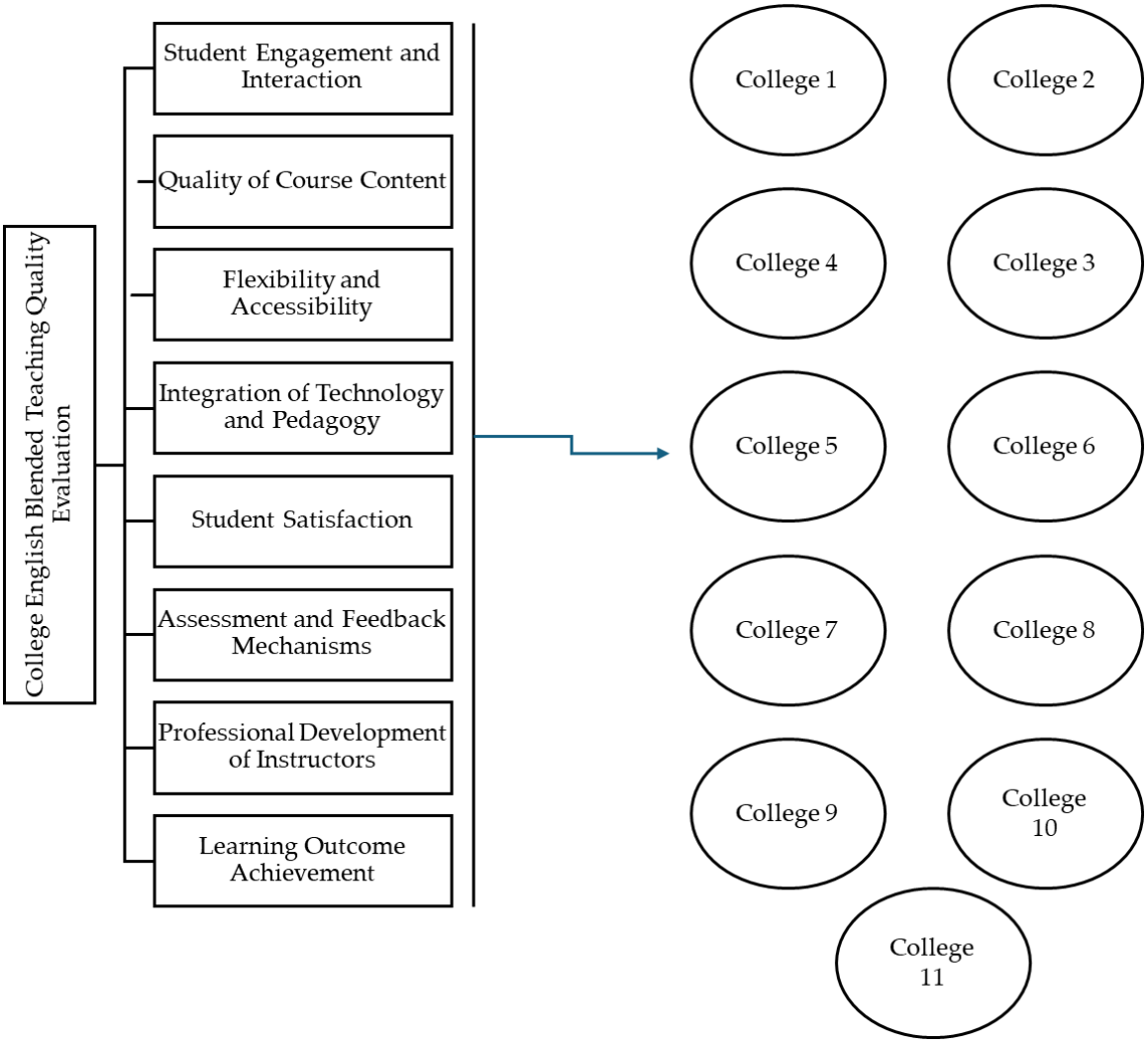

Figure 2. The list of criteria and alternatives.

Phase 1. The problem dimension criteria and alternatives are defined to compute the criteria weights and rank the alternatives.

Phase 2. The criteria weights are computed. This study invited five experts to evaluate the criteria. We used the plithogenic numbers to evaluate the criteria. Then we aggregate it using the plithogenic operations. Then we obtain crisp values. Then we normalize the crisp values to obtain the criteria weights as shown in Figure 3.

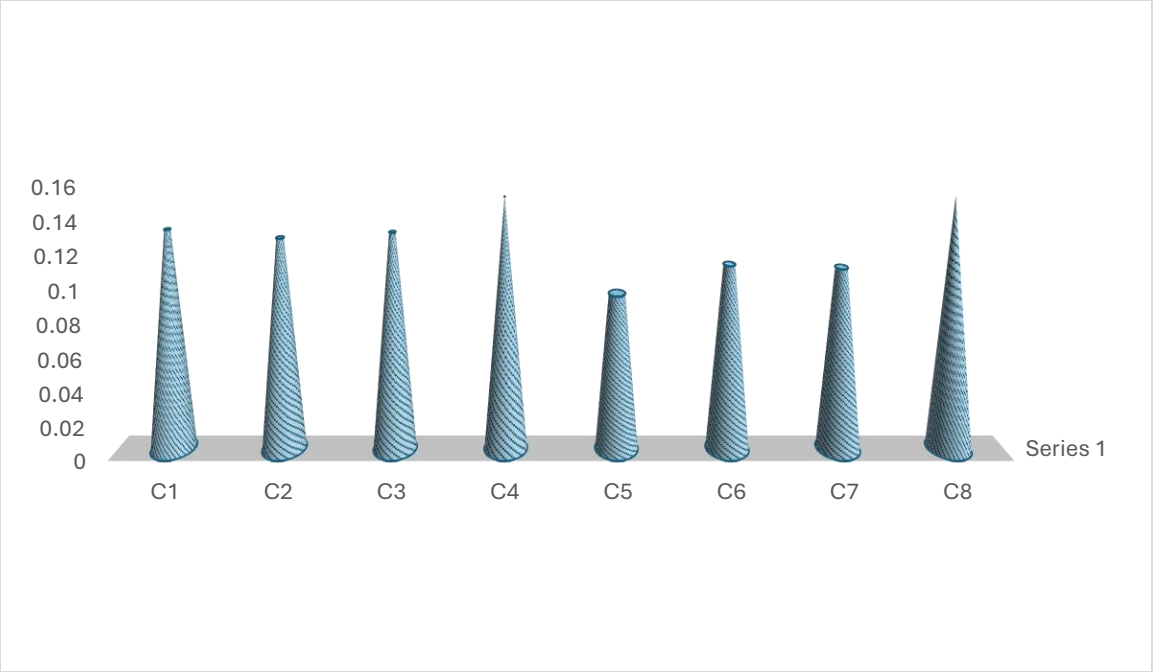

Figure 3. The criteria weights.

Phase 3. Rank the alternatives.

Five experts are building the decision matrix using plithogenic numbers as shown in Tables 1-5.

Table 1. The decision matrix by first expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_2	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_6	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_7	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_8	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_9	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)
A_{10}	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_{11}	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)

Table 2. The decision matrix by second expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_2	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_3	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_5	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_6	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_7	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_8	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_9	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_{10}	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_{11}	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)

Table 3. The decision matrix by third expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_2	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_3	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_5	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)
A_6	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_7	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)
A_8	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_9	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_{10}	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A_{11}	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)

Table 4. The decision matrix by fourth expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_2	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_3	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_5	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_6	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_7	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_8	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_9	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)
A_{10}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_{11}	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)

Table 5. The decision matrix by fifth expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_2	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_3	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_5	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_6	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_7	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_8	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_9	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_{10}	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_{11}	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)

We combine the decision matrix into a single matrix. We obtain crisp values. Then we normalized the decision matrix based on these crisp values as shown in Table 6. Then we computed the additive relative importance, and we computed the multiplicative relative importance. Then we compute the joint generalized criterion. We ranked the alternatives in Table 7.

Table 6. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.820411	0.934723	0.805462	0.579536	0.329652	0.259273	0.657752	1
A_2	0.752346	0.410648	0.625657	0.746087	0.476285	0.44686	0.428386	0.926284
A_3	0.636711	0.621069	0.616405	0.828923	0.263345	0.4703	0.275699	0.746087
A_4	0.523292	0.56088	0.473847	0.936904	1	0.7855	0.035164	0.603113
A_5	0.44192	0.523292	0.259	0.639379	0.859064	0.705153	0.544784	0.645077
A_6	0.954657	0.482832	0.877706	0.531349	0.740393	1	0.732557	0.690421
A_7	1	0.867441	0.828338	0.400686	0.573475	0.682439	1	0.502268
A_8	0.403831	1	0.805462	0.540093	0.371753	0.567133	0.846742	0.685191
A_9	0.432306	0.200923	0.625657	0.746087	0.36033	0.42767	0.786714	0.869962
A_{10}	0.422479	0.319448	0.407482	0.926284	0.031969	0.576466	0.73083	0.676456
A_{11}	0.545013	0.752884	1	1	0.031969	0.259273	0.428386	0.579536

Table 7. The rank of alternatives.

	Relative importance	*Multiplicative importance*	*Joint generalization criterion*	*Rank*
A_1	0.698532	0.642584	0.670558	9
A_2	0.623568	0.596971	0.610269	7
A_3	0.58514	0.547828	0.566484	6
A_4	0.616112	0.481414	0.548763	4
A_5	0.567951	0.540323	0.554137	5
A_6	0.742614	0.7207	0.731657	11
A_7	0.721075	0.685453	0.703264	10
A_8	0.657138	0.625461	0.6413	8
A_9	0.572695	0.52054	0.546618	3
A_{10}	0.535731	0.423582	0.479657	1
A_{11}	0.611654	0.467905	0.53978	2

5. Results Discussion and Sensitivity Analysis

Five experts are invited to evaluate the criteria and alternatives using the plithogenic numbers. Five decision matrices are obtained from opinions of experts. We combine these matrices into single matrixes. Then we obtain crisp values. Then we obtain the criteria, weights and rank of alternatives.

The results of the proposed methodology showed the Learning Outcome Achievement criterion has the highest weighted followed by the Integration of Technology and Pedagogy, and Student Engagement and Interaction. The worst criterion is Student Satisfaction.

In the rank of alternatives, we show alternative 6 is the best and alternative 10 is the worst.

We applied sensitivity analysis in this study to show different ranks of alternatives under different criteria weights. Figure 4 shows different criteria for weights. Then we applied the steps of the WASPAS method under different criteria weights. Then we ranked the alternatives as shown in Table 8. We show the rank of alternatives is stable under different criteria weights.

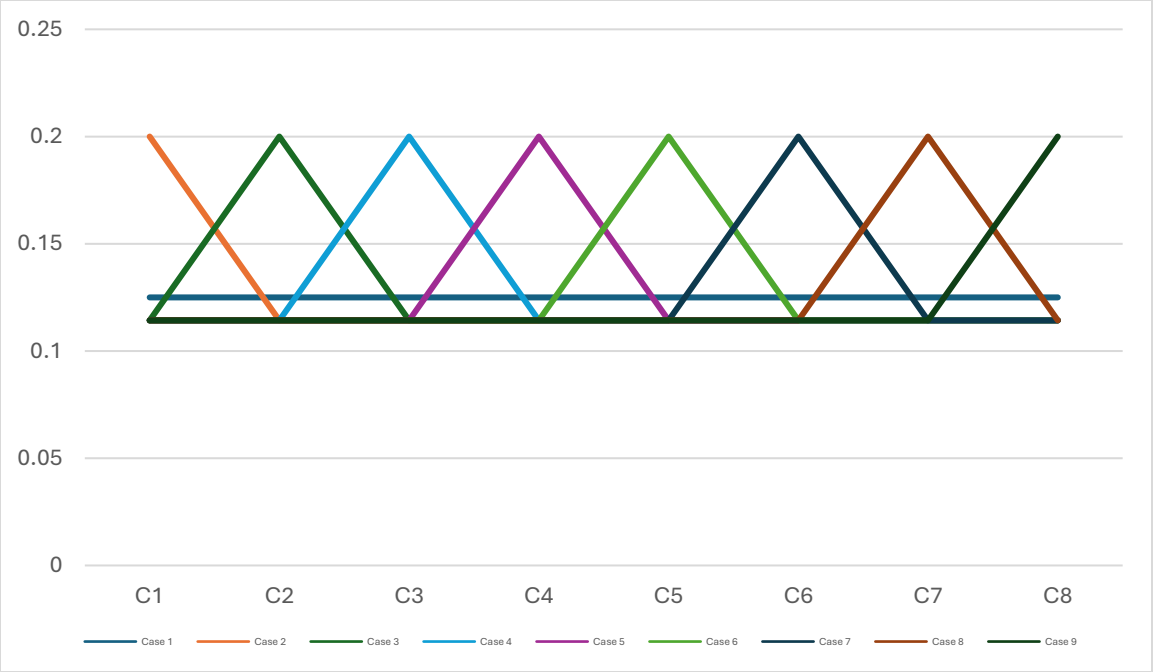

Figure 4. The different criteria weights.

Table 8. The rank of alternatives under different criteria weights.

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*	*Case 6*	*Case 7*	*Case 8*	*Case 9*
A_1	9	9	9	9	9	9	8	8	9
A_2	7	7	7	7	7	6	7	7	7
A_3	4	5	5	6	4	3	4	4	4
A_4	5	4	4	4	5	5	5	2	3
A_5	6	6	6	3	6	7	6	6	6
A_6	11	11	10	11	11	11	11	11	11
A_7	10	10	11	10	10	10	10	10	10
A_8	8	8	8	8	8	8	9	9	8
A_9	3	3	2	5	3	4	3	5	5
A_{10}	1	1	1	1	1	1	1	1	1
A_{11}	2	2	3	2	2	2	2	3	2

6. Conclusions

This study proposed a MCDM methodology for evaluation of College English Blended Teaching Quality under uncertainty framework. This study used the WASPAS method under plithogenic sets to rank the alternatives and deal with uncertain data. The criteria weights are computed to show the highest importance criteria. Eight criteria and eleven alternatives are used in this study. Five experts and decision makers are invited to evaluate the criteria and alternatives. Five decision matrices are obtained from opinions of experts. We used plithogenic operations to combine them. Then we obtain crisp values. Then we compute the criteria weights and rank of alternatives. The results show alternatives 6 is the best and alternative 10 is the worst. We conducted a sensitivity

analysis with nine cases to show different ranks of alternatives. The results show the rank of alternatives is stable under different criteria weights.

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An Uncertainty Framework for Development Level Evaluation of Preschool Education in Rural and Remote Communities

Jing Liu[1]*, Songlin Tang[1], Biqiong Luo[2]

[1]Changde College, Changde, 415000, Hunan, China

[2]Hunan University of Arts and Science, Changde, 415000, Hunan, China

*Corresponding author Email: 18229778969@163.com

Abstract: This study proposed an uncertainty framework for evaluation of Integrated Preschool with School in Rural and Remote Communities. This evaluation has various criteria so, we used the concept of multi-criteria decision making (MCDM). There are various MCDM methods to deal with decision making problems. We used the COPRAS method to rank the alternatives. COPRAS is a MCDM method dealing with various criteria and alternatives. This study used the Plithogenic sets to deal with uncertainty and vague information. Three experts are invited to evaluate the criteria and alternatives. Case study with an empirical example with nine criteria and nine alternatives is conducted. The results show the Accessibility and Enrollment Rates criterion has the highest rate and the criterion Financial and Resource Management has the lowest rate. The sensitivity analysis is conducted to show the different criteria weights and rank of alternatives.

Keywords: Uncertainty Framework; Multi-Criteria Decision Making; Rural Preschool Education.

1. Introduction

These days, there is no question about the strategic significance of education for the advancement of a country. Nonetheless, there is a great deal of disagreement over the best strategies to advance education. Critics point to the neoliberal education policies of the 1980s, which were marked by competition for school performance and the industrialization and marketization of education. In a similar vein, the grim globalization and uncertain global economic outlook have put state welfare education initiatives with

high welfare qualities under pressure. Today, public education policies in many nations face the challenges of striking a balance between neoliberal and state welfare education policies, as well as between those that prioritize economic growth and employment and those that prioritize public well-being[1], [2].

From a global standpoint, the requirement for experts in socialized mass production following the industrial revolution has compelled nations to prioritize and actively advance education. The state has progressively grown to be the biggest provider and investment in education due to the rise of pre-school education. Governments at all levels have deliberately viewed the development of rural pre-school education as one of the key strategies in the process of putting poverty alleviation policies into practice and completing the task of poverty alleviation, as this situation is more prevalent in certain underdeveloped rural areas[3], [4].

Welfare economics is where the idea of rural preschool education efficiency first emerged, and its main research areas are equity, the input-output ratio, and their interactions. Evaluation of rural preschool is an MCDM methodology.

Decision information in the external environment is becoming more ambiguous and unpredictable as objective things continue to advance. With exact numbers alone, it is challenging to adequately and accurately convey the DMs' cognitive preference for objective objects. As a result, several potent mathematical tools are constantly emerging to make these kinds of problems easier to solve. Following Zadeh's groundbreaking introduction of fuzzy set (FS) theory, a number of expanded techniques, including intuitionistic fuzzy set (IFS) and interval-valued fuzzy set (IVFS), have been widely investigated and effectively used to solve MCDM problem[5], [6]s.

However, in certain real-world scenarios, those technologies that consider membership level, non-membership, or a combination of both are unable to handle ambiguous data. To address this issue, Smarandache developed the idea of the neutrosophic set (NS), which extends the sets from a philosophical perspective and investigates the nature and origin of entities as well as their interactions with other intellectual views. To improve the evaluation accuracy of the DMs' subjective judgments, plithogenic sets—which are produced based on neutrosophic sets with a high degree of uncertainty and vagueness—have been widely used in many different fields[7], [8], [9].

The highly helpful method known as Complex Proportional Assessment (COPRAS) is typically applied to multi-attribute decision making. The primary goal of COPRAS is to determine the relative importance and usefulness of each option. To choose the best

option out of all of them, this utility can be translated into the ranks of all the alternatives. Only a small number of researchers have chosen the best machining parameters using COPRAS[10], [11]. One method for making decisions that considers multiple factors is the COPRAS technique. The available options are ranked using the COPRAS technique according to several factors, including associated weight criteria. Finding a solution with the ratio to the ideal solution and the ratio with the ideal-worst solution is the fundamental goal of COPRAS[12], [13].

1.1 Motivation

In summary, the plithogenic set has various advantages to deal with uncertainty information in the MCDM problems. However, few studies focus on computing criteria weights based on the plithogenic sets under uncertainty information from opinions of experts and decision makers. Validation of plithogenic MCDM method in this study is applied to in the education field.

1.2 Objective

To fill the studies gaps and overcome the limitations of previous studies, this paper proposes a MCDM model under plithogenic sets in education field. The aggregation of plithogenic numbers is computed and score function to obtain crisp values is calculated in this study. The criteria weights in this study are computed and rank of alternatives to select the best one is computed.

1.3 Novelty

The contribution and innovation of this study mainly in the field of education under uncertainty framework can be detailed below:

[1] Integrated the plithogenic numbers into one matrix instead of various decision makers and the score function is applied to obtain crisp values.
[2] The criteria weights are computed under the plithogenic numbers and normalized the crisp values.
[3] The plithogenic COPRAS method is applied in this study to solve ethe MCDM problem and rank the alternatives.
[4] The case study with an empirical example is conducted to show the validation of the proposed methodology.
[5] The sensitivity analysis is conducted in this study under ten criteria weights to show the rank of alternatives.

The reminder of this study is organized as follows: Section 2 shows the steps of the COPRAS methodology. Section 3 shows the case study results. Section 4 shows the sensitivity analysis. Section 5 shows the conclusions.

2. COPRAS Method for Plithogenic MCDM with Incomplete Weight Information

In this section, aiming to solve the decision-making problem for ranking the alternatives in this study and compute the criteria weights by rate criteria from the biggest to the smallest. We first outline the decision making methods and then discuss the steps of the proposed method to better understand and solve the MCDM problems. Figure 1 shows the steps of the proposed method.

2.1 Plithogenic Set

Plithogenic set (p, a, V, d, c), an extension of crisp values, fuzzy set, and neutrosophic sets[14], [15].

Let two Plithogenic numbers and their operations as $x = (x_1, x_2, x_3)\ and\ y = (y_1, y_2, y_3)$.

Plithogenic intersection

$$\big((x_{i1}, x_{i2}, x_{i3}), 1 \le i \le n\big) \wedge P\big((y_{i1}, y_{i2}, y_{i3}), 1 \le i \le n\big) = \left(\begin{matrix} (x_{i1} \wedge y_{i1}), \\ \left(\frac{1}{2}(x_{i2} \wedge y_{i2}) + \frac{1}{2}(x_{i2} \vee y_{i2})\right), \\ (x_{i3} \vee y_{i3}) \end{matrix} \right), 1 \le i \le n \tag{1}$$

$$\big((x_{i1}, x_{i2}, x_{i3}), 1 \le i \le n\big) \vee P\big((y_{i1}, y_{i2}, y_{i3}), 1 \le i \le n\big) = \left(\begin{matrix} (x_{i1} \vee y_{i1}), \\ \left(\frac{1}{2}(x_{i2} \wedge y_{i2}) + \frac{1}{2}(x_{i2} \vee y_{i2})\right), \\ (x_{i3} \wedge y_{i3}) \end{matrix} \right), 1 \le i \le n \tag{2}$$

2.2 Method Description

To solve the MCDM problem the opinions of experts are expressed by the Plithogenic numbers. Then we developed a COPRAS under the Plithogenic set to analysis the criteria

weights and rank the alternatives. The proposed methodology uses the MCDM methodology to compute the criteria weights and rank the alternatives. The steps of the proposed methodology are provided below.

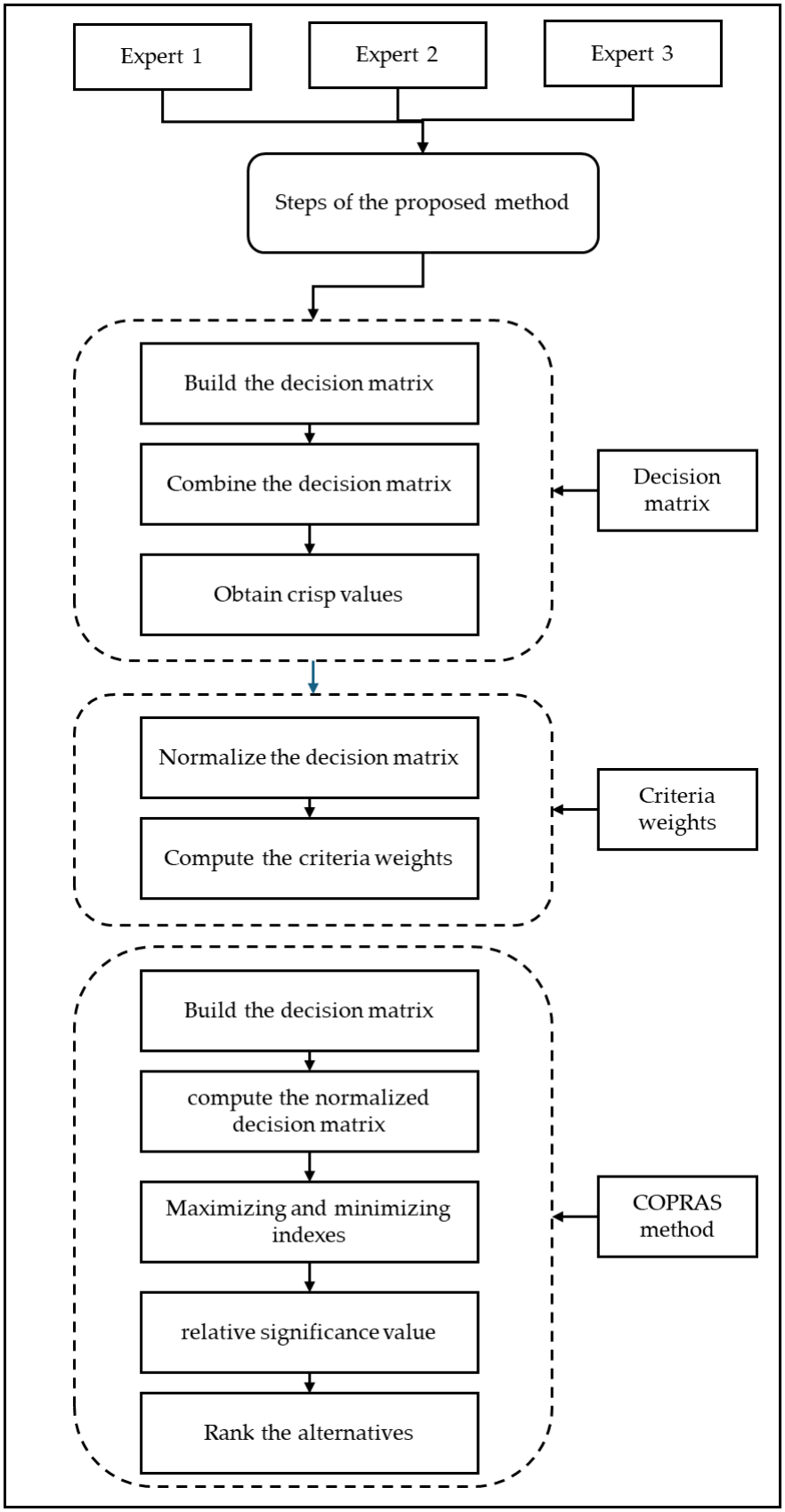
Expert 1
Expert 2
Expert 3
Steps of the proposed method
Build the decision matrix
Combine the decision matrix
Obtain crisp values
Decision matrix
Normalize the decision matrix
Compute the criteria weights
Criteria weights
Build the decision matrix
compute the normalized decision matrix
Maximizing and minimizing indexes
relative significance value
Rank the alternatives
COPRAS method

Figure 1. The conceptual framework.

2.3 The proposed methodology

The features and advantages of this study are to deal with uncertainty, compute the criteria weights and rank the alternatives. The Plithogenic set can deal with the uncertain and vague information in MCDM problems. This study combines the advantages of the Plithogenic set with COPRAS method to compute the criteria weights and rank the alternatives[16], [17].

Step 1. Problem description and determine the combined decision matrix.

Let experts evaluate the criteria and alternatives. Then we used the Plithogenic numbers to replace their opinions. Then we combine these numbers into a single matrix using the Plithogenic operations. Then we obtain crisp values.

Let $A = (A_1, \dots, A_m), m \geq 1$ be a set of alternatives, $C = (C_1, \dots, C_n), n \geq 1$ be a set of criteria. The sum of criteria weights must be equal 1 $0 \leq w_j \leq 1 \; and \; \sum_{j=1}^{n} w_j = 1$.

Step 2. Determine the comprehensive weights of criteria.

The comprehensive weights considering both subjective and objective elements are obtained by applying the modified coefficient to the objective weights of characteristics based on the integrated plithogenic set assessment matrix, which is derived by merging the initial subjective weights that were provided. The degree of consistency between the attribute and other attribute evaluation data is used to calculate the adjustment coefficient of weight.

Then we normalize the crisp values in the decision matrix to compute the criteria weights.

Step 3. Determine the order of alternatives based on COPRAS method under plithogenic set.

The COPRAS method is an MCDM method. It can be used to rank the alternatives. In this stage, we applied the COPRAS method to rank the alternatives and select the optimal one.

We started with the crisp values of the decision matrix.

Normalize the decision matrix.

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{3}$$

Weighted normalized decision matrix.

We compute the normalized decision matrix as:

$$q_{ij} = w_j * r_{ij} \tag{4}$$

Maximizing and minimizing indexes for positive and negative criteria.

$$S_{+i} = \sum_{j=1}^{g} q_{ij}\,; i = 1,..,m; j = 1,\dots,n \tag{5}$$

$$S_{+i} = \sum_{j=g+1}^{n} q_{ij}\,; i = 1,..,m; j = 1,\dots,n \tag{6}$$

The relative significance value.

$$U_i = S_{+i} + \frac{\min S_{-i} \sum_{i=1}^{m} S_{-i}}{S_{+i} \sum_{j=1}^{m} \frac{\min S_{-i}}{S_{-i}}} \tag{7}$$

$$U_i = S_{+i} + \frac{\sum_{i=1}^{m} S_{-i}}{S_{+i} \sum_{j=1}^{m} \frac{1}{S_{-i}}} \tag{8}$$

Final order of alternatives.

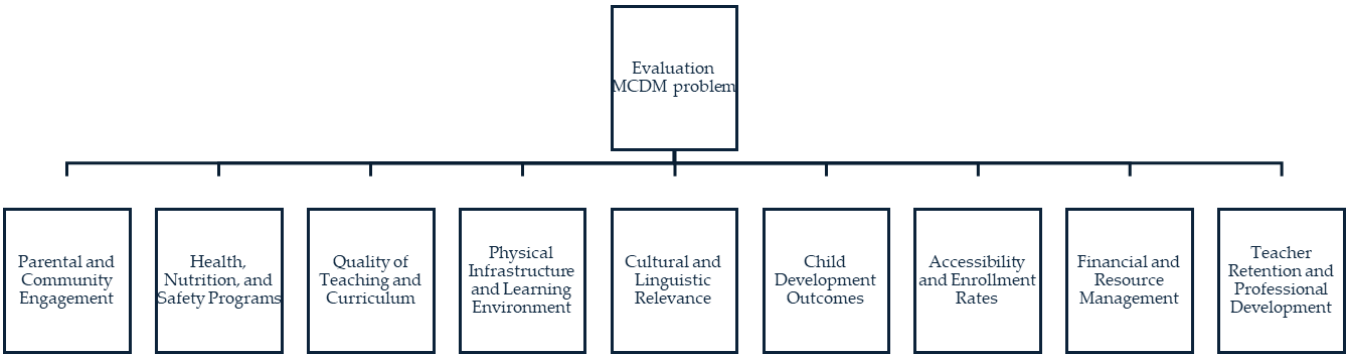
Evaluation MCDM problem
Parental and Community Engagement
Health, Nutrition, and Safety Programs
Quality of Teaching and Curriculum
Physical Infrastructure and Learning Environment
Cultural and Linguistic Relevance
Child Development Outcomes
Accessibility and Enrollment Rates
Financial and Resource Management
Teacher Retention and Professional Development

Figure 2. The set of criteria.

3. Case Study

In this section, to illustrate the effectiveness and feasibility of the proposed methodology, an application concerning with a set of criteria and alternates to compute the criteria weights and rank the alternatives.

This study invited three experts to evaluate the criteria and alternatives. We collected nine criteria and nine alternatives. Figure 2 shows the set of criteria.

Step 1. Problem description and determine the combined decision matrix.

Three experts use plithogenic linguistic terms to evaluate the criteria and alternatives. Then we replace these terms by plithogenic numbers as shown in Tables 1-3. Then we

combine them by using the operations of plithogenic sets using Eqs. (1 and 2). Then we obtain crisp values.

Table 1. The plithogenic numbers by E1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_2	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_3	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_4	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_5	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A_6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_7	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_8	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_9	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)

Table 2. The plithogenic numbers by E2.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_2	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_3	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_4	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_5	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_6	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_7	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_8	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A_9	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)

Table 3. The plithogenic numbers by E3.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_2	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_4	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_6	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_7	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)
A_8	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A_9	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)

Step 2. Determine the comprehensive weights of criteria.

We used the normalized opinions of experts to compute the criteria weights. As

W=(0.115638051, 0.118522883, 0.119418196, 0.118739547, 0.101092946, 0.09967489, 0.123297502, 0.093307968, 0.110308017).

We show the criterion 7 has the highest score and criterion 8 has the lowest score.

Step 3. Rank the alternatives.

Normalize the decision matrix using Eq. (3) as shown in Table 4.

Table 4. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.138099	0.137093	0.150426	0.143311	0.142166	0.117725	0.07293	0.064986	0.023274
A_2	0.121702	0.111307	0.142496	0.121038	0.116074	0.109651	0.03997	0.027514	0.162849

A_3	0.109285	0.110668	0.076566	0.098823	0.126086	0.136506	0.136646	0.027514	0.154265
A_4	0.094514	0.106625	0.146394	0.10198	0.059488	0.073837	0.127414	0.09105	0.104366
A_5	0.101474	0.079611	0.128683	0.14278	0.051471	0.099829	0.119632	0.15087	0.141451
A_6	0.110227	0.145556	0.106575	0.125469	0.159369	0.082808	0.137502	0.140059	0.118474
A_7	0.147155	0.132547	0.075299	0.092046	0.109224	0.125758	0.087974	0.145821	0.09374
A_8	0.155343	0.107544	0.098262	0.07573	0.093955	0.09087	0.139919	0.159667	0.054519
A_9	0.022201	0.069049	0.075299	0.098823	0.142166	0.163016	0.138013	0.192519	0.147063

Weighted normalized decision matrix is computed using Eq. (4) as in Table 5.

Table 5. The weighted normalized decision matrix

	C_1	**C_2**	**C_3**	**C_4**	**C_5**	**C_6**	**C_7**	**C_8**	**C_9**
A_1	0.015969	0.016249	0.017964	0.017017	0.014372	0.011734	0.008992	0.006064	0.002567
A_2	0.014073	0.013192	0.017017	0.014372	0.011734	0.010929	0.004928	0.002567	0.017964
A_3	0.012638	0.013117	0.009143	0.011734	0.012746	0.013606	0.016848	0.002567	0.017017
A_4	0.010929	0.012638	0.017482	0.012109	0.006014	0.00736	0.01571	0.008496	0.011512
A_5	0.011734	0.009436	0.015367	0.016954	0.005203	0.00995	0.01475	0.014077	0.015603
A_6	0.012746	0.017252	0.012727	0.014898	0.016111	0.008254	0.016954	0.013069	0.013069
A_7	0.017017	0.01571	0.008992	0.010929	0.011042	0.012535	0.010847	0.013606	0.01034
A_8	0.017964	0.012746	0.011734	0.008992	0.009498	0.009057	0.017252	0.014898	0.006014
A_9	0.002567	0.008184	0.008992	0.011734	0.014372	0.016249	0.017017	0.017964	0.016222

Maximizing and minimizing indexes for positive and negative criteria are computed using Eq. (5)

The relative significance value is computed using Eq. (7)

Final order of alternatives in Table 6. We show alternative 6 is the best and alternative 4 is the worst.

Table 6. The rank of alternatives.

	Rank
A_1	7
A_2	2
A_3	5
A_4	1
A_5	8
A_6	9
A_7	4
A_8	3
A_9	6

4. Sensitivity Analysis

A sensitivity analysis is a rerun of the initial study or meta-analysis that replaces arbitrary or ambiguous decisions with alternative choices or ranges of values. For instance, sensitivity analysis can entail conducting the meta-analysis twice: once with all studies

and again with only those that are unquestionably known to be eligible if the eligibility of some of the studies is questionable due to their incomplete information. "Are the findings robust to the decisions made in the process of obtaining them?" is the question posed by a sensitivity analysis.

This section shows the rank of alternatives under different criteria weights. We change the criteria weights by ten different cases as shown in Figure 3. Then we rank the alternatives under different criteria weights. We show the rank of alternatives is stable under different criteria weights. Figure 4 shows the rank of alternatives.

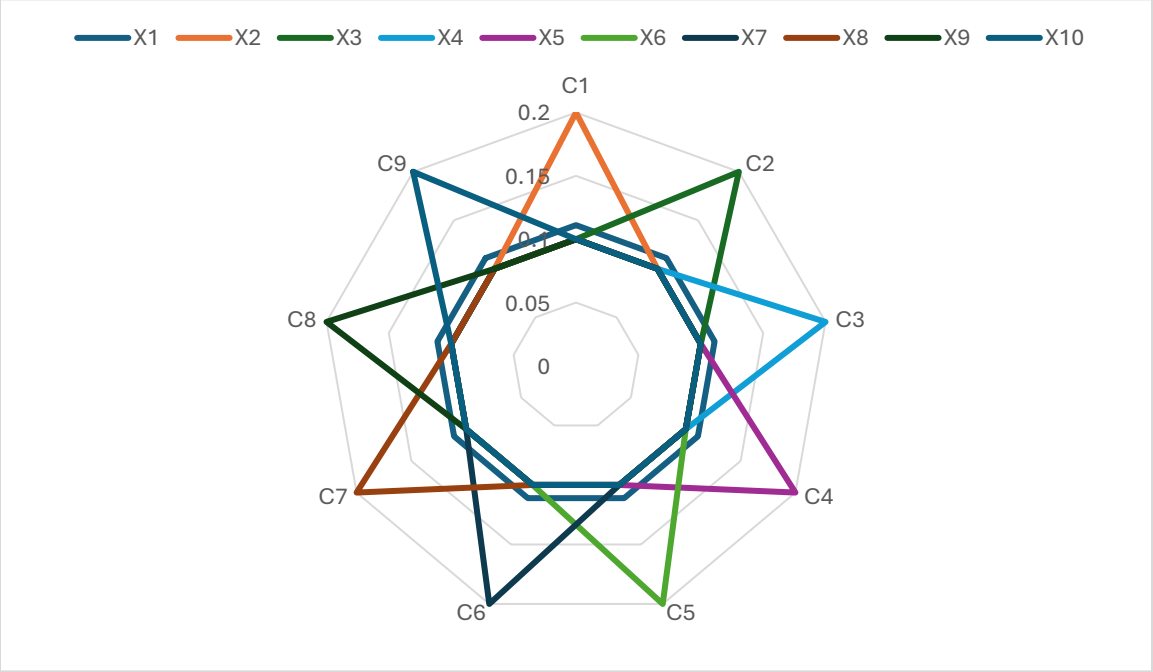

Figure 3. The criteria weights under different cases.

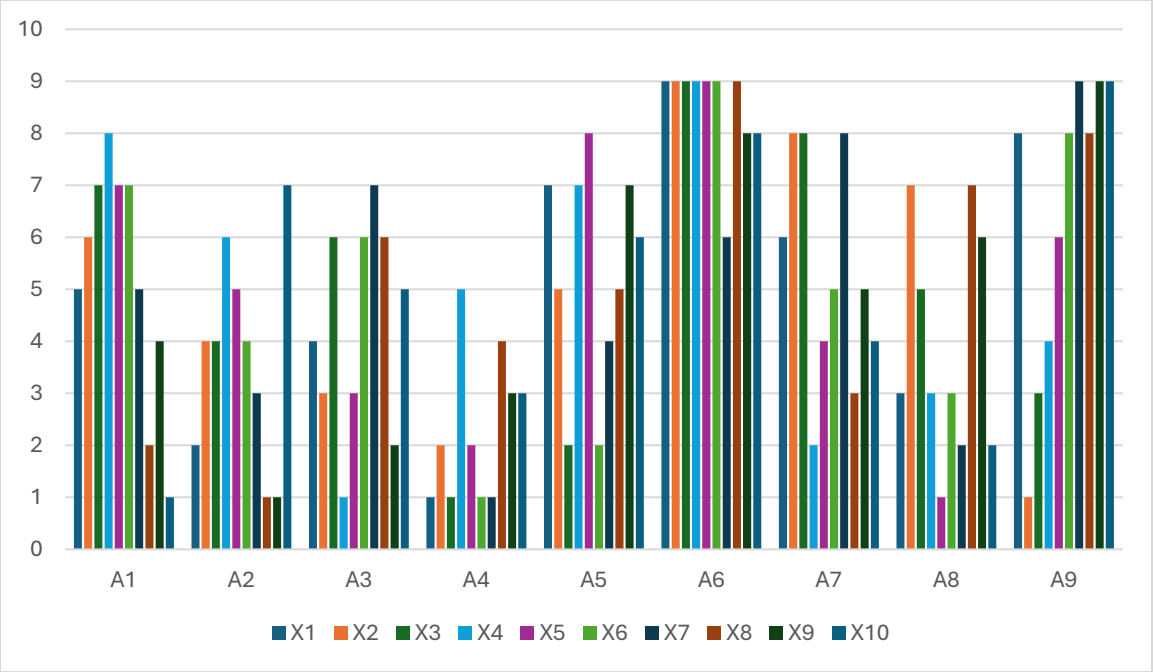

Figure 4. The rank of alternatives under different weights.

5. Conclusions

Evaluation of Rural and Remote Communities preschool is a MCDM problem due to it having various criteria. This study proposed decision making model for evaluating this MCDM problem to compute the criteria weights and rank the alternatives. The COPRAS method is a MCDM method used to rank alternatives. This method is used under the Plithogenic sets to deal with uncertainty and vague information. Nine criteria and nine alternatives are used in this study to be evaluated by three experts and decision makers. We used the Plithogenic numbers to evaluate the criteria and alternatives. Then we combine these numbers using the Plithogenic operations. Then we obtain crisp values. The results show the alternative 6 has the highest rate and alternative 4 has the lowest rate.

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Pentapartitioned Neutrosophic Soft Set with Interval Membership

Mithun Datta[1], Kalyani Debnath[2] and Surapati Pramanik[3,*]

[1]Department of Mathematics, ICFAI University Tripura, India, PIN-799210; mithunagt007@gmail.com

[2]Department of Mathematics, NIT Agartala, Tripura, India, PIN-799046; mou9900@live.in

[3]Nandalal Ghosh B.T. College, Panpur, Narayanpur, Dist-North 24 Parganas, West Bengal, India,PIN-743126; surapati.math@gmail.com

[*]Correspondence: surapati.math@gmail.com

Abstract. In this article, the concept of Interval valued pentapartitioned neutrosophic soft set (IVPNSS) has been introduced, which is a hybrid method by combining Pentapartitioned Neutrosophic set and Interval valued soft set. IVPNSS is an ordered pair of a function and an interval valued pentapartitioned neutrosophic set, which is the domain of the function and co-domain is the assembly of all the interval valued pentapartitioned neutrosophic sets (IVPNS). Additionally, some basic operations on IVPNSSs have been defined which plays an imperative role in decision making processes. Some useful properties of IVPNSS are also presented in the article. In the end, the article addresses the strength and drawbacks of the proposed method and the concluding remarks are presented to signify the future scope in this area.

2010 MSC: 94D05, 90C70.

Keywords: Neutrosophic set, pentapartitioned neutrosophic set, interval valued pentapartitioned neutrosophic set, soft set, interval valued pentapartitioned neutrosophic soft set.

1. Introduction

Numerous frequent issues in the fields of economics, science and engineering, arise in real life that cannot be conquered with classical mathematical methods. The introduction of Fuzzy Sets (FSs) (Zadeh, 1965) and Intuitionistic FSs (IFSs) (Atanassov, 1986) became crucial for addressing this kind of real-world problem. Since IFS (Atanassov, 1986) theory takes into account both membership and non-membership values, it has numerous of applications in the subject of uncertainty, whereas FS (Zadeh, 1965) primarily addresses membership values and

is therefore less significant in practical matters than IFS (Atanassov, 1986). However, IFS theory does not fully address indeterminacy as an independent factor or inconsistent observations in belief systems. To tackle these issues, Smarandache (1998) introduced the Neutrosophic Set (NS) in 1998, extending the IFS theory to handle indeterminacy. The foundational aspects of NS (Smarandache, 1998) and Single-Valued NS (SVNS) (Wang et al., 2010) are explored in the works of Broumi et al. (2018) and Pramanik (2022a) respec tively. Chatterjee et al. (2016) developed the concept of the Quadripartition Single-Valued NS (QSVNS), which was later extended to interval quadripartitioned neutrosophic sets by Pramanik (2022b). In 2013 Smarandache (2013) characterized indeterminacy into functions which are unknown membership function, contradiction membership function and ignorance membership function to propose Five Symbol Valued Neutrosophic Logic (FSVNL). Using the notion of FSVNL Pentapartitioned Neutrosophic Set (PNS) has been introduced by Mallick and Pramanik (2020), where truth, contradiction, ignorance, unknown and falsity membership functions have been considered. Molodtsov (1999) proposed soft set (SS) theory in 1999, a whole new way to model uncertainty and ambiguities. Shil et al. (2024) combined PNS (Mallick Pramanik, 2020) and soft sot (Molodtsov (1999) and developed Single Valued Pentapartitioned Neutrosophic Soft Set (SVPNSS). Giving the membership value in an interval becomes convenient since decision-makers are often finding themselves unable to provide a fixed membership value in real-life uncertain scenarios, and as a necessity, Interval Pentapartitioned Neutrosophic Set (IVPNS) was first developed by Pramanik (2023) and later Datta (2023) found some interesting results. The motivation for this work stems from the following fact:

• The existing fuzzy sets and soft sets are unable to address uncertainty in three different independent ways.

• In decision-making processes, experts often struggle to assign a precise value when rating an object using membership values. Therefore, to simplify the decision-making process, it is sometimes necessary to rate an object in terms of membership values that fall within a specific interval.

• Existing SS approaches are unable to address five membership values.

• It is crucial to examine the fundamental operational laws and properties of the proposed method to implement it effectively in real-life situations. The remaining portion of the article organized as follows: Section 2 comprises of some basic literature survey, section 3 includes the new notion of IVPNSS where IVPNS has been combined with SS theory and in which five significant membership values (truthiness, contradiction, ignorance, unknown/ lack of knowledge, falsity) are considered using soft set theory. Section 3 also includes some basic characteristic of IVPNSS, where soft set theory was used to examine various significant features

of IVPNSS. Section 4 presented the strength and drawbacks of the induced method and section 5 presented the concluding remarks and future scope of this study.

2. Preliminaries

In this section, we first lay out the fundamental ideas that serve as the basis for this analysis before getting into the primary sections of this research article. We hope to provide a clear grasp of the framework driving our inquiry by providing this foundation.

2.1. *Definition [15]*

A NS $\mathfrak{L}$ on the universe $\mathfrak{W}$ is $\mathfrak{L} = \{(\mathfrak{J}, T_{\mathfrak{L}}(\mathfrak{J}), I_{\mathfrak{L}}(\mathfrak{J}), F_{\mathfrak{L}}(\mathfrak{J})) : \mathfrak{J} \in \mathfrak{W}\}$, where $T_{\mathfrak{L}}, I_{\mathfrak{L}}, F_{\mathfrak{L}} : \mathfrak{W} \to {}^{-}0, 1^{+}[$ are functions such that the condition: $\forall \mathfrak{J} \in \mathfrak{W}$, ${}^{-}0 \leq T_{\mathfrak{L}}(\mathfrak{J}) + I_{\mathfrak{L}}(\mathfrak{J}) + F_{\mathfrak{L}}(\mathfrak{J}) \leq 3^{+}$ is satisfied. Here $T_{\mathfrak{L}}, I_{\mathfrak{L}}, F_{\mathfrak{L}}$ represent the truth, indeterminacy and falsity membership function respectively.

2.2. *Definition [4]*

A PNS $\mathfrak{L}$ on the universe $\mathfrak{W}$ is $\mathfrak{L} = \{(\mathfrak{J}, T_{\mathfrak{L}}(\mathfrak{J}), C_{\mathfrak{L}}(\mathfrak{J}), G_{\mathfrak{L}}(\mathfrak{J}), U_{\mathfrak{L}}(\mathfrak{J}), F_{\mathfrak{L}}(\mathfrak{J})) : \mathfrak{J} \in \mathfrak{W}\}$, where $T_{\mathfrak{L}}(\mathfrak{J}), C_{\mathfrak{L}}(\mathfrak{J}), G_{\mathfrak{L}}(\mathfrak{J}), U_{\mathfrak{L}}(\mathfrak{J}), F_{\mathfrak{L}}(\mathfrak{J}) : \mathfrak{J} \in \mathfrak{W} \to [0, 1]$ are functions such that the condition: $\forall \mathfrak{J} \in \mathfrak{W}$, $0 \leq T_{\mathfrak{L}}(\mathfrak{J}) + C_{\mathfrak{L}}(\mathfrak{J}) + G_{\mathfrak{L}}(\mathfrak{J}) + U_{\mathfrak{L}}(\mathfrak{J}) + F_{\mathfrak{L}}(\mathfrak{J}) \leq 5$ is satisfied. Here $T_{\mathfrak{L}}, C_{\mathfrak{L}}, G_{\mathfrak{L}}, U_{\mathfrak{L}}, F_{\mathfrak{L}}$ represent the truth, contradiction, ignorance, unknown and falsity membership functions respectively.

2.3. *Definition [3]*

An interval valued pentapartitioned NS $\mathfrak{L}$ on the universe $\mathfrak{W}$ is defined as $\mathfrak{L} = \{(\mathfrak{J}, T_{\mathfrak{L}}(\mathfrak{J}), C_{\mathfrak{L}}(\mathfrak{J}), G_{\mathfrak{L}}(\mathfrak{J}), U_{\mathfrak{L}}(\mathfrak{J}), F_{\mathfrak{L}}(\mathfrak{J})) : \mathfrak{J} \in \mathfrak{W}\}$, where $T_{\mathfrak{L}}, C_{\mathfrak{L}}, G_{\mathfrak{L}}, U_{\mathfrak{L}}, F_{\mathfrak{L}} : \mathfrak{W} \to \text{Int}([0, 1])$ are functions satisfying the condition: $\forall \mathfrak{J} \in \mathfrak{W}$, $0 \leq supT_{\mathfrak{L}}(\mathfrak{J}) + supC_{\mathfrak{L}}(\mathfrak{J}) + supG_{\mathfrak{L}}(\mathfrak{J}) + supU_{\mathfrak{L}}\mathfrak{J}) + supF_{\mathfrak{L}}(\mathfrak{J}) \leq 5$. Here $T_{\mathfrak{L}}(\mathfrak{J}) = [infT_{\mathfrak{L}}(\mathfrak{J}), supT_{\mathfrak{L}}(\mathfrak{J})]$ and similarly, $C_{\mathfrak{L}}(\mathfrak{J}), G_{\mathfrak{L}}(\mathfrak{J}), U_{\mathfrak{L}}(\mathfrak{J})$ and $F_{\mathfrak{L}}(\mathfrak{J})$ all are sub intervals of $[0, 1]$. Here $T_{\mathfrak{L}}, C_{\mathfrak{L}}, G_{\mathfrak{L}}, U_{\mathfrak{L}}, F_{\mathfrak{L}}$ represent the truth, contradiction, ignorance, unknown and falsity membership functions respectively.

2.4. *Definition [5]*

Let $\mathfrak{W}$ be an universe set and $\mathfrak{E}$ be a set of attributes. Now let us Consider $\mathfrak{L}$ as a non empty subset of $\mathfrak{E}$ and $P(\mathfrak{W})$ be power set of $\mathfrak{W}$. By soft set we mean the collection $(\mathfrak{f}, \mathfrak{L})$ over $\mathfrak{W}$, where $\mathfrak{f} : \mathfrak{L} \to P(\mathfrak{W})$.

3. **Interval Valued Pentapartitioned Neutrosophic Soft Set**

3.1. *Definition:*

Let $\mathfrak{W}$ be the universe whereas $\mathfrak{E}$ be a set of attributes. Let $IVPNS^{\mathfrak{W}}$ be the collection of all interval valued pentapartitioned neutrosophic set on $\mathfrak{W}$. The pair $(\mathfrak{f}, \mathfrak{L})$, $\mathfrak{L} \subseteq \mathfrak{E}$ is called a interval valued pentapartitioned neutrosophic soft set over $\mathfrak{W}$, $\mathfrak{f} : \mathfrak{L} \to IVPNS^{\mathfrak{W}}$ is a mapping. The collection of all $IVPNSS$ over $\mathfrak{W}$ is denoted by $IVPNSS^{\mathfrak{W}}$.

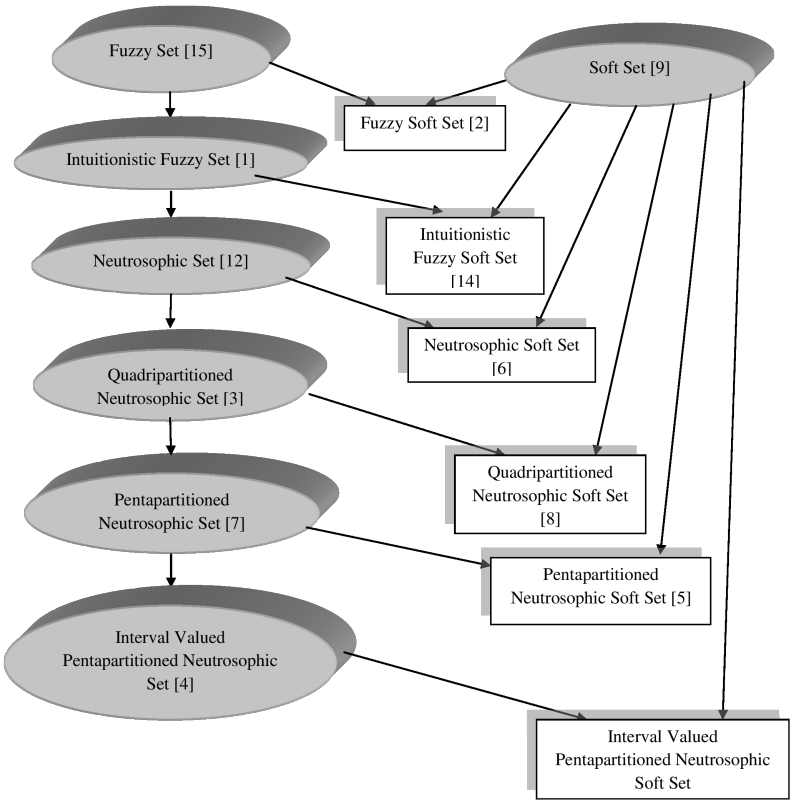
Fuzzy Set [15]
Soft Set [9]
Fuzzy Soft Set [2]
Intuitionistic Fuzzy Set [1]
Intuitionistic Fuzzy Soft Set [14]
Neutrosophic Set [12]
Neutrosophic Soft Set [6]
Quadripartitioned Neutrosophic Set [3]
Quadripartitioned Neutrosophic Soft Set [8]
Pentapartitioned Neutrosophic Set [7]
Pentapartitioned Neutrosophic Soft Set [5]
Interval Valued Pentapartitioned Neutrosophic Set [4]
Interval Valued Pentapartitioned Neutrosophic Soft Set

FIGURE 1. Different types of hybrid models

3.2. *Definition:*

Let $\mathfrak{W}$ be the universe $\mathfrak{E}$ be a set of attributes. Let $\mathfrak{g} : \mathfrak{L} \to IVPNS^{\mathfrak{W}}$ and $\mathfrak{h} : \mathfrak{M} \to IVPNS^{\mathfrak{W}}$ defined by $\mathfrak{g}(a) = \{(\mathfrak{J}, T_{\mathfrak{g}(a)}(\mathfrak{J}), C_{\mathfrak{g}(a)}(\mathfrak{J}), G_{\mathfrak{g}(a)}(\mathfrak{J}), U_{\mathfrak{g}(a)}(\mathfrak{J}), F_{G(a)}(\mathfrak{J})) : \mathfrak{J} \in \mathfrak{W}\}$, $\mathfrak{h}(b) = \{(\mathfrak{J}, T_{\mathfrak{h}(b)}(\mathfrak{J}), C_{\mathfrak{h}(b)}(\mathfrak{J}), G_{\mathfrak{h}(b)}(\mathfrak{J}), U_{\mathfrak{h}(b)}(\mathfrak{J}), F_{\mathfrak{h}(b)}(\mathfrak{J})) : \mathfrak{J} \in \mathfrak{W}\}$, $a \in \mathfrak{L} \subseteq \mathfrak{E}$, $b \in \mathfrak{M} \subseteq \mathfrak{E}$ are two $IVPNSS$ over $\mathfrak{W}$ and its denoted by $(\mathfrak{g}, \mathfrak{L}), (\mathfrak{h}, \mathfrak{M})$ respectively. Here $T_{\mathfrak{g}(a)}(\mathfrak{J}), C_{\mathfrak{g}(a)}(\mathfrak{J}), G_{\mathfrak{g}(a)}(\mathfrak{J}), U_{\mathfrak{g}(a)}(\mathfrak{J}), F_{\mathfrak{g}(a)}(\mathfrak{J}), T_{\mathfrak{h}(b)}(\mathfrak{J}), C_{\mathfrak{h}(b)}(\mathfrak{J}), G_{\mathfrak{h}(b)}(\mathfrak{J}), U_{\mathfrak{h}(b)}(\mathfrak{J}), F_{\mathfrak{h}(b)}(\mathfrak{J}) \in$

$Int([0,1])$. Also $0 \leq supT_{\mathfrak{g}(a)}(\beth) + supC_{\mathfrak{g}(a)}(\beth) + supG_{\mathfrak{g}(a)}(\beth) + supU_{\mathfrak{g}(a)}(\beth) + supF_{\mathfrak{g}(a)}(\beth) \leq 5$
$0 \leq supT_{\mathfrak{h}(b)}(\beth) + supC_{\mathfrak{h}(b)}(\beth) + supG_{\mathfrak{h}(b)}(\beth) + supU_{\mathfrak{h}(b)}(\beth) + supF_{\mathfrak{h}(b)}(\beth) \leq 5$.

(i) The union is $(\mathfrak{g},\mathfrak{L}) \cup (\mathfrak{h},\mathfrak{M}) = (\mathfrak{K},\mathfrak{N})$, is an $IVPNSS$ over $\mathfrak{W}$, where $\mathfrak{N} = \mathfrak{L} \cup \mathfrak{M}$ and for $\vartheta \in \mathfrak{N}$, $\mathfrak{K} : \mathfrak{N} \to IVPNS^{\mathfrak{W}}$ is defined by $\mathfrak{K}(\vartheta) = \{\beth, T_{\mathfrak{K}(\vartheta)}\beth), C_{\mathfrak{K}(\vartheta)}(\beth), G_{\mathfrak{K}(\vartheta)}(\beth), U_{\mathfrak{K}(\vartheta)}(\beth), F_{\mathfrak{K}(\vartheta)}(\beth) : \beth \in \mathfrak{W}\}$, where for $\beth \in \mathfrak{W}$

$$T_{\mathfrak{K}(\vartheta)}(\beth)=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L}-\mathfrak{M} \\ T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{M}-\mathfrak{L} \\ T_{\mathfrak{g}(\vartheta)}(\beth) \vee T_{\mathfrak{h}(\vartheta)}(\beth) = max\{T_{\mathfrak{g}(\vartheta)}(\beth), T_{\mathfrak{h}(\vartheta)}(\beth)\} & if\ \vartheta \in \mathfrak{L} \cap \mathfrak{M} \end{cases}$$

$$C_{\mathfrak{K}(\vartheta)}(\beth)=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L}-\mathfrak{M} \\ C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{M}-\mathfrak{L} \\ C_{\mathfrak{g}(\vartheta)}(\beth) \vee C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L} \cap \mathfrak{M} \end{cases}$$

$$G_{\mathfrak{K}(\vartheta)}(\beth)=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L}-\mathfrak{M} \\ G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{M}-\mathfrak{L} \\ G_{\mathfrak{g}(\vartheta)}(\beth) \square G_{\mathfrak{h}(\vartheta)}(\beth) = \frac{G_{\mathfrak{g}(\vartheta)}(\beth)+G_{\mathfrak{h}(\vartheta)}\beth)}{2} & if\ \vartheta \in \mathfrak{L} \cap \mathfrak{M} \end{cases}$$

$$U_{\mathfrak{K}(\vartheta)}(\beth)=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L}-\mathfrak{M} \\ U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{M}-\mathfrak{L} \\ U_{\mathfrak{g}(\vartheta)}(\beth) \wedge U_{\mathfrak{h}(\vartheta)}(\beth) = min\{U_{\mathfrak{g}(\vartheta)}(\beth), U_{\mathfrak{h}(\vartheta)}(\beth)\} & if\ \vartheta \in \mathfrak{L} \cap \mathfrak{M} \end{cases}$$

$$F_{\mathfrak{K}(\vartheta)}(\beth)=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L}-\mathfrak{M} \\ F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{M}-\mathfrak{L} \\ F_{\mathfrak{g}(\vartheta)}(\beth) \wedge F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta \in \mathfrak{L} \cap \mathfrak{M} \end{cases}.$$

(ii) The intersection is $(\mathfrak{g},\mathfrak{L}) \cap (\mathfrak{h},\mathfrak{M}) = (\mathfrak{K},\mathfrak{N})$ (say), is an $IVPNSS$ over $\mathfrak{W}$, where $\mathfrak{N} = \mathfrak{L} \cap \mathfrak{M}$ and for $\vartheta \in \mathfrak{N}$, $\mathfrak{K} : \mathfrak{N} \to IVPNS^{\mathfrak{W}}$ is defined by $\mathfrak{K}(\vartheta)=\{\beth, T_{\mathfrak{K}(\vartheta)}(\beth), C_{\mathfrak{K}(\vartheta)}(\beth), G_{\mathfrak{K}(\vartheta)}(\beth), U_{\mathfrak{K}(\vartheta)}(\beth), F_{\mathfrak{K}(\vartheta)}(\beth) : \beth \in \mathfrak{W}\}$, where for $\beth \in \mathfrak{W}$
$T_{\mathfrak{K}(\vartheta)}(\beth)=T_{\mathfrak{g}(\vartheta)}(\beth) \wedge T_{\mathfrak{h}(\vartheta)}(\beth)$, $C_{\mathfrak{K}(\vartheta)}(\beth)=C_{\mathfrak{g}(\vartheta)}(\beth) \wedge C_{\mathfrak{h}(\vartheta)}(\beth)$, $G_{\mathfrak{K}(\vartheta)}(\beth)=G_{\mathfrak{g}(\vartheta)}(\beth)\square G_{\mathfrak{h}(\vartheta)}(\beth)$, $U_{\mathfrak{K}(\vartheta)}(\beth)=U_{\mathfrak{g}(\vartheta)}(\beth) \vee U_{\mathfrak{h}(\vartheta)}(\beth)$, $F_{\mathfrak{K}(\vartheta)}(\beth)=F_{\mathfrak{g}(\vartheta)}(\beth) \vee F_{\mathfrak{h}(\vartheta)}(\beth)$.

(iii) The complement of $(\mathfrak{g},\mathfrak{L})$, $(\mathfrak{g},\mathfrak{L})^c$ is an $IVPNSS$ over $\mathfrak{W}$ and defined as $(\mathfrak{g},\mathfrak{L})^c=(\mathfrak{g}^c, \rceil\mathfrak{L})$, where $\mathfrak{g}^c : \rceil\mathfrak{L} \to IVPNS^{\mathfrak{W}}$ is defined by
$\mathfrak{g}^c(\neg a) = \big\{\big(\beth, T_{\mathfrak{g}^c(\neg a)}(\beth), C_{\mathfrak{g}^c(\neg a)}(\beth), G_{\mathfrak{g}^c(\neg a)}(\beth), U_{\mathfrak{g}^c(\neg a)}(\beth), F_{\mathfrak{g}^c(\neg a)}(\beth)\big) : \beth \in \mathfrak{W}\big\}$ for $a \in \mathfrak{L}$, where $T_{\mathfrak{g}^c(\neg a)}(\beth) = F_{\mathfrak{g}(a)}(\beth)$, $C_{\mathfrak{g}^c(\neg a)}(\beth) = U_{\mathfrak{g}(a)}(\beth)$, $G_{\mathfrak{g}^c(\neg a)}(\beth) = 1 - G_{\mathfrak{g}(a)}(\beth)$, $U_{\mathfrak{g}^c(\neg a)}(\beth) = C_{\mathfrak{g}(a)}(\beth)$ and $F_{\mathfrak{g}^c(\neg a)}(\beth) = T_{\mathfrak{g}(a)}(\beth)$.
We can write, $\mathfrak{g}^c(\neg a)=\big\{\big(\beth, F_{\mathfrak{g}(a)}(\beth), U_{\mathfrak{g}(a)}(\beth), 1 - G_{\mathfrak{g}(a)}(\beth), C_{\mathfrak{g}(a)}(\beth), T_{\mathfrak{g}(a)}(\beth)\big) : \beth \in \mathfrak{W}\big\}$. Here $\neg a$ means 'not a', for example if a is 'beautiful' then $\neg a$ will be 'not beautiful'. Similarly, $\rceil\mathfrak{L}$ stands for the 'NOT set of $\mathfrak{L}$'.

3.3. *Example*

Let $\mathfrak{W}$ be the set of houses that are being studied and $\mathfrak{E}$ is the collection of attributes. Consider $\mathfrak{E}$ ={beautiful, wooden, costly, moderate}. In this instance, to define a $IVPNSS$ means to point out beautiful houses, Costly houses. let, there are three houses in the universe $\mathfrak{W}$ given by, $\mathfrak{W} = \{h_1, h_2, h_3\}$ and the set of attributes $\mathfrak{L} = \{e_1, e_2\}$, where e_1 and e_2 denote the attributes 'beautiful' and 'costly' respectively.

The tabular depiction of $(\mathfrak{g}, \mathfrak{L})$

$\mathfrak{W}$	e_1	e_2
h_1	([.1,.2], [0,.2], [.3,.4], [.5,.6], [.7,.9])	([.2,.3], [.1,.2], [.4,.5], [.6,.7], [.8,.9])
h_2	([.7,.8], [.1,.2], [.3,.4], [.5,.6], [.8,.9])	([.1,.2], [.3,.5], [.4,.6], [.7,.9], [.5,.8])
h_3	([.2,.3], [.4,.5], [.6,.7], [.1,.2], [.8,.9])	([.2,.4], [.1,.3], [.6,.7], [.8,.9], [.5,.6])

The tabular depiction of $(\mathfrak{h}, \mathfrak{M})$

$\mathfrak{W}$	e_2
h_1	([.6,.7], [.1,.2], [.8,.9], [.3,.4], [.4,.5])
h_2	([.4,.5], [.2,.3],[.5,.6], [.8,.9],[.3,.7])
h_3	([.2,.4], [.1,.2], [.4,.6], [.2,.9], [.1,.7])

Union of $(\mathfrak{g}, \mathfrak{L})$ and $(\mathfrak{h}, \mathfrak{M})$ is as follows:

$\mathfrak{W}$	e_1	e_2
h_1	([.1,.2], [0,.2], [.3,.4], [.5,.6], [.7,.9])	([.6,.7], [.1,.2], [.6,.7], [.3,.4], [.4,.5])
h_2	([.7,.8], [.1,.2], [.3,.4], [.5,.6], [.8,.9])	([.4,.5], [.3,.5], [.45,.6], [.7,.9], [.3,.7])
h_3	([.2,.3], [.4,.5], [.6,.7], [.1,.2], [.8,.9])	([.2,.4], [.1,.3], [.5,.65], [.2,.9], [.1,.6])

Intersection of $(\mathfrak{g}, \mathfrak{L})$ and $(\mathfrak{h}, \mathfrak{M})$ is as follows:

$\mathfrak{W}$	e_2
h_1	([.2,.3], [.1,.2], [.6,.7], [.6,.7], [.8,.9])
h_2	([.1,.2], [.2,.3],[.45,.6], [.8,.9],[.5,.8])
h_3	([.2,.4], [.1,.2], [.5,.65], [.8,.9], [.5,.7])

The tabular depiction of $(\mathfrak{g}, \mathfrak{L})^c$

$\mathfrak{W}$	e_1	e_2
h_1	([.7,.9], [.5,.6], [.6,.7], [0,.2], [.1,.2])	([.8,.9], [.6,.7], [.5,.6], [.1,.2], [.2,.3])
h_2	([.8,.9], [.5,.6], [.6,.7], [.1,.2], [.7,.8])	([.5,.8], [.7,.9], [.4,.6], [.3,.5], [.1,.2])
h_3	([.8,.9], [.1,.2], [.3,.4], [.4,.5], [.2,.3])	([.5,.6], [.8,.9], [.3,.4], [.1,.3], [.2,.4])

3.4. *Theorem*

For any three $IVPNSS$s $(\mathfrak{g}, \mathfrak{L}), (\mathfrak{h}, \mathfrak{M})$ and $(\mathfrak{K}, \mathfrak{N})$ the followings are true:

(i) Associative Law:

$$((\mathfrak{g}, \mathfrak{L}) \cup (\mathfrak{h}, \mathfrak{M})) \cup (\mathfrak{K}, \mathfrak{N}) = (\mathfrak{g}, \mathfrak{L}) \cup ((\mathfrak{h}, \mathfrak{M}) \cup (\mathfrak{K}, \mathfrak{N}))$$

$((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M}))\cap(\mathfrak{K},\mathfrak{N})=(\mathfrak{g},\mathfrak{L})\cap((\mathfrak{h},\mathfrak{M})\cap(\mathfrak{K},\mathfrak{N}))$

(ii) Distributive law:

$(\mathfrak{g},\mathfrak{L})\cap((\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N}))=((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M}))\cup((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{K},\mathfrak{N}))$

$(\mathfrak{g},\mathfrak{L})\cup((\mathfrak{h},\mathfrak{M})\cap(\mathfrak{K},\mathfrak{N}))=((\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M}))\cap((\mathfrak{g},\mathfrak{L})\cup(\mathfrak{K},\mathfrak{N}))$

(iii) Commutative Law:

$(\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M})=(\mathfrak{h},\mathfrak{M})\cup(\mathfrak{g},\mathfrak{L})$

$(\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M})=(\mathfrak{h},\mathfrak{M})\cap(\mathfrak{g},\mathfrak{L})$

(iv) De Morgan's Law:

De Morgan's law holds only when $\mathfrak{M}=\mathfrak{L}$ as in such cases only we will get $\mathfrak{L}\cup\mathfrak{M}=\mathfrak{L}\cap\mathfrak{M}$

$((\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{L}))^c=(\mathfrak{g},\mathfrak{L})^c\cap(\mathfrak{h},\mathfrak{L})^c.$

$((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{L}))^c=(\mathfrak{g},\mathfrak{L})^c\cup(\mathfrak{h},\mathfrak{L})^c.$

(v) Idempotent Law:

$(\mathfrak{g},\mathfrak{L})\cup(\mathfrak{g},\mathfrak{L})=(\mathfrak{g},\mathfrak{L})$

$(\mathfrak{g},\mathfrak{L})\cap(\mathfrak{g},\mathfrak{L})=(\mathfrak{g},\mathfrak{L})$

(vi) Absorption Law:

$(\mathfrak{g},\mathfrak{L})\cup((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M}))=(\mathfrak{g},\mathfrak{L})$

$(\mathfrak{g},\mathfrak{L})\cap((\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M}))=(\mathfrak{g},\mathfrak{L})$

(vii) Involution Law:

$((\mathfrak{g},\mathfrak{L})^c)^c=(\mathfrak{g},\mathfrak{L})$

Proof: Let $\mathfrak{W}$ be the universe and $\mathfrak{E}$ be a set of attributes. Let us also consider $(\mathfrak{g},\mathfrak{L})$ and $(\mathfrak{h},\mathfrak{M})$ be defined as in definition 3.2 and $(\mathfrak{K},\mathfrak{N})$ be another $IVPNSS$ where $\mathfrak{K}:\mathfrak{N}\to IVPNS^{\mathfrak{W}}$, $\mathfrak{K}(c)=\{(\mathfrak{J},T_{\mathfrak{K}(c)}(\mathfrak{J}),C_{\mathfrak{K}(c)}(\mathfrak{J}),G_{\mathfrak{K}(c)}(\mathfrak{J}),U_{k(c)}(\mathfrak{J}),F_{k(c)}(\mathfrak{J})):\mathfrak{J}\in\mathfrak{W}\}$, $c\in\mathfrak{N}\subseteq\mathfrak{E}$, over $\mathfrak{W}$. Here, $T_{\mathfrak{K}(c)}(\mathfrak{J}),C_{\mathfrak{K}(c)}(\mathfrak{J}),G_{\mathfrak{K}(c)}(\mathfrak{J}),U_{\mathfrak{K}(c)}(\mathfrak{J}),F_{\mathfrak{K}(c)}(\mathfrak{J})\in Int([0,1])$ satisfying the condition $0\leq supT_{\mathfrak{K}(a)}(\mathfrak{J})+supC_{\mathfrak{K}(a)}(\mathfrak{J})+supG_{\mathfrak{K}(a)}(\mathfrak{J})+supU_{\mathfrak{K}(a)}(\mathfrak{J})+supF_{\mathfrak{K}(a)}(\mathfrak{J})\leq 5$.

(i) Associative Law:

Let $(\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M})=(\mathfrak{s},\mathfrak{D})$, where $\mathfrak{D}=\mathfrak{L}\cup\mathfrak{M}$ and $\mathfrak{s}:\mathfrak{D}\to IVPNS^{\mathfrak{W}}$ is given by, for $\vartheta\in\mathfrak{D}$, $\mathfrak{s}(\vartheta)=\{(\mathfrak{J},T_{\mathfrak{s}(\vartheta)}(\mathfrak{J}),C_{\mathfrak{s}(\vartheta)}(\mathfrak{J}),G_{\mathfrak{s}(\vartheta)}(\mathfrak{J}),U_{\mathfrak{s}(\vartheta)}(\mathfrak{J}),F_{\mathfrak{s}(\vartheta)}(\mathfrak{J})):\mathfrak{J}\in\mathfrak{W}\}$

$$T_{\mathfrak{s}(\vartheta)}(\mathfrak{J})=\begin{cases}T_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M}\\ T_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L}\\ T_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\vee T_{h(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\end{cases}$$

$$C_{\mathfrak{s}(\vartheta)}(\mathfrak{J})=\begin{cases}C_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M}\\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L}\\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\vee C_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\end{cases}$$

$$G_{\mathfrak{s}(\vartheta)}(\beth)=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ G_{\mathfrak{g}(\vartheta)}(\beth)\square G_{h(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$U_{\mathfrak{s}(\vartheta)}(\beth)=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ U_{\mathfrak{g}(\vartheta)}(\beth)\wedge U_{h(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$F_{\mathfrak{s}(\vartheta)}(\beth)=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ F_{\mathfrak{g}(\vartheta)}(\beth)\wedge F_{\beth(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

Now let $((\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M}))\cup(\mathfrak{K},\mathfrak{N})=(\mathfrak{s},\mathfrak{D})\cup(\mathfrak{K},\mathfrak{N})=(\mathfrak{t},\mathfrak{E})$, where $\mathfrak{E}=\mathfrak{D}\cup\mathfrak{N}=\mathfrak{L}\cup\mathfrak{M}\cup\mathfrak{N}$ and $\mathfrak{t}:\vartheta\to IVPNS^{\mathfrak{W}}$ is given by, for $\vartheta\in\mathfrak{E}$,

$\mathfrak{t}(\vartheta)=\{(\beth,T_{\mathfrak{t}(\vartheta)}(\beth),C_{\mathfrak{t}(\vartheta)}(\beth),G_{\mathfrak{t}(\vartheta)}(\beth),U_{\mathfrak{t}(\vartheta)}(\beth),F_{\mathfrak{t}(\vartheta)}(\beth)):\beth\in\mathfrak{W}\}$.

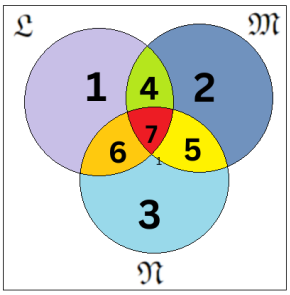

FIGURE 2. Venn diagram of seven disjoint regions

Here $\mathfrak{E}$ can be divided into seven disjoint regions and the membership values will be different in each of these seven regions. The seven disjoint regions are

1. $\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N})$
2. $\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L})$
3. $\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M})$
4. $(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}$
5. $(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L}$
6. $(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}$
7. $\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}$

$$T_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N}) \\ T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L}) \\ T_{\mathfrak{K}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M}) \\ T_{\mathfrak{g}(\vartheta)}(\beth)\vee T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ T_{\mathfrak{h}(\vartheta)}(\beth)\vee T_{\mathfrak{K}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L} \\ T_{\mathfrak{K}(\vartheta)}(\beth)\vee T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ T_{\mathfrak{g}(\vartheta)}(\beth)\vee T_{\mathfrak{h}(\vartheta)}(\beth)\vee T_{\mathfrak{K}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap N \end{cases}$$

$$C_{\mathfrak{t}(\vartheta)}(\mathfrak{I})=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N}) \\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L}) \\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M}) \\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee C_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\vee C_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L} \\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\vee C_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee C_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\vee C_{k(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{C} \end{cases}$$

$$G_{\mathfrak{t}(\vartheta)}(\mathfrak{I})=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N}) \\ G_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L}) \\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M}) \\ G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ G_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L} \\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$U_{\mathfrak{t}(\vartheta)}(\mathfrak{I})=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N}) \\ U_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L}) \\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M}) \\ U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ U_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L} \\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$F_{\mathfrak{t}(\vartheta)}(\mathfrak{I})=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N}) \\ F_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L}) \\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M}) \\ F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ F_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L} \\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

Again assume that $(\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N})=(\mathfrak{u},\mathfrak{M})$, where $\mathfrak{M}=\mathfrak{M}\cup\mathfrak{N}$ and $\mathfrak{u}:\mathfrak{M}\to IVPNS^{\mathfrak{W}}$ is given by, for $\vartheta\in\mathfrak{M}$, $\mathfrak{u}(\vartheta)=\mathfrak{s}(\vartheta)=\{(\mathfrak{I},T_{\mathfrak{u}(\vartheta)}(\mathfrak{I}),C_{\mathfrak{u}(\vartheta)}(\mathfrak{I}),G_{\mathfrak{u}(\vartheta)}(\mathfrak{I}),U_{\mathfrak{u}(\vartheta)}(\mathfrak{I}),F_{\mathfrak{u}(\vartheta)}(\mathfrak{I})):\mathfrak{I}\in\mathfrak{W}\}$

$$T_{\mathfrak{u}(\vartheta)}(\mathfrak{I})=\begin{cases} T_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ T_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ T_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$C_{\mathfrak{u}(\vartheta)}(\mathfrak{I})=\begin{cases} C_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\vee C_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$G_{\mathfrak{u}(\vartheta)}(\mathfrak{I})=\begin{cases} G_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ G_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$U_{\mathfrak{u}(\vartheta)}(\mathfrak{I})=\begin{cases} U_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ U_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$F_{\mathfrak{u}(\vartheta)}(\mathfrak{I})=\begin{cases} F_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ F_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

Now if $(\mathfrak{g},\mathfrak{L})\cup((\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N}))=(\mathfrak{g},\mathfrak{L})\cup(\mathfrak{u},\mathfrak{M})=(\mathfrak{v},\mathfrak{E})$, where $\mathfrak{E}=\mathfrak{L}\cup\mathfrak{M}=\mathfrak{L}\cup\mathfrak{M}\cup\mathfrak{N}$ and $\mathfrak{v}:\vartheta\to IVPNS^{\mathfrak{W}}$ is given by, for $\vartheta\in\mathfrak{E}$,

$\mathfrak{v}(\vartheta)=\{(\mathfrak{I},\mathfrak{I}_{\mathfrak{v}(\vartheta)}(\mathfrak{I}),C_{\mathfrak{v}(\vartheta)}(\mathfrak{I}),G_{\mathfrak{v}(\vartheta)}(\mathfrak{I}),U_{\mathfrak{v}(\vartheta)}(\mathfrak{I}),F_{\mathfrak{v}(\vartheta)}(\mathfrak{I})):\mathfrak{I}\in\mathfrak{W}\}$

$$T_{\mathfrak{v}(\vartheta)}(\mathfrak{I})=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N}) \\ T_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L}) \\ T_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M}) \\ T_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ T_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L} \\ T_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ T_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{K}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$C_{\mathfrak{v}(\vartheta)}(\mathfrak{J})=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N})\\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L})\\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M})\\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\vee C_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\vee C_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L}\\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{J})\vee C_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\vee C_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\vee C_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$G_{\mathfrak{v}(\vartheta)}(\mathfrak{J})=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N})\\ G_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L})\\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M})\\ G_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ G_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L}\\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{J})\square G_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ G_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$U_{\mathfrak{v}(\vartheta)}(\mathfrak{J})=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N})\\ U_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L})\\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M})\\ U_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\wedge U_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ U_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L}\\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{J})\wedge U_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ U_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\wedge U_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$F_{\mathfrak{v}(\vartheta)}(\mathfrak{J})=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}-(\mathfrak{M}\cup\mathfrak{N})\\ F_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{M}-(\mathfrak{N}\cup\mathfrak{L})\\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{N}-(\mathfrak{L}\cup\mathfrak{M})\\ F_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\wedge F_{\mathfrak{h}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ F_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\wedge F_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{M}\cap\mathfrak{N})-\mathfrak{L}\\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{J})\wedge F_{\mathfrak{g}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ F_{\mathfrak{g}(\vartheta)}(\mathfrak{J})\wedge F_{\mathfrak{h}(\vartheta)}(\mathfrak{J})\wedge F_{\mathfrak{K}(\vartheta)}(\mathfrak{J}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$\therefore (\mathfrak{t},\mathfrak{E})=(\mathfrak{v},\mathfrak{E})$

i.e, $((\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M}))\cup(\mathfrak{K},\mathfrak{N})=(\mathfrak{g},\mathfrak{L})\cup((\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N}))$

The second one can be proved similarly.

(ii) Distributive Law:

Let $(\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N})=(\mathfrak{s},\mathfrak{D})$ (say), is an $IVPNSS$ over $\mathfrak{W}$, where $\mathfrak{D}=\mathfrak{M}\cup\mathfrak{N}$ and for $\vartheta\in\mathfrak{D}$, $\mathfrak{s}:\mathfrak{D}\to IVPNS^{\mathfrak{W}}$ is given by

$\mathfrak{s}(\vartheta)=\{\big(\mathfrak{Z},T_{\mathfrak{s}(\vartheta)}(\mathfrak{Z}),C_{\mathfrak{s}(\vartheta)}(\mathfrak{Z}),G_{\mathfrak{s}(\vartheta)}(\mathfrak{Z}),U_{\mathfrak{s}(\vartheta)}(\mathfrak{Z}),F_{\mathfrak{s}(\vartheta)}(\mathfrak{Z})\big):\mathfrak{Z}\in\mathfrak{W}\}$

$$T_{\mathfrak{s}(\vartheta)}(\mathfrak{Z})=\begin{cases}T_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{N}\\ T_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{N}-\mathfrak{M}\\ T_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\vee T_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$C_{\mathfrak{s}(\vartheta)}(\mathfrak{Z})=\begin{cases}C_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{N}\\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{N}-\mathfrak{M}\\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\vee C_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$G_{\mathfrak{s}(\vartheta)}(\mathfrak{Z})=\begin{cases}G_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{N}\\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{N}-\mathfrak{M}\\ G_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$U_{\mathfrak{s}(\vartheta)}(\mathfrak{Z})=\begin{cases}U_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{N}\\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{N}-\mathfrak{M}\\ U_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$F_{\mathfrak{s}(\vartheta)}(\mathfrak{Z})=\begin{cases}F_{\mathfrak{g}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{N}\\ F_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{N}-\mathfrak{M}\\ F_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\wedge F_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

Now, let $(\mathfrak{g},\mathfrak{L})\cap((\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N}))=(\mathfrak{g},\mathfrak{L})\cap(\mathfrak{s},\mathfrak{D})=(\mathfrak{t},\mathfrak{E})$ where $\mathfrak{E}=\mathfrak{L}\cap\mathfrak{D}=\mathfrak{L}\cap(\mathfrak{M}\cup\mathfrak{N})=(\mathfrak{L}\cap\mathfrak{M})\cup(\mathfrak{L}\cap\mathfrak{N})$ and $\mathfrak{t}:\vartheta\to IVPNS^{\mathfrak{W}}$ is given by for $\vartheta\in E$,

$\mathfrak{t}(\vartheta)=\{\big(x,T_{\mathfrak{t}(\vartheta)}(x),C_{\mathfrak{t}(\vartheta)}(x),G_{\mathfrak{t}(\vartheta)}(x),U_{\mathfrak{t}(\vartheta)}(x),F_{\mathfrak{t}(\vartheta)}(x):\big)\ \mathfrak{Z}\in\mathfrak{W}\}$

$$T_{\mathfrak{t}(\vartheta)}(\mathfrak{Z})=\begin{cases}T_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\wedge T_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ T_{\mathfrak{K}(\vartheta)}(\mathfrak{Z})\wedge T_{\mathfrak{g}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ T_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\wedge\big(T_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\vee T_{\mathfrak{K}(\vartheta)}(\mathfrak{Z})\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$C_{\mathfrak{t}(\vartheta)}(\mathfrak{Z})=\begin{cases}C_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\wedge C_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ C_{\mathfrak{K}(\vartheta)}(\mathfrak{Z})\wedge C_{\mathfrak{g}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\wedge\big(C_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\vee C_{\mathfrak{K}(\vartheta)}(\mathfrak{Z})\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$G_{\mathfrak{t}(\vartheta)}(\mathfrak{Z})=\begin{cases}G_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N}\\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{Z})\square G_{\mathfrak{g}(\vartheta)}(\mathfrak{Z}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M}\\ G_{\mathfrak{g}(\vartheta)}(\mathfrak{Z})\square\big(G_{\mathfrak{h}(\vartheta)}(\mathfrak{Z})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{Z})\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N}\end{cases}$$

$$U_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\beth)\vee U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ U_{\mathfrak{K}(\vartheta)}(\beth)\vee U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ U_{\mathfrak{g}(\vartheta)}(\beth)\vee\big(U_{\mathfrak{h}(\vartheta)}(\beth)\wedge U_{\mathfrak{K}(\vartheta)}(\beth)\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$F_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\beth)\vee F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ F_{\mathfrak{K}(\vartheta)}(\beth)\vee F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ F_{\mathfrak{g}(\vartheta)}(\beth)\vee\big(F_{\mathfrak{h}(\vartheta)}(\beth)\wedge F_{\mathfrak{K}(\vartheta)}(\beth)\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

Again assume that $(\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M})=(\mathfrak{u},\mathfrak{M})$ (say), is an $IVPNSS$ over $\mathfrak{W}$,where $\mathfrak{M}=\mathfrak{L}\cap\mathfrak{M}$ and for $\vartheta\in\mathfrak{M}$, $\mathfrak{u}:\mathfrak{M}\to IVPNS^{\mathfrak{W}}$ is given by

$\mathfrak{u}(\vartheta)=\big\{\big(\beth,T_{\mathfrak{u}(\vartheta)}(\beth),C_{\mathfrak{u}(\vartheta)}(\beth),G_{\mathfrak{u}(\vartheta)}(\beth),U_{\mathfrak{u}(\vartheta)}(\beth),F_{\mathfrak{u}(\vartheta)}(\beth)\big):\beth\in\mathfrak{W}\big\}$

where, $T_{\mathfrak{u}(\vartheta)}(\beth)$=$T_{\mathfrak{g}(\vartheta)}(\beth)\wedge T_{\mathfrak{h}(\vartheta)}(\beth)$, $C_{\mathfrak{u}(\vartheta)}(\beth)$=$C_{\mathfrak{g}(\vartheta)}(\beth)\wedge C_{\mathfrak{h}(\vartheta)}(\beth)$, $G_{\mathfrak{u}(\vartheta)}(\beth)$=$G_{\mathfrak{g}(\vartheta)}(\beth)\square G_{\mathfrak{h}(\vartheta)}(\beth)$, $U_{\mathfrak{u}(\vartheta)}(\beth)$=$U_{\mathfrak{g}(\vartheta)}(\beth)\vee U_{\mathfrak{h}(\vartheta)}(\beth)$, $F_{\mathfrak{u}(\vartheta)}(\beth)$=$F_{\mathfrak{g}(\vartheta)}(\beth)\vee F_{\mathfrak{h}(\vartheta)}(\beth)$

Again $(\mathfrak{g},\mathfrak{L})\cap(\mathfrak{K},\mathfrak{N})=(\mathfrak{v},\mathfrak{N})$ (say), is an $IVPNSS$ over $\mathfrak{W}$, where $\mathfrak{N}=\mathfrak{L}\cap\mathfrak{N}$ and for $\vartheta\in\mathfrak{N}$, $\mathfrak{v}:\mathfrak{N}\to IVPNS^{\mathfrak{W}}$ is given by

$\mathfrak{v}(\vartheta)$=$\big\{\big(\beth,T_{\mathfrak{v}(\vartheta)}(\beth),C_{\mathfrak{v}(\vartheta)}(\beth),G_{\mathfrak{v}(\vartheta)}(\beth),U_{\mathfrak{v}(\vartheta)}(\beth),F_{\mathfrak{v}(\vartheta)}(\beth)\big):\beth\in\mathfrak{W}\big\}$

where, $T_{\mathfrak{v}(\vartheta)}(\beth)$=$T_{\mathfrak{g}(\vartheta)}(\beth)\wedge T_{\mathfrak{K}(\vartheta)}(\beth)$, $C_{\mathfrak{v}(\vartheta)}(\beth)$=$C_{\mathfrak{g}(\vartheta)}(\beth)\wedge C_{\mathfrak{K}(\vartheta)}(\beth)$, $G_{\mathfrak{v}(\vartheta)}(\beth)$=$G_{\mathfrak{g}(\vartheta)}(\beth)\square G_{\mathfrak{K}(\vartheta)}(\beth)$, $U_{\mathfrak{v}(\vartheta)}(\beth)$=$U_{\mathfrak{g}(\vartheta)}(\beth)\vee U_{\mathfrak{K}(\vartheta)}(\beth)$, $F_{(\vartheta)}(\beth)$=$F_{\mathfrak{g}(\vartheta)}(\beth)\vee F_{\mathfrak{K}}(\vartheta)(\beth)$

Now, $((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M}))\cup((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{K},\mathfrak{N}))$=$(\mathfrak{u},\mathfrak{M})\cup(\mathfrak{v},\mathfrak{N})=(\mathfrak{j},\mathfrak{E})$ where $\mathfrak{E}=\mathfrak{M}\cup\mathfrak{N}=(\mathfrak{L}\cap\mathfrak{M})\cup(\mathfrak{L}\cap\mathfrak{N})$ and $\mathfrak{j}:\vartheta\to IVPNS^{\mathfrak{W}}$ is given by for $\vartheta\in\mathfrak{E}$, $\mathfrak{j}(\vartheta)$=$\big\{\big(\beth,T_{\mathfrak{j}(\vartheta)}(\beth),C_{\mathfrak{j}(\vartheta)}(\beth),G_{\mathfrak{j}(\vartheta)}(\beth),U_{\mathfrak{j}(\vartheta)}(\beth),F_{\mathfrak{j}(\vartheta)}(\beth):\big)\beth\in\mathfrak{W}\big\}$

where,

$$T_{\mathfrak{j}(\vartheta)}(\beth)=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\beth)\wedge T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ T_{\mathfrak{K}(\vartheta)}(\beth)\wedge T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ \big(T_{\mathfrak{g}(\vartheta)}(\beth)\wedge T_{\mathfrak{g}(\vartheta)}(\beth)\big)\vee\big(T_{\mathfrak{g}(\vartheta)}(\beth)\wedge T_{\mathfrak{h}(\vartheta)}(\beth)\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\beth)\wedge T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ T_{\mathfrak{K}(\vartheta)}(\beth)\wedge T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ T_{\mathfrak{g}(\vartheta)}(\beth)\wedge\big(T_{\mathfrak{h}(\vartheta)}(\beth)\vee T_{\mathfrak{K}(\vartheta)}(\beth)\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$C_{\mathfrak{j}(\vartheta)}(\beth)=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\beth)\wedge C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ C_{\mathfrak{K}(\vartheta)}(\beth)\wedge C_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ \big(C_{\mathfrak{g}(\vartheta)}(\beth)\wedge C_{\mathfrak{h}(\vartheta)}(\beth)\big)\vee\big(C_{\mathfrak{g}(\vartheta)}(\beth)\wedge C_{\mathfrak{K}(\vartheta)}(\beth)\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\beth)\wedge C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ C_{\mathfrak{K}(\vartheta)}(\beth)\wedge C_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ C_{\mathfrak{g}(\vartheta)}(\beth)\wedge\big(C_{\mathfrak{h}(\vartheta)}(\beth)\vee C_{\mathfrak{K}(\vartheta)}(\beth)\big) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$G_{\mathfrak{j}(\vartheta)}(\mathfrak{I})=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ \left(G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\right)\square\left(G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\right) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ G_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ G_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\square\left(G_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\square G_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\right) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$U_{\mathfrak{j}(\vartheta)}(\mathfrak{I})=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee U_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\vee U_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ \left(U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee U_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\right)\wedge\left(U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee U_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\right) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee U_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ U_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\vee U_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ U_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee\left(U_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge U_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\right) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$F_{\mathfrak{j}(\vartheta)}(\mathfrak{I})=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee F_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\vee F_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ \left(F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee F_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\right)\wedge\left(F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee F_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\right) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$$=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee F_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{L}\cap\mathfrak{M})-\mathfrak{N} \\ F_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\vee F_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in(\mathfrak{N}\cap\mathfrak{L})-\mathfrak{M} \\ F_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee\left(F_{\mathfrak{h}(\vartheta)}(\mathfrak{I})\wedge F_{\mathfrak{K}(\vartheta)}(\mathfrak{I})\right) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M}\cap\mathfrak{N} \end{cases}$$

$\therefore \mathfrak{t}(\vartheta)=\mathfrak{j}(\vartheta)$

i.e, $(\mathfrak{g},\mathfrak{L})\cap((\mathfrak{h},\mathfrak{M})\cup(\mathfrak{K},\mathfrak{N}))=((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{h},\mathfrak{M}))\cup((\mathfrak{g},\mathfrak{L})\cap(\mathfrak{K},\mathfrak{N}))$

The other one can be proved in similar way.

(iii) Commutative Law:

Let $(\mathfrak{g},\mathfrak{L})\cup(\mathfrak{h},\mathfrak{M})=(\mathfrak{s},\mathfrak{D})$, where $\mathfrak{D}=\mathfrak{L}\cup\mathfrak{M}$ and $\mathfrak{s}:\mathfrak{D}\to IVPNS^{\mathfrak{W}}$ is given by, for $\vartheta\in\mathfrak{D}$, $\mathfrak{s}(\vartheta)=\left\{\left(\mathfrak{I},T_{\mathfrak{s}(\vartheta)}(\mathfrak{I}),C_{\mathfrak{s}(\vartheta)}(\mathfrak{I}),G_{\mathfrak{s}(\vartheta)}(\mathfrak{I}),U_{\mathfrak{s}(\vartheta)}(\mathfrak{I}),F_{\mathfrak{s}(\vartheta)}(\mathfrak{I})\right):\mathfrak{I}\in\mathfrak{W}\right\}$

$$T_{\mathfrak{s}(\vartheta)}(\mathfrak{I})=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ T_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ T_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee T_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$C_{\mathfrak{s}(\vartheta)}(\mathfrak{I})=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ C_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ C_{\mathfrak{g}(\vartheta)}(\mathfrak{I})\vee C_{\mathfrak{h}(\vartheta)}(\mathfrak{I}) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$G_{\mathfrak{s}(\vartheta)}(\beth)=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ G_{\mathfrak{g}(\vartheta)}(\beth)\square G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$U_{\mathfrak{s}(\vartheta)}(\beth)=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ U_{\mathfrak{g}(\vartheta)}(\beth)\wedge U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$F_{\mathfrak{s}(\vartheta)}(\beth)=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ F_{\mathfrak{g}(\vartheta)}(\beth)\wedge F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

Again assume that, Let $(\mathfrak{h},\mathfrak{M})\cup(\mathfrak{g},\mathfrak{L})=(\mathfrak{t},\mathfrak{D})$, where $\mathfrak{D}=\mathfrak{M}\cup\mathfrak{L}$ and $\mathfrak{t}:\mathfrak{D}\to IVPNS^{\mathfrak{W}}$ is given by, for $\vartheta\in\mathfrak{W}$, $\mathfrak{t}(\vartheta)=\{(\beth,T_{\mathfrak{t}(\vartheta)}(\beth),C_{\mathfrak{t}(\vartheta)}(\beth),G_{\mathfrak{t}(\vartheta)}(\beth),U_{\mathfrak{t}(\vartheta)}(\beth),F_{\mathfrak{t}(\vartheta)}(\beth\beth)):\beth\in\mathfrak{W}\}$

$$T_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ T_{\mathfrak{h}(\vartheta)}(\beth)\vee T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{L} \end{cases}=\begin{cases} T_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ T_{\mathfrak{g}(\vartheta)}(\beth)\vee T_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$C_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ C_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ C_{\mathfrak{h}(\vartheta)}(\beth)\vee C_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{L} \end{cases}=\begin{cases} C_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ C_{\mathfrak{g}(\vartheta)}(\beth)\vee C_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$G_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ G_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ G_{\mathfrak{h}(\vartheta)}(\beth)\square G_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{L} \end{cases}=\begin{cases} G_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ G_{\mathfrak{g}(\vartheta)}(\beth)\square G_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$U_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ U_{\mathfrak{h}(\vartheta)}(\beth)\wedge U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{L} \end{cases}=\begin{cases} U_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ U_{\mathfrak{g}(\vartheta)}(\beth)\wedge U_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$$F_{\mathfrak{t}(\vartheta)}(\beth)=\begin{cases} F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ F_{\mathfrak{h}(\vartheta)}(\beth)\wedge F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}\cap\mathfrak{L} \end{cases}=\begin{cases} F_{\mathfrak{g}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}-\mathfrak{M} \\ F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{M}-\mathfrak{L} \\ F_{\mathfrak{g}(\vartheta)}(\beth)\wedge F_{\mathfrak{h}(\vartheta)}(\beth) & if\ \vartheta\in\mathfrak{L}\cap\mathfrak{M} \end{cases}$$

$\therefore \mathfrak{s}(\vartheta) = \mathfrak{t}(\vartheta)$

$i.e, (\mathfrak{s}, \mathfrak{D}) = (\mathfrak{t}, \mathfrak{D})$

$i.e, (\mathfrak{g}, \mathfrak{L}) \cup (\mathfrak{h}, \mathfrak{M}) = (\mathfrak{h}, \mathfrak{M}) \cup (\mathfrak{g}, \mathfrak{L})$

Similarly the remaining one can be proved.

(iv) De Morgans Law

Let, $\mathfrak{g} : \mathfrak{L} \to IVPNS^{\mathfrak{W}}$ and $\mathfrak{h} : \mathfrak{L} \to IVPNS^{\mathfrak{W}}$ and $(\mathfrak{g}, \mathfrak{L}) \cup (\mathfrak{h}, \mathfrak{L}) = (\mathfrak{s}, \mathfrak{L})$, where $\mathfrak{s} : \mathfrak{L} \to IVPNS^{\mathfrak{W}}$ is given by, for $a \in \mathfrak{L}$,

$\mathfrak{s}(a)=\{(\mathfrak{J}, T_{\mathfrak{s}(a)}(\mathfrak{J}), C_{\mathfrak{s}(a)}(\mathfrak{J}), G_{\mathfrak{s}(a)}(\mathfrak{J}), U_{\mathfrak{s}(a)}(\mathfrak{J}), F_{\mathfrak{s}(a)}(\mathfrak{J})) : \mathfrak{J} \in \mathfrak{W}\}$

where, $T_{\mathfrak{s}(a)}(\mathfrak{J}) = T_{\mathfrak{g}(a)}(\mathfrak{J}) \vee T_{\mathfrak{h}(a)}(\mathfrak{J})$,

$C_{\mathfrak{s}(a)}(\mathfrak{J}) = C_{\mathfrak{g}(a)}(\mathfrak{J}) \vee C_{\mathfrak{h}(a)}(\mathfrak{J})$,

$G_{\mathfrak{s}(a)}(\mathfrak{J})=G_{\mathfrak{g}(a)}(\mathfrak{J})\square G_{\mathfrak{h}(a)}(\mathfrak{J}), U_{\mathfrak{s}(a)}(\mathfrak{J})=U_{\mathfrak{g}(a)}(\mathfrak{J}) \wedge U_{\mathfrak{h}(a)}(\mathfrak{J}), F_{\mathfrak{s}(a)}(\mathfrak{J})=F_{\mathfrak{g}(a)}(\mathfrak{J}) \wedge F_{\mathfrak{h}(a)}(\mathfrak{J})$

Now,$((\mathfrak{g}, \mathfrak{L}) \cup (\mathfrak{h}, \mathfrak{M}))^c = (\mathfrak{s}, \mathfrak{L})^c = (\mathfrak{s}^c, \rceil\mathfrak{L})$, where for $a \in \mathfrak{L}$,

$\mathfrak{s}^c(\neg a)= \{(\mathfrak{J}, F_{\mathfrak{s}(a)}(\mathfrak{J}), U_{\mathfrak{s}(a)}(\mathfrak{J}), G_{\mathfrak{s}(a)}(\mathfrak{J}), C_{\mathfrak{s}(a)}(\mathfrak{J}), T_{\mathfrak{s}(a)}(\mathfrak{J})) : \mathfrak{J} \in W\}$

Now, $(\mathfrak{g}, \mathfrak{L})^c = (\mathfrak{g}^c, \rceil\mathfrak{L})$ where

$\mathfrak{g}^c(\neg a)=\{(\mathfrak{J}, F_{\mathfrak{g}(a)}(\mathfrak{J}), U_{\mathfrak{g}(a)}(\mathfrak{J}), G_{\mathfrak{g}(a)}(\mathfrak{J}), C_{\mathfrak{g}(a)}(\mathfrak{J}), T_{\mathfrak{g}(a)}(\mathfrak{J}) : \mathfrak{J} \in \mathfrak{W})\}$

$(\mathfrak{h}, \mathfrak{L})^c = (\mathfrak{h}^c, \rceil\mathfrak{L})$ wherϑ

$\mathrm{h}^c(\neg a)=\{(\mathfrak{J}, F_{\mathfrak{h}(a)}(\mathfrak{J}), U_{\mathfrak{h}(a)}(\mathfrak{J}), G_{\mathfrak{h}(a)}(\mathfrak{J}), C_{\mathfrak{h}(a)}(\mathfrak{J}), T_{\mathfrak{h}(a)}(\mathfrak{J}) : \mathfrak{J} \in \mathfrak{W})\}$

$\therefore (\mathfrak{g}, \mathfrak{L})^c \cap (\mathfrak{h}, \mathfrak{L})^c = (\mathfrak{g}^c, \rceil\mathfrak{L}) \cap (\mathfrak{h}^c, \rceil\mathfrak{L}) = (t^c, \rceil\mathfrak{L})$ (say) where,

$\mathfrak{t}(\neg a) =$

$\{(\mathfrak{J}, F_{\mathfrak{g}(a)}(\mathfrak{J}) \wedge F_{\mathfrak{h}(a)}(\mathfrak{J}), U_{\mathfrak{g}(a)}(\mathfrak{J}) \wedge U_{\mathfrak{h}(a)}(\mathfrak{J}), G_{\mathfrak{g}(a)}(\mathfrak{J})\square G_{\mathfrak{h}(a)}(\mathfrak{J}), C_{\mathfrak{g}(a)}(\mathfrak{J}) \vee C_{\mathfrak{h}(a)}(\mathfrak{J}), T_{\mathfrak{g}(a)}(\mathfrak{J}) \vee T_{\mathfrak{h}(a)}(\mathfrak{J}) : \mathfrak{J} \in \mathfrak{W})\}$

$=\mathfrak{s}^c(\neg a)$

Therefore, $(\mathfrak{s}, \mathfrak{L})^c = (\mathfrak{t}^c, \rceil\mathfrak{L})$

Consequently, $((\mathfrak{g}, \mathfrak{L}) \cup (\mathfrak{h}, \mathfrak{L}))^c = (\mathfrak{g}, \mathfrak{L})^c \cap (\mathfrak{h}, \mathfrak{L})^c$.

The other part can be proved similarly.

(v) Idempotent Law:

The proof follows directly.

(vi) Absorption Law:

The proof follows directly.

(vii) Involution Law:

Follows directly from definition.

4. Strength and drawbacks of the induced method:

Identifying the strength helps an individual to leverage the method effectively in a specific suitable situation whereas drawbacks of a method provide opportunities for refinement and development of a better method. So to ensure the capability of the method strength and

drawbacks of the induced method is highlighted below:

Strength:

• Strength of the method lies in the fact that it can addresses five significant membership values which are very much essential during rating of an object or to describe a object efficiently.

• By using intervals, IVPNSS provides the decision-maker flexibility by allowing membership values to fall within a range.

• IVPNSS can capture contradictory and incomplete information effectively and making it suitable for complex systems where uncertainty is inherent.

Drawbacks:

• Compared to other current approaches, the calculating process employing basic operating laws is time-consuming because IVPNSS includes five membership functions.

• During decision making process, some expert may get confused when they have to provide a rating using IVPNSS.

However, the aforementioned disadvantage can be eliminated if the equation for transforming linguistic terms into IVPNS is created. In this case, decision-makers can express their opinions in linguistic terms, which can then be converted into IVPNS.

5. Conclusions

In conclusion, while Pentapartitioned Fuzzy Soft Sets ($PNSS$) offer a valuable tool for decision making problems, their effectiveness is limited when dealing with membership values that fall outside a crisp number. This paper addressed this limitation by proposing a new upgraded soft set theory specifically designed to handle such scenarios. By incorporating this new theory, decision makers can leverage $IVPNSS$ to achieve a higher degree of precision and accuracy in situations involving uncertain or interval-based membership values. Relations on $IVPNSS$s can be defined as [7] along with a new hybrid model by combining this concept with rough set as established in [6]. Topological structures can also be defined as in [8], [9], [14]. The decision-making difficulty can be resolved with the help of the recently developed approach. It can be utilized directly in multi-criteria decision-making problems, and the generalized form can also be used by employing an aggregation operator. This conclusion reiterates the importance of $IVPNSS$ and highlights the need for this upgraded soft set theory. It emphasizes the benefit of using the new theory for achieving more precise decisions.

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Plithogenic Sets-Based MABAC Framework for Evaluating the Quality of Elderly Care Security Services for Left-Behind Elderly in Rural Areas of Western China: A Healthcare Industry Perspective

Chenghua Wang[1]*, Xiaogang Liu[2], Jiyun Yao[3]
[1]School of Civil and Social Governance, Chongqing City Management College, Chongqing 401331, China
[2]School of Digital Intelligence Finance and Economics, Chongqing City Management College, Chongqing, 401331, China
[3]School of International Business and management, Chongqing Institute of Foreign Studies, Chongqing, 401120, China
*Corresponding author, e-mail: wangwan198199@163.com

Abstract:

This study proposed a multi-criteria decision-making (MCDM) methodology for evaluating the elderly care security services. These services consist various criteria, so the MCDM methodology is used. MABAC methodology is a MCDM method used to rank the alternatives. The MABAC methodology is integrated under the plithogenic sets to deal with vague and uncertainty data. The plithogenic sets are an extension of neutrosophic sets to deal with inconsistencies data. The criteria weights are computed in this study. Seven criteria and 13 alternatives are collected in this study. An application with opinions of three experts evaluate the criteria and alternatives. The results show the criterion Accessibility of Services has the highest weights and the criterion Technological Integration in Care Services has the lowest weights. The sensitivity analysis is performed to show the stability of rank of alternatives.

Keywords: Elderly Care Security Services; Plithogenic Sets; Healthcare Industry; MABAC Framework; Left-Behind Elderly in Rural Areas.

1. Introduction

Elcare is a start-up providing socially conscious healthcare services. Our goal is to offer senior parents top-notch security, convenience, and care services. Elcare, which is run by a group of highly skilled professionals, provides a comprehensive range of services

aimed at meeting all the needs of senior citizens and enhancing their general quality of life as well as their mental, physical, and emotional well-being[1], [2].

Elcare works hard to deliver the best senior care possible. Our staff of CareMates and CareAngels is amiable and well-trained, and they are prepared to handle any need that an older person may have. The caregivers at Elcare are the partner your elderly parents need at every stage to make their life better and easier, from attending to their health and wellness needs to help them and making it convenient for them to carry out daily tasks to handling any emergencies or crises[3], [4].

Elcare is only a phone call away if you're searching for a trustworthy and qualified senior care service provider! Elcare employs a team of knowledgeable, kind, and highly qualified people who are committed to helping the elderly with everything from everyday activities and outings to financial, legal, and medical advice. Elcare provides a range of senior care packages designed to meet the different needs of senior citizens and support them in living a stress-free, comfortable, and peaceful life[5], [6].

Making decisions is common in many domains, such as the management, social, and economic sectors. It includes tasks like business location, program selection, quality assessment, project evaluation, investment choice, resource allocation, staff evaluation, scientific research assessment, and overall grading of economic benefits. Scientific decision analysis is essential to modern management, and scientific decision making is the key to management success. An essential component of decision analysis, MCDM helps DMs score many options based on a predetermined set of criteria[7], [8].

DMs often bring in subject-matter experts throughout the assessment process. Expert evaluations are often uncertain since DMs are often evaluated using fuzzy numbers rather than precise ones due to the complicated environment and human preferences[9], [10], [11], [12]. To deal with this, Zadeh developed the idea of fuzzy sets, which are a useful tool for managing imprecise data. Since then, scholarly literature has proposed a number of expanded classifications for fuzzy sets[13], [14]. This research considers plithogenic sets in performance analysis to eliminate ambiguity in MCDM. In short, the plithogenic set is the neutrosophic set generalized[15], [16].

Ranking the options is the next crucial step in the MADM process. An essential instrument in this process is the widely used classical MABAC technique, which was developed by Pamučar & Ćirović. Determining how far each option is from the Border Approximation Area (BAA) is a crucial step. Among the many benefits that MABAC provides are simple computations, lucid reasoning, reliable results, and smooth

interoperability with other algorithms. Researchers have used MABAC in a variety of domains because of its advantages[17], [18].

The following are the primary ways in which this study has contributed:

- We introduce an evaluation framework under the plithogenic sets to deal with vague and uncertainty information. This model uses the opinions of experts to evaluate the criteria and alternatives.
- The MABAC method is used as a MCDM method to rank the alternatives based on plithogenic numbers.
- Application with seven criteria and 13 alternatives are considered to validate the proposed methodology.
- A sensitivity analysis is conducted to show the rank of alternatives is stable under different criteria weights.

The rest of this paper is organized as follows: Section 2 shows the plithogenic sets operations. Section 3 shows the steps of the MABAC method under the plithogenic sets. Section 4 shows the application of this study. Section 5 shows the sensitivity analysis. Section 6 shows the conclusions.

2. Plithogenic Sets

Plithogenic set is a generalization of the neutrosophic sets and the attribute values are the component of Plithogenic set. The Plithogenic set (P, A, V, d, c) is a set that has several components defined by number of criteria $A = \{a_1, \dots, a_m\}, m \geq 1$ and their values are $V = \{v_1, \dots, v_n\}, n \geq 1$. Contradiction $[c(v, D)]$ and appurtenance degrees functions $[d(x, v)]$ are the advantages elements of the Plithogenic sets[19].

In this study, the Plithogenic aggregation operation is applied to combine various evaluations of decision-making groups on input and output performance values with respect to all alternatives. This approach includes the following steps.

Step 1. In this study the evaluations of group decision making are used as input and output with review of experts. Their evaluations include the linguistic terms between criteria and alternatives. These terms are converted to the Plithogenic numbers to reflect uncertainty in this study.

Step 2. The dominant criterion between inputs and outputs is computed and the contradiction degrees of each input and output are defined. Then all decision-making evaluations by experts and decision makers are combined with the Plithogenic operator as:

$$\left((a_{i1}, a_{i2}, a_{i3}) \Lambda_p (b_{i1}, b_{i2}, b_{i3})\right) = \begin{pmatrix} a_{i1} \Lambda_F b_{i1}, \\ \frac{1}{2}(a_{i2} \Lambda_F b_{i2}) + \frac{1}{2}(a_{i2} \vee_F b_{i2}), \\ (a_{i3} \vee_F b_{i3}) \end{pmatrix} \tag{1}$$

Step 3. The combined evaluations are Plithogenic numbers are they are converted to crisp values such as:

$$s(A) = \frac{(2 + a_1 - b_1 - c_1)}{3} \tag{2}$$

Step 4. Normalize crisp values to compute the criteria weights.

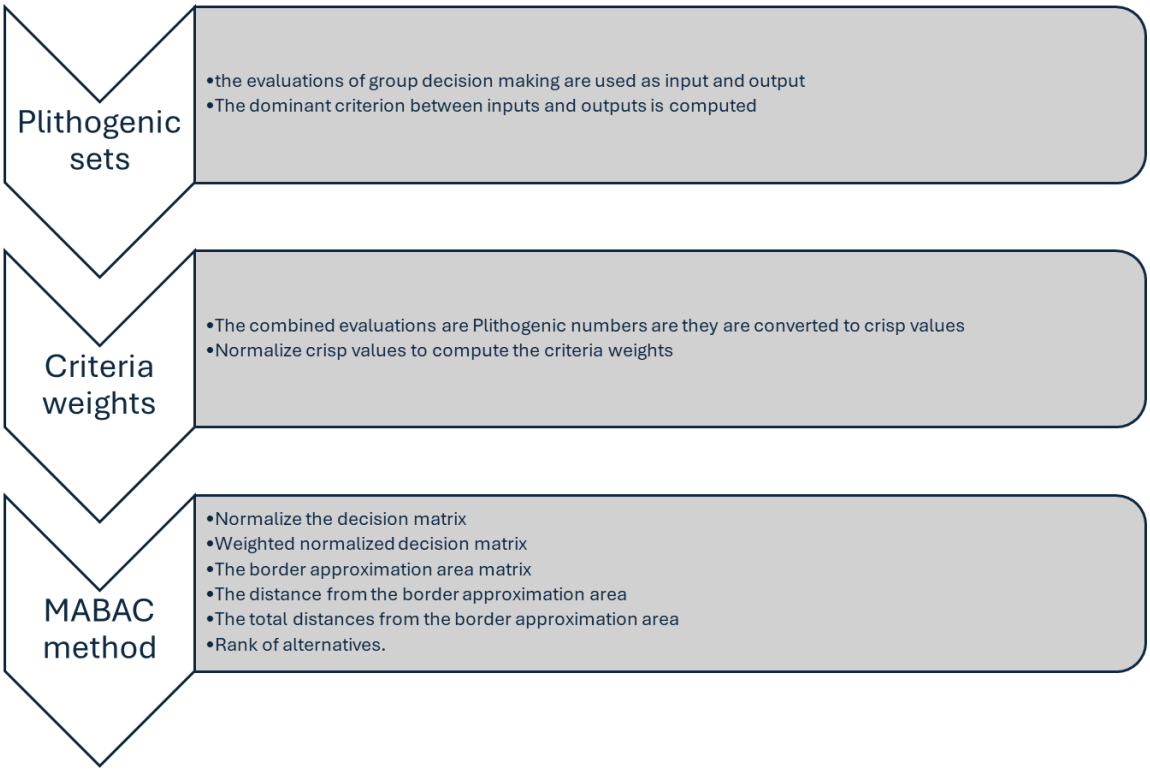
Plithogenic sets
•the evaluations of group decision making are used as input and output
•The dominant criterion between inputs and outputs is computed
Criteria weights
•The combined evaluations are Plithogenic numbers are they are converted to crisp values
•Normalize crisp values to compute the criteria weights
MABAC method
•Normalize the decision matrix
•Weighted normalized decision matrix
•The border approximation area matrix
•The distance from the border approximation area
•The total distances from the border approximation area
•Rank of alternatives.

Figure 1. The steps of MABAC method under uncertainty environment.

3. The Multi-Attributive Border Approximation area Comparison (MABAC) method

This section shows the steps of the MABAC under the Plithogenic numbers to rank the alternatives and deal with vague information[20]. Figure 1 shows the steps of MABAC method under Plithogenic sets.

Normalize the decision matrix.

The normalized decision matrix is computed for positive and negative criteria such as:

$$l_{ij} = \frac{x_{ij} - \min x_i}{\max x_i - \min x_i}; i = 1, \dots, m; j = 1, \dots, n \quad (3)$$

$$l_{ij} = \frac{x_{ij} - \max x_i}{\min x_i - \max x_i}; i = 1, \dots, m; j = 1, \dots, n \quad (4)$$

Weighted normalized decision matrix

$$u_{ij} = w_j + l_{ij} w_j \quad (5)$$

The border approximation area matrix

$$g_j = \left(\prod_{i=1}^{m} u_{ij}\right)^{\frac{1}{m}} \quad (6)$$

The distance from the border approximation area

$$d_{ij} = u_{ij} - g_j \quad (7)$$

The total distances from the border approximation area.

$$K_i = \sum_{j=1}^{n} d_{ij} \quad (8)$$

The final rank of alternatives.

4. Application

This study evaluates the criteria and alternatives performance and importance with Plithogenic sets and MABAC method. The performance indicators are evaluated from a set of experts. Three experts are invited to evaluate seven criteria and 13 alternatives. The sets of criteria as:

C1: Financial Security and Assistance

C2: Quality of Care Facilities and Personnel

C3: Social Inclusion and Community Engagement

C4: Technological Integration in Care Services

C5: Accessibility of Services

C6: Health and Medical Support

C7: Emotional and Psychological Support

Step 1. Three experts evaluated the criteria and alternatives using the linguistic term. These terms are transformed into Plithogenic numbers as shown in Tables 1-3.

Step 2. The dominant criterion between inputs and outputs is computed. Step 3. We obtain the crisp values between inputs and outputs. Step 4. Normalize crisp values to compute the criteria weights. The criteria weights are shown in Figure 2.

Table 1. The opinion of first expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_2	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A_3	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_4	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_5	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)
A_6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)
A_7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_8	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)
A_9	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)
A_{10}	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_{11}	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_{12}	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_{13}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)

Table 2. The opinion of second expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_2	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_4	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A_5	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_6	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_7	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_8	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_9	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A_{10}	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_{11}	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_{12}	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_{13}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)

Table 3. The opinion of third expert.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_2	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)

A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_4	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_5	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)
A_6	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_7	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_8	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_9	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A_{10}	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_{11}	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_{12}	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)
A_{13}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)

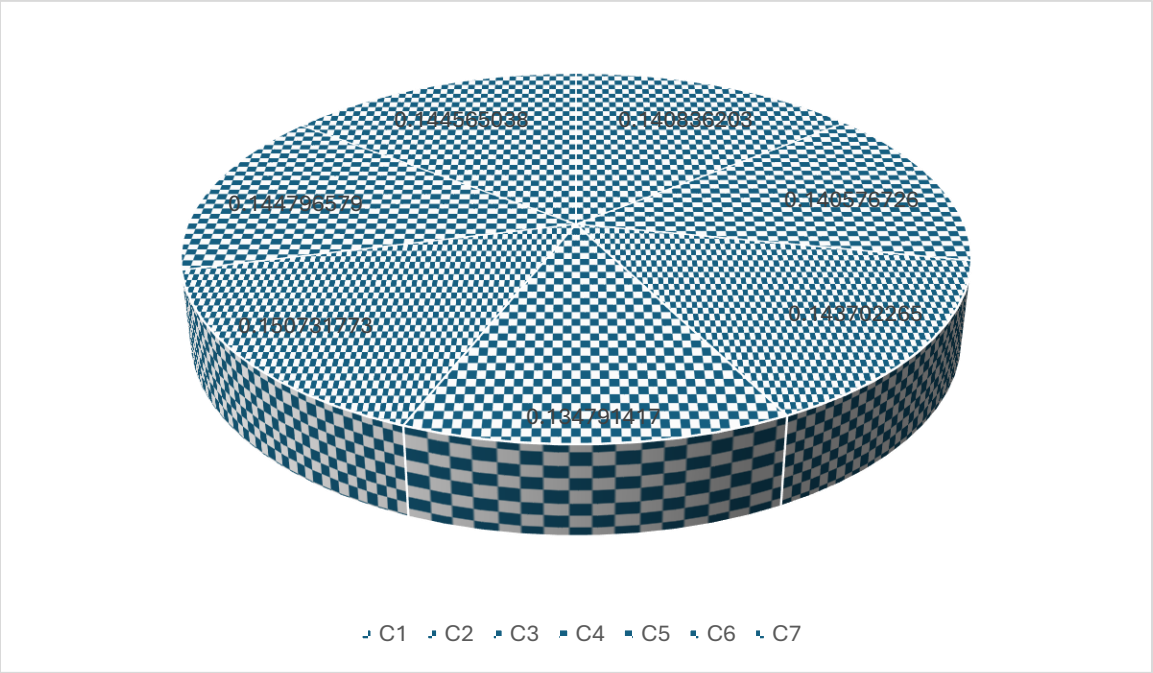

Figure 2. The criteria weights.

We normalize the decision matrix as shown in Table 4. Then we compute the weighted normalized matrix as shown in Table 5.

Table 4. The normalized matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0	0.162322	0.336178	0.591114	0.736936	1	1
A_2	0.510073	1	1	0.729845	0.495968	0.444643	0.158812
A_3	0.42076	0.250807	0.563016	0.666129	0.859148	0.241977	0.642511
A_4	0.365119	0.658812	0	0	0.530213	0	0.558586
A_5	0.802475	0.717398	0.677383	0	0.488952	0.927526	0.465946
A_6	0.48868	0.69327	0.74623	1	0.81108	0.960332	0.833844

A_7	0.475249	0.655423	0.496444	0.982584	0.768248	0.782141	0.784216
A_8	0.341328	0.642511	0.269322	0.91547	0.506859	0.600698	0.769246
A_9	0.833125	0	0.02276	0.167191	0.265996	0.39355	0
A_{10}	1	0.556085	0.947416	0.266927	0	0.71407	0.082392
A_{11}	0.254701	0.589897	0.762352	0.33659	0.488952	0.520806	0.658812
A_{12}	0.291587	0.631375	0.781223	0.399116	0.25144	0.796303	0.717398
A_{13}	0.249233	0.561491	0.781223	0.982584	1	1	0.747176

Table 5. The weighted normalized matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.140836	0.163395	0.192012	0.214468	0.261811	0.289593	0.28913
A_2	0.212673	0.281153	0.287405	0.233168	0.22549	0.209179	0.167524
A_3	0.200094	0.175834	0.224609	0.22458	0.280233	0.179834	0.23745
A_4	0.192258	0.23319	0.143702	0.134791	0.230652	0.144797	0.225317
A_5	0.253854	0.241426	0.241044	0.134791	0.224432	0.279099	0.211925
A_6	0.20966	0.238034	0.250937	0.269583	0.272987	0.283849	0.26511
A_7	0.207769	0.232714	0.215042	0.267235	0.266531	0.258048	0.257935
A_8	0.188908	0.230899	0.182404	0.258189	0.227132	0.231776	0.255771
A_9	0.25817	0.140577	0.146973	0.157327	0.190826	0.201781	0.144565
A_{10}	0.281672	0.218749	0.279848	0.170771	0.150732	0.248192	0.156476
A_{11}	0.176707	0.223502	0.253254	0.180161	0.224432	0.220208	0.239806
A_{12}	0.181902	0.229333	0.255966	0.188589	0.188632	0.260099	0.248276
A_{13}	0.175937	0.219509	0.255966	0.267235	0.301464	0.289593	0.252581

The border approximation area matrix is computed and the distance from the border approximation area is computed. We compute the total distances from the border approximation area. Final the rank of alternatives in Figure 3.

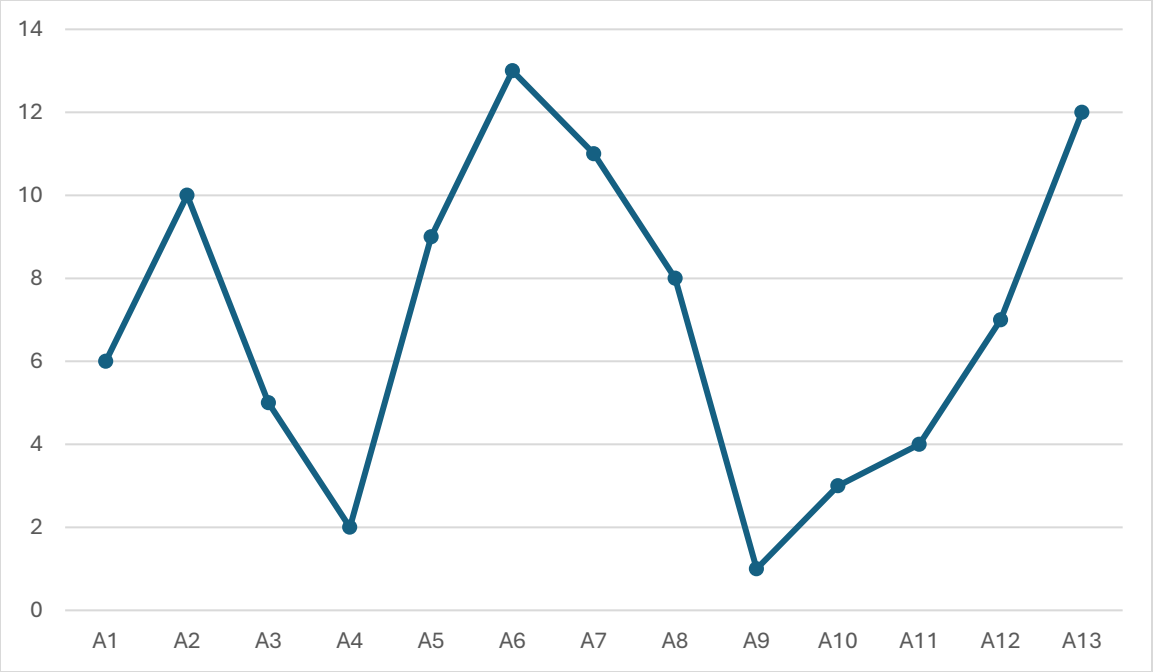

Figure 3. The rank of alternatives.

5. Analysis

This section shows the sensitivity analysis to show the different rank of alternatives under different weights. We proposed five cases in criteria weights in Figure 4. Then we applied the MABAC method under these weights. Table 6 shows the rank of alternatives. The results show the rank of alternatives is stable under different criteria weights.

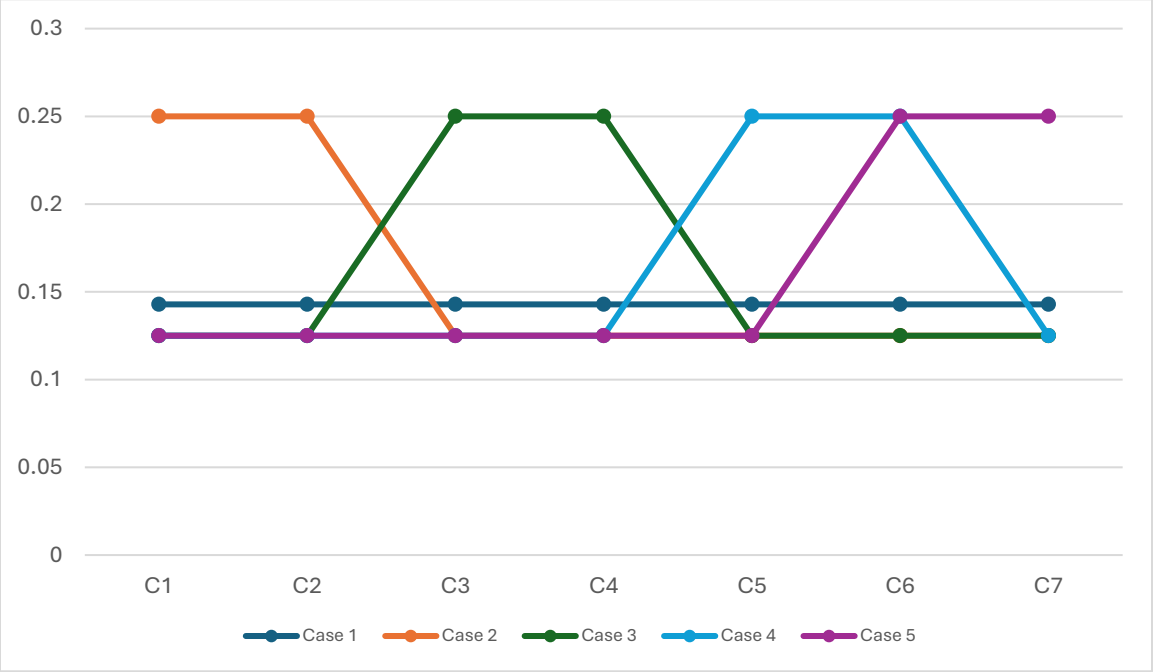

Figure 4. The different criteria weights.

Table 6. The rank of alternatives under sensitivity analysis.

	Case 1	Case 2	Case 3	Case 4	Case 5
A_1	6	3	4	10	10
A_2	10	10	10	8	6
A_3	5	4	7	5	4
A_4	2	2	2	2	2
A_5	9	9	5	9	9
A_6	13	13	13	12	13
A_7	11	11	11	11	11
A_8	8	7	9	7	8
A_9	1	1	1	1	1
A_{10}	3	8	6	3	3
A_{11}	4	5	3	4	5
A_{12}	7	6	8	6	7
A_{13}	12	12	12	13	12

6. Conclusions

In this paper a performance evaluation methodology for ranking the elderly care security services with a set of criteria and alternatives. Three experts have evaluated the criteria and alternatives. Then we used the plithogenic numbers instead of the linguistic terms. Then these numbers are combined into a single matrix. Then we obtain crisp values. The Plithogenic numbers are combined with the MCDM methodology such as MABAC method to rank the alternatives. Seven criteria and 13 alternatives are collected to select the best alternatives. The results show the alternative 13 has the highest rank and alternative 9 has the lowest rank. The sensitivity analysis was conducted to show the different ranks of alternatives under different weights. Five cases of criteria weights are conducted. The results show the rank of the proposed methodology is stable under different weights.

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Neutrosophic-Supported Machine Learning Models for Oral Disease Classification

Ibrahim M. Elezmazy[1] and Doaa El-Shahat[2]
[1,2]Zagazig University, Faculty of Computers and Informatics, Computer Science, Egypt
ib.elazmazy024@fci.zu.edu.eg; doaazidan@fci.zu.edu.eg

Abstract
This study is presented to investigate the influence of the neutrosophic (NS) domain on the performance of the most common machine learning (ML) models. Specifically, it evaluates the effectiveness of Random Forest (RF), Extra Trees (ET), K-Neighbors (KNN), Gaussian Naive Bayes (GaussianNB), and Decision Tree (DT) classifiers in detecting oral diseases. The NS domain divides any image into three membership components: falsity (F), indeterminacy (I) and truth (T), where T denotes the degree to which each pixel in an image is a member of a particular class or category, F denotes the degree to which the pixel is not a member of that class or category and I denotes the degree of uncertainty or indeterminacy. This domain is herein employed to divide the images into three datasets (T, I, and F). Those three datasets, in addition to the original dataset, one by one, are divided into training and testing datasets with a splitting ratio of 80% and 20%, respectively. Afterwards, the five studied ML models are trained on the training dataset before being assessed on the testing dataset to show their ability to generalize using four performance indicators, such as precision, accuracy, F1-score and recall. According to the experimental results, the I-domain, when compared to the other domains, could considerably increase the performance of four of the five studied ML models, meaning that the NS domain might improve the ML models' performance in categorizing oral diseases more correctly. Among all studied models, the RF classifier with the I-domain has the highest classification accuracy, indicating that it is a strong alternative for oral disease classification.

Keywords: Neutrosophic domain; Machine learning; Oral diseases; Image classification.

1. Introduction

Problems affecting the mouth include, but are not limited to, tooth decay, arthritis, and bad breath. Because treatment improves a patient's prognosis, it is an important metric for early diagnosis [1]. Most traditional diagnostic methods consist of clinical tests, although they can be lacking in quality and accuracy [2]. In recent years, the use of machine learning for the diagnosis of facial diseases has grown exponentially. By identifying the complex relationships found in medical data, machine learning models improve overall patient care, diagnostic knowledge and quality treatment planning [3].

There are several issues that may hinder the application of ML in oral disease classification. First, oral disease data usually take several forms, such as photographs (radiographs, photographs) clinical

records genetic data [4]. Therefore, the challenge will be combining or integrating, as well as analyzing the disparate data sources. Great variability exists between clinical presentation, patient, and imaging modality making it especially very hard to work with standardized diagnostic criteria. Incomplete and/or uncertain clinical data are where standards may or may not present, and resources are vague as to what is published, and interpretations are subjective.

NS sets by Florentin Smarandache are a very powerful frame in which uncertainties, imprecisions, and inconsistencies are represented and managed. Situations in a real-world context that might involve missing, vague, or contradictory information can thus be modeled using NS sets [5]. Bringing uncertainty into the epistemic domain of machine learning models may improve their robustness and reliability within medical diagnosis relative to the result provided [6].

NS Logic may have great integration possibilities for various ML models with respect to enhancing the efficiency of the approached process in classifying oral diseases. Its common methods embody NS Feature Selection, which may lead to better accuracy and interpretability of models [7]. NS-supported ML models have shown great promises in the classification of various oral diseases, such as caries detection, periodontal disease diagnosis, and oral cancer screening.

The rest of this paper is classified as follows. Section 2 provides the background needed for this study. Section 3 presents proposed model. Section 4 presents experimental results. Section 5 illustrates the conclusion and future directions of this proposal.

2. Related work

There is a growing interest within the application of NS idea to the dental disease, offering robust solutions to address uncertainties inherent in diagnostic and treatment processes. This research highlights vast research the use of NS strategies in dental technological know-how. To triumph over the sensitivity drawbacks of dental imaging, Datta et al. [8] proposed a NS set-based totally approach for removal of caries lesions. The approach decreased the ambiguities and indeterminacies in medical photos and progressed the visible accuracy.

According to Lalama et al. [9] they made a NS approach for evaluating dental implant success rates by qualifying and quantifying paired and related uncertainties. The study included emaciating criteria such as absence of pain and functionality and further-estimated maintenance duration greatness, exhibiting a feasible application of NS theory within holistic assessments. Datta et al.[10] have put forwarded a Marker based watershed Digital X-ray image segmentation technique to tackle the problems caused due to the complex internal structures of teeth and variable intensity of images in order to achieve better segmentation accuracy.

A mobile network V2-and-NS semantic multi-label framework was introduced by Alsakar et al. [11] to diagnose different dental disorders. This paper explains the nearness logic in handling the diagnostic uncertainty for increasing reliability of diagnosis. Chen et al. [12] have applied a NS logic framework

to diagnosis dental diseases from periapical radiographs. The methodology combines convolutional neural networks (CNNs) with NS logic in addressing the uncertainties in the process of lesion detection.

Recalde et al. [13] made use of NS logic in assessing the efficacy of preventive dental education in a primary school atmosphere. The study adopted a quantitative assessment under various dental prevention education schemes that had been modified for the TOPSIS method and provided further evidence for the applicability of NS logic in the evaluation of educational programs. Table 1 shows the related works.

Table 1. Summarization of some recent related work

Author and Ref.	Year	Technique/model	Evaluation Metric (Accuracy)
Datta et al. [8]	2020	NS sets for caries lesion detection	95%
Lalama et al. [9]	2023	NS evaluation	93%
Datta et al.[10]	2020	Marker-based watershed segmentation	89.3%
Alsakar et al. [11]	2024	MobileNetV2 with NS sets	94%
Chen et al [12]	2021	CNNs with NS logic	93%
Recalde et al [13]	2023	NS Logic with TOPSIS	90%

3. The proposed models

3.1. Neutrosophic (NS) domain

Neutrosophy was proposed by F. Smarandache [14] as a more advanced way than fuzzy theory for assessing situations involving uncertainty. This approach separates the image into three subsets: falsehood ($\boldsymbol{F}$), indeterminacy ($\boldsymbol{I}$), and true ($\boldsymbol{T}$), transforming it from the spatial domain into the NS domain $\boldsymbol{P_{NS}}$. In a broader context, every pixel in image $\boldsymbol{B}$ is separated into three membership components: $\boldsymbol{I}$, $\boldsymbol{T}$, and $\boldsymbol{F}$. In the NS set, those three membership components fall between 0 and 1. In contrast, the fuzzy set has just two components—$\boldsymbol{F}$ and $\boldsymbol{T}$; in the crisp set, $\boldsymbol{I}$ is equal to 0, and $\boldsymbol{T}$ **and** $\boldsymbol{F}$ for each pixel are either 1 or 0. The NS domain, $\boldsymbol{P_{NS}} = \{\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F}\}$, for each pixel in the image $\boldsymbol{B}$ can be obtained according to the following formula:

$$\boldsymbol{P_{NS}(k,j)} = \{\boldsymbol{T_{k,j}}, \boldsymbol{F_{k,j}}, \boldsymbol{I_{k,j}}\} \tag{1}$$

$$\boldsymbol{T_{k,j}} = \frac{\boldsymbol{g(\bar{k},j) - g_{min}}}{\boldsymbol{g_{max} - g_{min}}} \tag{2}$$

$$\boldsymbol{g(\bar{k},j)} = \frac{1}{\boldsymbol{w \times w}} \sum_{\boldsymbol{x=k-\frac{w}{2}}}^{\boldsymbol{k+\frac{w}{2}}} \sum_{\boldsymbol{y=j-\frac{w}{2}}}^{\boldsymbol{j+\frac{w}{2}}} \boldsymbol{g(x,y)} \tag{3}$$

$$\boldsymbol{I_{k,j}} = \mathbf{1} - \frac{\boldsymbol{Hg_{k,j} - Hg_{min}}}{\boldsymbol{Hg_{max} - Hg_{min}}} \tag{4}$$

$$\boldsymbol{Hg_{k,j}} = \left|\boldsymbol{g(k,j) - g(\bar{k},j)}\right| \tag{5}$$

$$\boldsymbol{F_{k,j}} = \mathbf{1} - \boldsymbol{T_{k,j}} \tag{6}$$

where $\boldsymbol{P_{NS}(k,j)}$ represents the NS domain for the $\boldsymbol{(k,j)}$ pixel in the image $\boldsymbol{B}$, $\boldsymbol{g(k,j)}$ represent the intensity value of the $\boldsymbol{(k,j)}$ pixel in the image $\boldsymbol{B}$, $\boldsymbol{g(\bar{k},j)}$ represents the window size pixels' local mean value, $\boldsymbol{w}$ represents the window size, $\boldsymbol{g_{max}}$ and $\boldsymbol{g_{min}}$ represent the last and first peaks, respectively, that are measured from pixels whose values exceed the histogram's maximum local average, and $\boldsymbol{Hg_{min}}$ and $\boldsymbol{Hg_{max}}$ are the first and last peaks, respectively, determined from the homogeneity image.

3.2. The proposed NS-supported ML models

In this study, we integrate the NS domain with the ML model to classify the dental diseases more accurately. The NS domain is first applied to convert the dental disease images from the spatial domain into the NS domain using Eqs. (1-6) resulting in three newly generated datasets representing T, I and F components. These three datasets, one by one, are then divided into training and testing datasets. The former is employed to train five popular ML models, such as random forest (RF) [15], extra trees (ET) classifier [16], K neighbors (KNN) classifier [17], Gaussian Naive Bayes (GaussianNB) [18] and decision tree (DT) classifier [19], while the latter is used to test the generalization capability of those models after completing the training process. In the next list, we illustrate the main steps followed to observe the influence of the NS domain on the performance of the five studied ML models when applied to classify the dental diseases:

- **Preprocessing step**: Using the NS theory, the original datasets are separated into three different datasets (T, F and I). The datasets are then normalized between 0 and 1 by dividing all pixel intensities in images by 255. The normalization step's objective is to improve the performance and stability of models.

- **Building the ML models**: During this stage, the specified ML models are built with Python's Scikit-learn packages [20]. It provides easy and efficient methods for predictive data analysis and is developed on top of SciPy, NumPy and Matplotlib.

- **Training the ML models**: The three datasets of the NS domain (I, T and F), in addition to the original dataset, are then separated into training and testing datasets. The former is used to train the five studied ML models: RF, ET, KNN, GaussianNB and DT, while the latter is utilized for evaluating the models' generalization capability after training. In Fig. 1, we depict the main steps for training and testing the studied ML algorithms using the NS domains for classifying dental diseases.

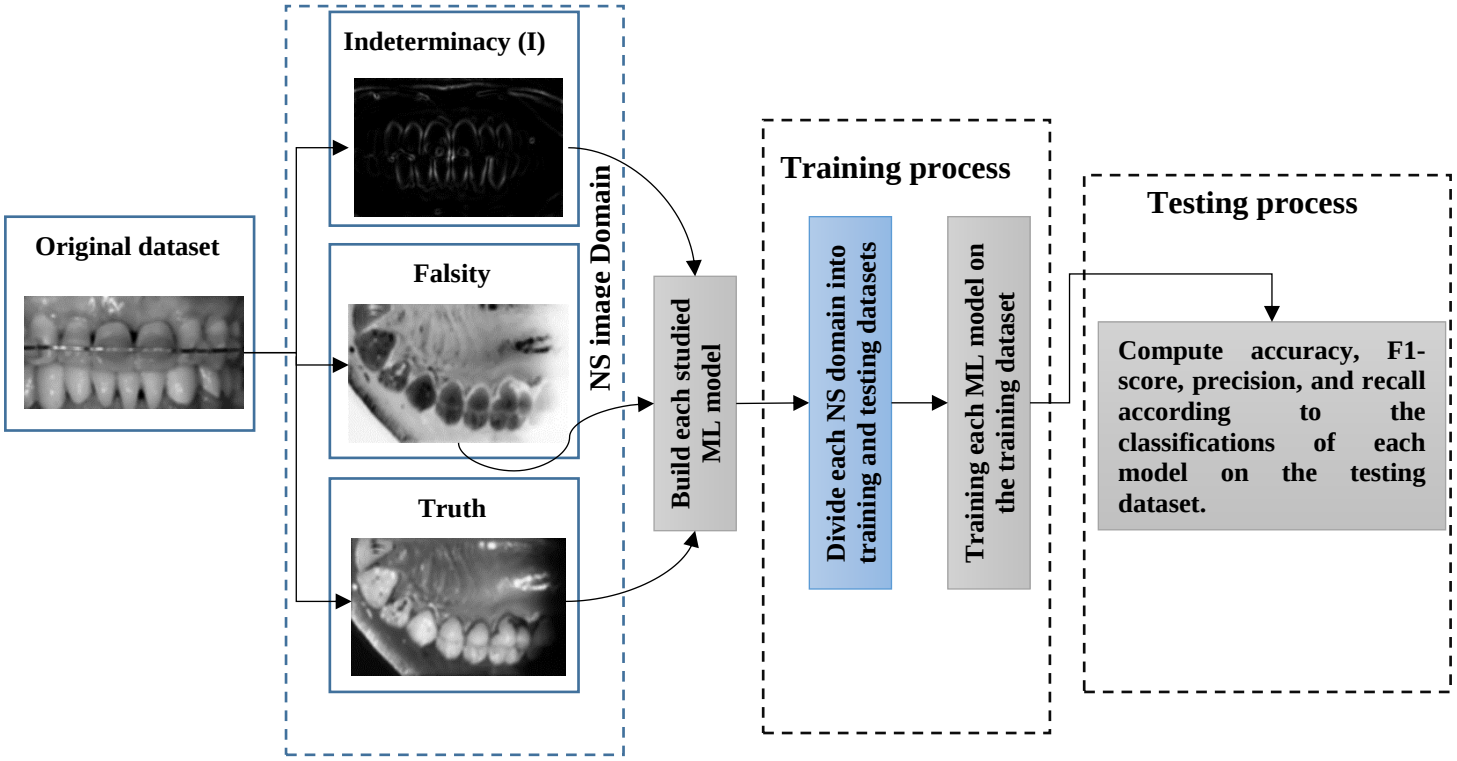

Original dataset
Indeterminacy (I)
Falsity
Truth
NS image Domain
Build each studied ML model
Training process
Divide each NS domain into training and testing datasets
Training each ML model on the training dataset
Testing process
Compute accuracy, F1-score, precision, and recall according to the classifications of each model on the testing dataset.

Fig. 1 Flowchart of the main steps for training and testing the studied ML algorithms under the NS domains when used to classify dental diseases

4. Results and discussion

4.1 Dataset description

In this section, the influence of the NS approach on the classification accuracy of five ML models is assessed using the publicly-available dental condition dataset [21]. This dataset includes a variety of dental disorders, like caries, calculus, gingivitis, ulcers and hypodontia. In this study, we used a subset of this dataset to test the influence of the NS approach when the dataset size is low. For example, for the category "caries", all images for this category in this dataset were considered in this study; for the category "calculus", 376 images were randomly selected from 1296; and for the category "gingivitis", 276 images were randomly chosen from 2349. In total, in the conducted experiments, we used around 1600 for all categories to test the performance of the studied ML models with the NS approach.

4.2 Dataset split

Next, all those images were first converted into the NS domain, which creates three new datasets: T, I and F, as depicted in Fig. 2. Then, the NS and original datasets, one by one, are divided into testing and training datasets with a splitting ratio of 20% and 80%, where 20% of all images are considered for testing the proposed model, while the rest are used for the training process.

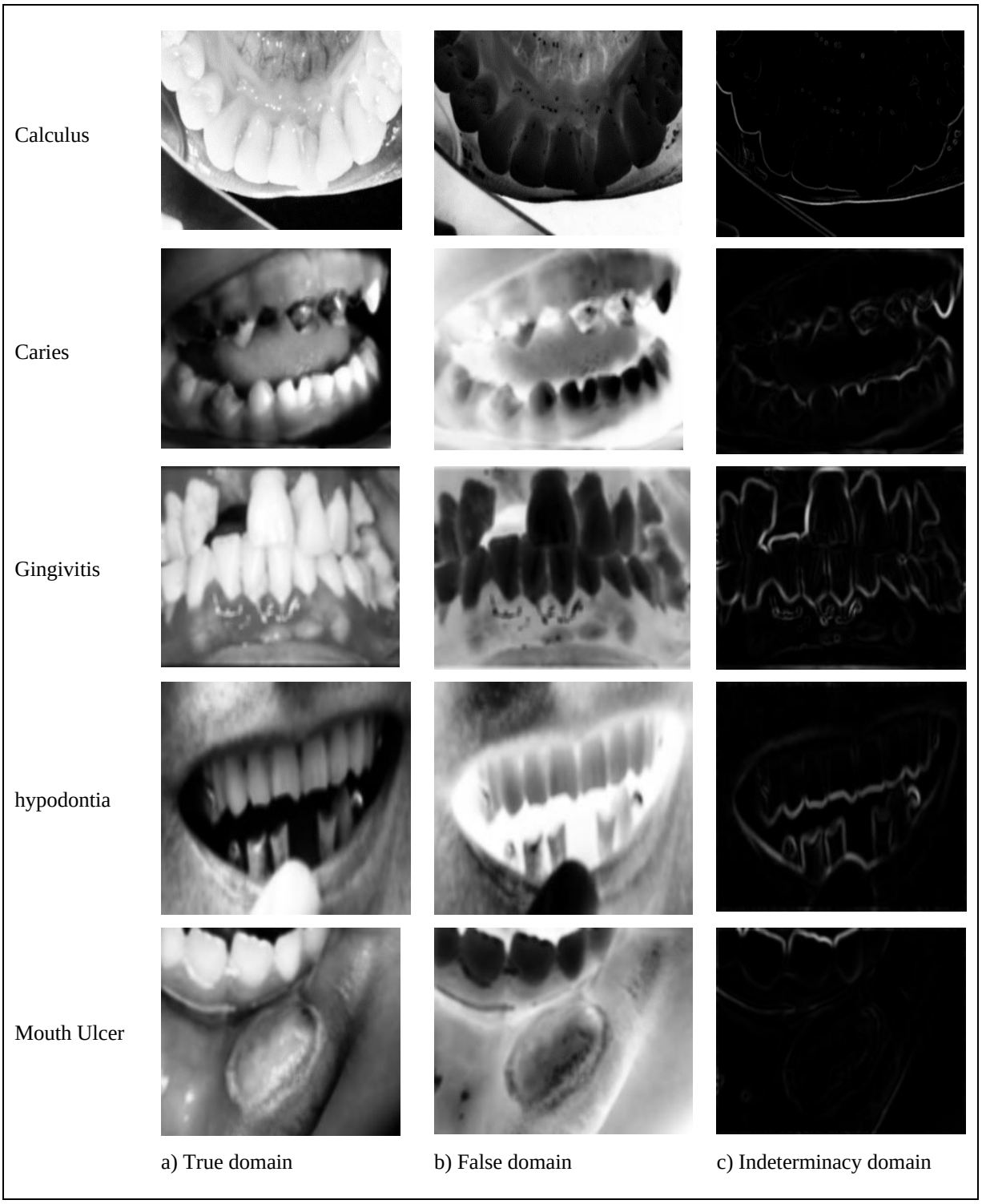


Fig. 2 Depiction of three NS domains for five oral diseases

4.3 Performance metrics

To measure the performance of various models, four common performance measures, namely accuracy, F1-score, precision, and recall, were used in this study and are defined as follows:

- **Accuracy** is defined as the ratio of correctly anticipated images to total images:

$$\textbf{Accuracy} = \frac{\text{TP+TN}}{\text{TP+FP+TN+FN}}. \tag{7}$$

  where FN stands for false negative, TN for true negative, FP for false positive, and TP for true positive.

- **Precision** is calculated as the ratio of correctly anticipated images as A to all images predicted as A:

$$\textbf{Precision} = \frac{\text{TP}}{\text{TP+FP}}. \tag{8}$$

- **Recall** is calculated as the ratio of correctly anticipated images as A to all images labeled as A in the dataset:

$$\textbf{Recall} = \frac{\text{TP}}{\text{TP+FN}}. \tag{8}$$

- **F1-score** is estimated by the next equation:

$$\textbf{F1} - \textbf{score} = 2 * \frac{Precision \cdot Recall}{Precision + Recal}. \tag{9}$$

4.4 Investigation of ML models for the oral diseases classification under neutrosophic domain

In addition, this section evaluates the effectiveness of several DL models for classifying different oral diseases in both the spatial and NS domains. As previously noted, each image is separated into three parts using the NS approach: T, I and F. Following the creation of three NS components for each image in the original dataset, five ML algorithms (RF, DT, KNN, GaussianNB and ET) are fitted and tested on the original and NS datasets to show how the NS domain affects their ability to identify the oral diseases. More broadly, the original dataset (O) is divided into two subsets: 80% for training and 20% for testing. After fitting the ML models on the training dataset, they are evaluated on the test dataset to demonstrate their capacity for generalization.

To verify the algorithm's stability, the splitting process is independently carried out five times, with training and testing conducted at each time, and the average of precision, accuracy, F1-score, and recall values obtained by each model within those independent runs are reported in Table 1. Similarly, the same table includes the average performance of the studied models for the NS datasets (T, I, and F). This table reveals that all ML models, except KNN, could achieve classification accuracies under the I datasets significantly better than those under the original and other NS datasets, while the KNN performs better under the T dataset. From that, it is concluded that the NS approach could improve the performance of the ML models when applied to classify oral diseases. Among the NS datasets, the I-

component may be able to extract the most effective features that could help those models classify oral diseases more accurately.

Finally, the accuracy values that each model achieved on the four datasets are displayed in Fig. 3 to visually represent the performance of different models under each dataset. According to this figure, ET with the I dataset has the highest accuracy of 0.793, followed by the RT with the same dataset, while GaussianNB and DT perform poorly with all datasets. As demonstrated in Fig. 4, the performance of the RF classifier with the I domain is further examined using the confusion matrix to disclose its accuracy for each classification label. As shown in the figure, for the majority of classes, the number of images correctly classified by this algorithm under the I-domain is greater than those correctly predicted under the other domains, demonstrating that the I-domain with the RF classifier is a strong alternative for detecting oral diseases.

Table 1 Comparison among ML models under the NS and original domains

		KNN	RF	DT	GaussianNB	ET
Accuracy	**O**	0.63036	0.7368	0.56683	0.55941	0.74175
	I	0.50743	**0.7731**	**0.60231**	**0.59076**	**0.79373**
	T	**0.64604**	0.72442	0.56271	0.4769	0.74752
	F	0.64026	0.7566	0.59158	0.54868	0.74917
Precision	**O**	0.61208	0.72224	0.55552	0.54885	0.71511
	I	0.49816	**0.77165**	**0.5788**	**0.56462**	**0.78365**
	T	**0.62927**	0.70369	0.54552	0.48809	0.72822
	F	0.62658	0.7418	0.57491	0.53834	0.73422
Recall	**O**	0.60121	0.70756	0.54868	0.54724	0.70714
	I	0.45567	**0.72916**	**0.57544**	**0.56041**	**0.74741**
	T	0.61627	0.70042	0.54424	0.46499	0.72371
	F	**0.61767**	0.73843	0.5764	0.54226	0.73245
F1-Score	**O**	0.59823	0.70672	0.54854	0.53839	0.70387
	I	0.44059	**0.72604**	**0.57489**	**0.55489**	**0.73797**
	T	**0.61631**	0.69822	0.54384	0.45465	0.72269
	F	0.61398	0.73725	0.5734	0.53476	0.73087

Bold values indicate the best results

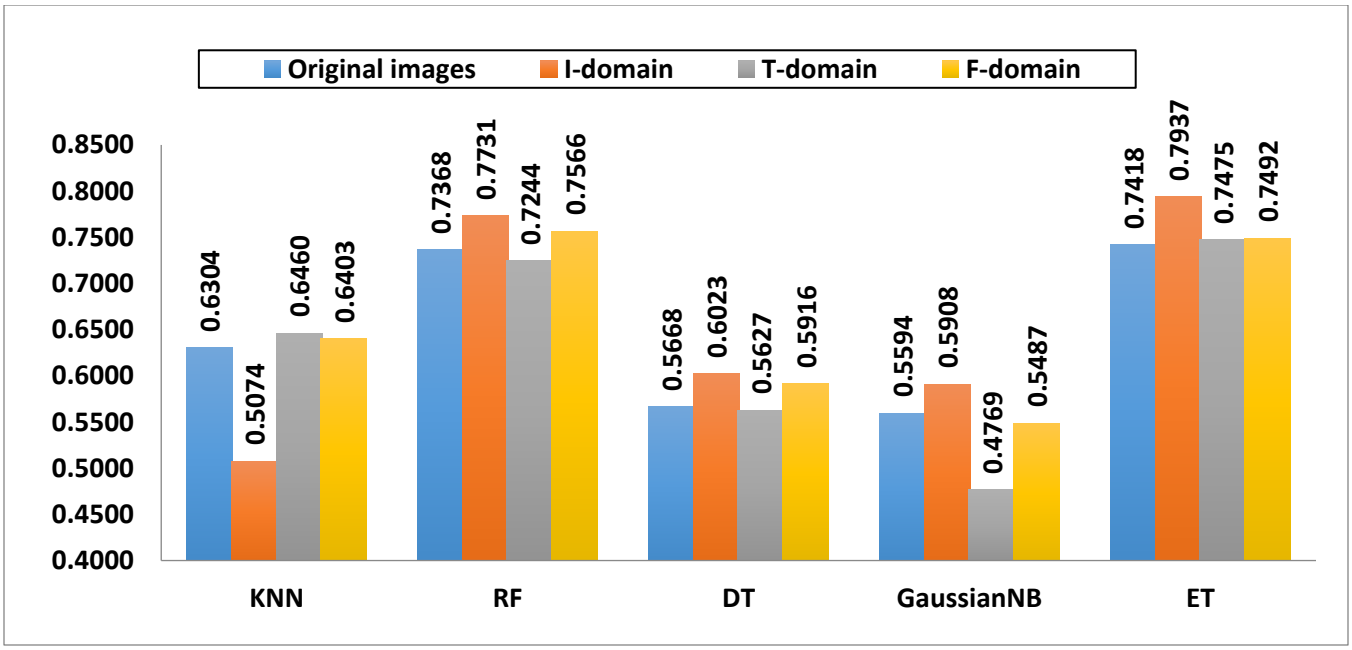
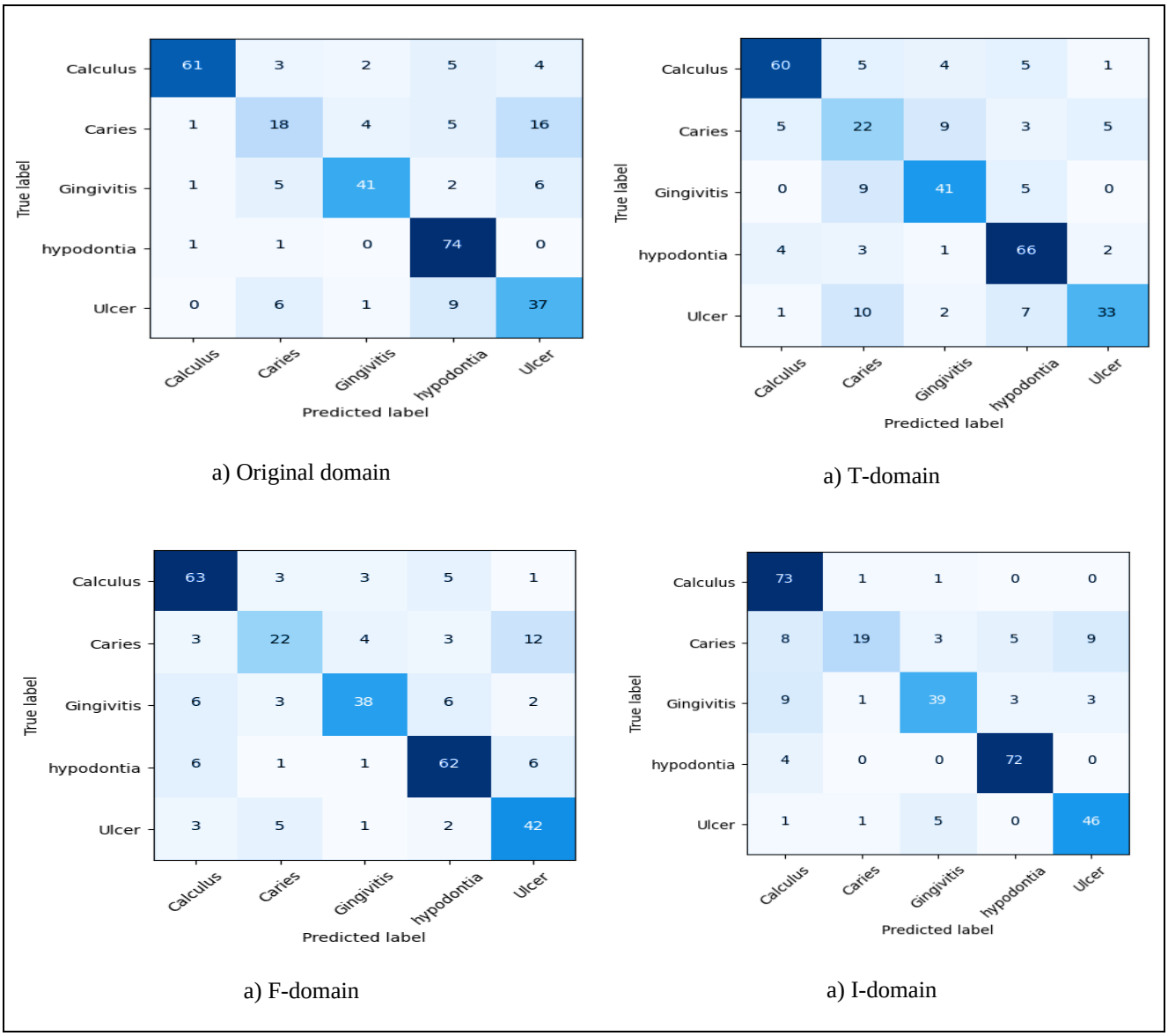


Fig. 3 classification accuracy attained by various models for four domains (O, T, I and F)



a) Original domain

a) T-domain

a) F-domain

a) I-domain

Fig. 4 Confusion matrices attained by the ET classifier as the best-performing algorithm under various domains

5. Conclusions and uture work

This study aims to observe how the NS domain affects the performance of several ML models, including the RF, ET, KNN, GaussianNB and DT, when used to detect oral diseases. Any image is split into three membership components by the NS domain: falsity (F), indeterminacy (I) and truth (T). In this study, the images are separated into three datasets (T, I and F) using this domain. One by one, those three datasets—along with the original dataset—are separated into training and testing datasets using an 80:20% splitting ratio. Following training on the training dataset, the five ML models are evaluated on the testing dataset to demonstrate their capacity for generalization using four performance metrics, including recall, precision, accuracy and F1-score. The experimental results indicate that the I-domain might significantly improve the performance of four of the ML models when compared to the other domains. This suggests that the NS domain may help the ML models perform better in accurately

classifying oral diseases. Of all the studied models, the best classification accuracy is achieved by the RF classifier with the I-domain, suggesting that it is a good substitute for classifying oral diseases.

In future work, we will investigate the impact of the NS domain on ML models for a number of additional image classification tasks, such as the classification of breast cancer and stroke disease. Furthermore, using larger datasets, we will assess how well the proposed ML models perform.

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Mappings Of Plithogenic Cubic Sets

S.P.Priyadharshini [1*], J.Ramkumar [2]

[1] Assistant Professor in Mathematics, Dr.N.G.P. Institute of Technology, Coimbatore, India.priyadharshini125@gmail.com

[2] Assistant Professor in Information Technology and Cognitive Systems; Sri Krishna Arts and Science College, India.

jramkumar1986@gmail.com;

* Correspondence: priyadharshini125@gmail.com

Abstract:

Objectives:

To introduce novel mappings for plithogenic fuzzy cubic structures and derive core properties of stable and almost stable plithogenic neutrosophic cubic mappings.

Methods:

This study employs a theoretical approach to develop mappings for plithogenic cubic sets. We conducted a comprehensive analysis of stable and almost stable mappings, utilizing comparative techniques to assess their properties. Unique modifications include the introduction of specific stability criteria and a detailed examination of their implications in decision-making contexts.

Findings:

Our findings reveal that the newly introduced mappings exhibit distinct stability characteristics, enhancing the theoretical framework of Plithogenic cubic mapping. These mappings not only align with previous reports but also provide novel insights into their applications in optimization and data analysis. The advancements made in this study contribute valuable knowledge to the field, addressing gaps in literature and offering new avenues for future research.

Novelty:

This study presents innovative mappings and stability properties, advancing the theoretical understanding of Plithogenic cubic sets.

Keywords: Plithogenic cubic sets, Mappings, Stable mappings, Almost stable mappings, Decision-making.

1. Introduction

The Plithogenic Set (PS), a generalized extension of neutrosophic sets, was first proposed by Smarandache [10] in 2017. One or more characteristics each with a wide range of potential values, determine the components of a PS. There are appurtenance degrees linked to a certain element (like the PS) for every attribute value. Smarandache additionally improved the accuracy of plithogenic aggregation techniques by introducing for the first time the idea of a variation degree between each attribute value and the most dominating attribute value. Later, cubic sets and stable cubic sets—which blend fuzzy sets with interval-valued fuzzy sets—were created by Y.B. Jun et al. [2,4]. Additionally, they investigated and examined some of the features of internal as well as external cubic sets.

In addition to the notions of internal as well as external truth, falsehood, and indeterminacy values, Y.B. Jun, Smarandache, and Kim [3] developed neutrosophic cubic sets. Additionally, for both internal and external neutrosophic cubic sets, they suggested a variety of P and R order operations.

Recently, Priyadharshini et al. proposed the concept of plithogenic cubic sets [5, 6]. Examples are given to discuss the related internal and external cubic sets after a summary of the concepts of the Plithogenic fuzzy cubic set, Plithogenic intuitionistic fuzzy cubic set, and Plithogenic neutrosophic cubic set. The main features of the Plithogenic neutrosophic cubic sets were also studied, along with their uses.

The concepts of P and R order of Plithogenic neutrosophic cubic sets were studied by Priyadharshini et al. [7, 8]. The aforementioned principle plays significant importance in the analysis of problems involving multi-attribute decision making as Plithogenic neutrosophic cubic sets are characterized by a multitude of attribute values and very accurate data consistency.

Stable plithogenic cubic sets, which define stable, unstable, stable cut, and unstable cut, were also presented by Priyadharshini et al. [9]. Stable degrees were specified for plithogenic cubic sets. The fundamental characteristics of almost stable plithogenic cubic sets under P and R orders were also studied.

Cubic mappings, which are a type of mapping between cubic sets, have been deliberated by many researchers. However, the study of cubic mappings is still in its early stages and many questions remain unanswered.

The literature review highlights the evolution of uncertainty modeling from fuzzy and intuitionistic fuzzy sets to more advanced frameworks like cubic sets and plithogenic sets. Fuzzy sets introduced the concept of partial truth, while intuitionistic fuzzy sets added non-membership degrees. Cubic sets combined fuzzy and interval-valued fuzzy sets, providing a more comprehensive approach to ambiguity. Neutrosophic cubic sets further incorporated truth, falsity, and indeterminacy

components, expanding analytical capabilities. Plithogenic cubic sets enhanced these frameworks by introducing variation degrees, enabling precise multi-attribute decision-making. Despite these advancements, the study of cubic mappings and their integration with plithogenic principles remains underdeveloped. Practical applications, such as supply chain risk management, showcase the potential of plithogenic cubic sets in addressing complex, real-world problems through accurate modeling and analysis of multi-dimensional uncertainties.

Motivation and Novelties

The study of cubic mappings is motivated by the need to develop more advanced mathematical tools for dealing with multipart and ambiguous systems. Traditional fuzzy sets and intuitionistic fuzzy sets have limitations in handling complex systems, and cubic mappings offer a more general and flexible framework for modeling and analyzing such systems.

Despite the recent developments in cubic mappings, there are still several research gaps that need to be addressed. One of the main gaps is the lack of a comprehensive study on the properties and behavior of cubic mappings. Another research gap is the need for more advanced mathematical tools for analyzing and processing cubic mappings.

The objective of this work is to fill in these research gaps by introducing new concepts and developments in cubic mappings. Specifically, we will study the properties and behavior of cubic mappings, and develop new mathematical tools for analyzing and processing these mappings.

Research Gaps

The paper aims to address the following gaps:

i. A lack of in-depth studies focusing on the properties, structures, and behaviors of cubic mappings.

ii. The absence of advanced mathematical tools to analyze and process cubic mappings, especially in complex or uncertain systems.

iii. Limited research on the interrelationships between cubic mappings and other mapping types, such as fuzzy mappings and intuitionistic fuzzy mappings.

These gaps highlight the need for further exploration to enhance both the theoretical understanding and practical applications of cubic mappings.

Novelties

The key novelties of the paper include:

i. The introduction of new concepts related to plithogenic cubic mappings.

ii. An analytical study of the unique properties and behaviors of plithogenic cubic mappings.

iii. The development of mathematical methods specifically designed for the analysis and processing of plithogenic cubic mappings.

These contributions aim to provide a foundation for advancing the theory and applications of Plithogenic cubic mappings in addressing complex mathematical and real-world problems.

The comparison study highlights that the incorporation of variation degrees in plithogenic sets and mappings significantly enhances their flexibility and accuracy, making them more effective for multi-attribute decision-making. Cubic mappings, as an extension of cubic sets, provide new avenues for exploring relationships and transformations between sets, with plithogenic cubic mappings offering greater precision and adaptability due to their unique structural characteristics. The proposed analytical tools address existing limitations by enabling more accurate analysis of variation-based mappings, particularly in complex and uncertain systems. These advancements demonstrate the applicability of plithogenic cubic mappings across diverse domains in multi-criteria decision-making, underscoring their potential for solving real-world problems involving complex attribute relationships.

2. Preliminaries

Definition 2.1 **[9]** Let $\Gamma =< X, \mu >$ be a Plithogenic fuzzy cubic sets in A, then the evaluation set of $\Gamma =< X, \mu >$ is specified to be a structure $\Xi_\Gamma = \{(a, E_\Gamma(a))|a \in A\}$ where $\Xi_\Gamma(a) = \langle L(E_\Gamma(a)), R(E_\Gamma(a))\rangle$ with $L(E_\Gamma(a)) = \mu(a) - X(a)^-$ and $R(E_\Gamma(a)) = X(a)^+ - \mu(a)$ which are called the left evaluation point and the right evaluation point respectively of $\Gamma =< X, \mu >$ at $a \in A$. We say that $\Xi_\Gamma(a)$ is the evaluation point of $\Gamma =< X, \mu >$ at $a \in A$.

Definition 2.2 [9] Let $\Gamma =< X, \mu >$ be a Plithogenic fuzzy cubic sets in A with the evaluation set $\Xi_\Gamma = \{(a, E_\Gamma(a))|a \in A\}$. An element $a \in A$ is called a stable element of $\Gamma =< X, \mu >$ in A if it satisfies $L(E_\Gamma(a)) = \mu(a) - X(a)^- \geq 0$ and $R(E_\Gamma(a)) = X(a)^+ - \mu(a) \geq 0$. Otherwise we say that 'a' is an unstable element of $\Gamma =< X, \mu >$ in A.

The set of all stable elements of $\Gamma =< X, \mu >$in A is called the stable cut of $\Gamma =< X, \mu >$in A and is denoted by SC_Γ.The set of all unstable elements of $\Gamma =< X, \mu >$in A is called the unstable cut of $\Gamma =< X, \mu >$in A and is denoted by USC_Γ. We say that $\Gamma =< X, \mu >$ is a stable Plithogenic fuzzy cubic set if $SC_\Gamma = A$. Otherwise $\Gamma =< X, \mu >$ is called an unstable Plithogenic fuzzy cubic set.

Definition 2.3 [9] Let $\Psi =< Y, \delta >$be a Plithogenic intuitionistic fuzzy cubic sets in A, then the evaluation set of $\Psi =< Y, \delta >$ is specified to be a structure $\Xi_\Psi = \{(a, E_\Psi(a))|a \in A\}$ where $\Xi_\Psi(a) = \langle L(E_\Psi(a)), R(E_\Psi(a))\rangle$ with $L(E_\Psi(a)) = \delta(a) - Y(a)^-$ and $R(E_\Psi(a)) = Y(a)^+ - \delta(a)$ which are called the left evaluation point and the right evaluation point respectively of $\Psi =< Y, \delta >$at $a \in A$. We say that $\Xi_\Psi(a)$ is the evaluation point of $\Psi =< Y, \delta >$at $a \in A$.

Definition 2.4 [9] Let $\Psi =< Y, \delta >$be a Plithogenic intuitionistic fuzzy cubic sets in A with the evaluation set $\Xi_\Psi = \{(a, E_\Psi(a))|a \in A\}$. An element $a \in A$ is called a stable element of $\Psi =< Y, \delta >$ in A if it satisfies $L(E_C(a)) = \delta(a) - Y(a)^- \geq 0$ and $R(E_\Psi(a)) = Y(a)^+ - \delta(a) \geq 0$. Otherwise we say that it is an unstable element of $\Psi =< Y, \delta >$in A.

The set of all stable elements of $\Psi =< Y, \delta >$in A is called the stable cut of $\Psi =< Y, \delta >$in A and is denoted by SC_Ψ.The set of all unstable elements of $\Psi =< Y, \delta >$in A is called the unstable cut of $\Psi =< Y, \delta >$in A and is denoted by USC_Ψ. We say that $\Psi =< Y, \delta >$ is a stable plithogenic intuitionistic fuzzy cubic set if $SC_\Psi = A$. Otherwise $\Psi =< Y, \delta >$ is called an unstable plithogenic intuitionistic fuzzy cubic set.

Definition 2.5 [9] Let $\Lambda =< Z, \lambda >$be a Plithogenic neutrosophic cubic sets in A, then the evaluation set of $\Lambda =< Z, \lambda >$ is specified to be a structure $\Xi_\Lambda = \{(a, E_\Lambda(a))|a \in A\}$ where $E_\Lambda(a) = \langle L(E_\Lambda(a)), R(E_\Lambda(a))\rangle$ with $L(E_\Lambda(a)) = \lambda(a) - Z(a)^-$ and $R(E_\Lambda(a)) = Z(a)^+ - \lambda(a)$ which are called the left evaluation point and the right evaluation point respectively of $\Lambda =< Z, \lambda >$at $a \in A$.

We say that $\Xi_\Lambda(a)$ is the evaluation point of $\Lambda =< Z, \lambda >$at $a \in A$.

Definition 2.6 [9] Let $\Lambda =< Z, \lambda >$be a Plithogenic neutrosophic cubic sets in A with the evaluation set $\Xi_\Lambda = \{(a, E_\Lambda(a))|a \in A\}$. An element $a \in A$ is called a stable element of $\Lambda =< Z, \lambda >$ in A if it satisfies $L(E_\Lambda(a)) = \lambda(a) - Z(a)^- \geq 0$ and $R(E_\Lambda(a)) = Z(a)^+ - \lambda(a) \geq 0$. Otherwise, we refer to unstable element of $\Lambda =< Z, \lambda >$in A.

The collection of all stable elements of $\Lambda =< Z, \lambda >$in A is named the stable cut of $\Lambda =< Z, \lambda >$in A and is denoted by SC_Λ.The set of all unstable elements of $\Lambda =< Z, \lambda >$in A is termed the unstable cut of $\Lambda =< Z, \lambda >$in A and is represented by USC_Λ. We say that $\Lambda =< Z, \lambda >$ is a stable Plithogenic neutrosophic cubic set if $SC_\Lambda = A$. Else $\Lambda =< Z, \lambda >$ is called an unstable Plithogenic neutrosophic cubic set.

3. Mappings of Plithogenic cubic set

Definition 3.1 Let $p : H \to N$ be a mapping and let $C =< K, \Phi >$ be a Plithogenic fuzzy Cubic set in H. then the image of $C =< K, \Phi >$under p is represented by $p(C) = \langle p(K), p(\Phi)\rangle$and is specified by $(\forall h \in H)(p(C)(h) = \langle p(K)(h), p(\Phi)(h)\rangle)$.

Definition 3.2 Let $p : H \to N$ be a mapping and let $D =< M, \Omega >$ be a Plithogenic fuzzy Cubic set in H. Then the image of inverse $D =< M, \Omega >$ under p is denoted by

$p^{-1}(D) = \langle p^{-1}(M), p(\Omega) \rangle$ and is specified by $(\forall h \in H)(p^{-1}(C)(h) = \langle p^{-1}(M)(h), p(\Omega)(h) \rangle)$

Example 3.3

Let $C =< K, \Phi >$ be a Plithogenic fuzzy Cubic set in N. For a set

Measures of contradiction	0	0.3	0.7	1
Value of traits	Frictional	Structural	Cyclical	Seasonal
Appurtenance Measure K(h)	[0.5,0.6]	[0.8,0.9]	[0.2,0.8]	[0.1,0.3]
$\Phi(h)$	0.2	0.5	0.1	0.6

Let g be a function given as $p{:}H \to N,\ h \mapsto \begin{cases} c & if \quad h = a \\ d & if \quad h = b \\ a & if \quad h = c \\ b & if \quad h = d \end{cases}$

Measures of contradiction	0	0.3	0.7	1
Value of traits	Frictional	Structural	Cyclical	Seasonal
Appurtenance Measure $p^{-1}(K(h))$	[0.2,0.8]	[0.1,0.3]	[0.5,0.6]	[0.8,0.9]
$p^{-1}(\Phi(h))$	0.1	0.6	0.2	0.5

Definition 3.4 Let $p : H \to N$ be a mapping and let $C =< K, \Phi >$ be a Plithogenic Intuitionistic fuzzy Cubic set in H. then the image of $C =< K, \Phi >$ under g is denoted by $p(C) = \langle p(K), p(\Phi) \rangle$ and is specified by $(\forall h \in H)(p(C)(h) = \langle p(K)(h), p(\Phi)(h) \rangle)$.

Definition 3.5 Let $g : H \to N$ be a mapping and let $C =< \mathrm{K}, \Phi >$ be a Plithogenic Intuitionistic fuzzy Cubic set in H. then the image of $C =< \mathrm{K}, \Phi >$ under g is denoted by $g(C) = \langle g(K), g(\Phi) \rangle$ and is specified by $(\forall h \in H)(g(C)(h) = \langle g(K)(h), g(\Phi)(h) \rangle)$.

Example 3.6

Let $C =< K, \Phi >$ be a Plithogenic Intuitionistic fuzzy Cubic set in N. For a set

Measures of contradiction	0	0.3	0.7	1
Value of traits	Frictional	Structural	Cyclical	Seasonal
Appurtenance Measure K(h)	([0.3, 0.5], [0.7, 0.8])	([0.2, 0.9], [0.1, 0.2])	([0.5, 0.7], [0.6, 0.8])	([0.1, 0.5], [0.2, 0.8])
$\Phi(h)$	([0.1,0.4])	([0.4,0.1])	([0.6, 0.1])	([0.2, 0.7])

Let g be a function given as $g: H \to N,\ h \mapsto \begin{cases} b & if \quad h = a \\ c & if \quad h = b \\ d & if \quad h = c \\ a & if \quad h = d \end{cases}$

Measures of contradiction	0	0.3	0.7	1
Value of traits	Frictional	Structural	Cyclical	Seasonal
Appurtenance Measure $g^{-1}(K(h))$	([0.2, 0.9], [0.1, 0.2])	([0.5, 0.7], [0.6, 0.8])	([0.1, 0.5], [0.2, 0.8])	([0.3, 0.5], [0.7, 0.8])
$g^{-1}(\Phi(h))$	([0.4,0.1])	([0.6, 0.1])	([0.2, 0.7])	([0.1,0.4])

Definition 3.7 Let $g : H \to N$ be a mapping and let $C =< K, \Phi >$ be a Plithogenic Neutrosophic Cubic set in H. then the image of $C =< K, \Phi >$under g is denoted by $g(C) = \langle g(K), g(\Phi)\rangle$and is specified by $(\forall h \in H)(g(C)(h) = \langle g(K)(h), g(\Phi)(h)\rangle)$.

Definition 3.8 Let $g : H \to N$ be a mapping and let $D =< M, \Omega >$ be a Plithogenic Neutrosophic Cubic set in H. Then the image of inverse $D =< M, \Omega >$ under $g$is denoted by

$g^{-1}(D) = \langle g^{-1}(M), g(\Omega)\rangle$ and is specified by $(\forall h \in H)(g^{-1}(C)(h) = \langle g^{-1}(M)(h), g(\Omega)(h)\rangle)$

Example 3.9

Let $C =< K, \Phi >$ be a Plithogenic Neutrosophic Cubic set in N.For a set

Measures of contradiction	0	0.3	0.7	1
Value of traits	Frictional	Structural	Cyclical	Seasonal
Appurtenance Measure K(h)	([0.5, 0.1], [0.3, 0.6], [0.4, 0.8])	([0.3, 0.5], [0.2, 0.8], [0.1, 0.4])	([0.1, 0.4], [0.2, 0.3], [0.6, 0.8])	([0.3, 0.5], [0.1,0.5], [0.6, 0.8])
$\Phi(h)$	([0.1,0.2, 0.4])	([0.4,0.6, 0.7])	([0.5,0.6,0.1])	([0.2,0.5,0.3])

Let g be a function given as $g: H \to N,\ h \mapsto \begin{cases} d & if \quad h = a \\ a & if \quad h = b \\ b & if \quad h = c \\ c & if \quad h = d \end{cases}$

Measures of contradiction	0	0.3	0.7	1
Value of traits	Frictional	Structural	Cyclical	Seasonal
Appurtenance Measure	([0.3, 0.5], [0.1,0.5],	([0.5, 0.1], [0.3, 0.6],	([0.3, 0.5], [0.2, 0.8],	([0.1, 0.4], [0.2, 0.3],

$g^{-1}(K(h))$	[0.6, 0.8])	[0.4, 0.8])	[0.1, 0.4])	[0.6, 0.8])
$g^{-1}(\Phi(h))$	([0.2,0.5,0.3])	([0.1,0.2, 0.4])	([0.4,0.6, 0.7])	([0.5,0.6,0.1])

Theorem 3.10 Let $g: H \to N$ be a mapping. If $C =< K, \Phi >$ is a stable Plithogenic Neutrosophic cubic set in H, then $g(C) = \langle g(K), g(\Phi) \rangle$ is a stable Plithogenic Neutrosophic cubic set in N.

Proof: Let $C =< K, \Phi >$ be a Plithogenic neutrosophic cubic sets in H. Then $H = S_C = \{h \in H | L(\Xi_C(H)) \geq 0,\ R(\Xi_C(H)) \geq 0\}$.Hence $\Phi(h) - K(h)^- \geq 0$ and $K(h)^+ - \Phi(h) \geq 0$ for all $h \in H$.

For each $n \in N,\ if\quad g^{-1}(n) \neq \phi$, then

$$g(\Phi)(n) = \underset{n=g(h)}{Sup}\ \Phi(h) \geq \underset{n=g(h)}{Sup}\ (\Phi(h))^- = g(K)(n)^-$$

And

$$g(K)(n) = \underset{n=g(h)}{Sup}\ (K(h))^+ \geq \underset{n=g(h)}{Sup}\ \Phi(h) = g(\Phi)(n).$$

Obviously $g(\Phi)(n) \geq g(K)(n)^-$ and $g(K)(n)^+ \geq g(\Phi)(n)$ if $g^{-1}(n) \neq \phi$.This shows that $S_{g(C)} = N,$ and therefore $g(C) = \langle g(K), g(\Phi) \rangle$ is a stable Plithogenic Neutrosophic cubic set in N.

Theorem 3.11 Let $g: H \to N$ be a mapping. If $C =< K, \Phi >$ is a stable Plithogenic Neutrosophic cubic set in H, then the negation of $g(C) = \langle g(K), g(\Phi) \rangle$ is a stable Plithogenic Neutrosophic cubic set in N.

Theorem 3.12 Let $g : H \to N$ be a mapping. If $C =< K, \Phi >$and $D =< M, \Omega >$ are stable Plithogenic Neutrosophic cubic set in H, then the P-order of $g(C) = \langle g(K), g(\Phi)\rangle$and $g(D) = \langle g(M), g(\Omega)\rangle$ is a stable Plithogenic Neutrosophic cubic set in N.

Theorem 3.13 Let $g : H \to N$ be a mapping. If $C =< K, \Phi >$and $D =< M, \Omega >$ are stable Plithogenic Neutrosophic cubic set in H, then $g(C \cup D)$ and $g(C \cap D)$ are stable Plithogenic Neutrosophic cubic set in N.

Theorem 3.14 Let $g : H \to N$ be an 1-1 mapping. If $C =< K, \Phi >$is an almost stable Plithogenic Neutrosophic cubic set in H, then $g(C) = \langle g(K), g(\Phi)\rangle$is an almost stable Plithogenic Neutrosophic cubic set in N.

Proof Let $C =< K, \Phi >$be an almost stable Plithogenic Neutrosophic cubic set in H, then there exists a stable degree SDG_C such that

$$\sum_{h\in H} L(\Xi_C)(h) = \sum_{h\in H} (\Phi(h) - K(h)^-) \geq 0,$$

and

$$\sum_{h\in H} R(\Xi_C)(h) = \sum_{h\in H} \left(K(h)^+ - \Phi(h)\right) \geq 0.$$

We have to prove that $\sum_{h\in H} L(\Xi_C)(h) \geq 0$ and $\sum_{h\in H} R(\Xi_C)(h) \geq 0$ in the stable degree $SDG_{g(C)}$

For every $n \in N$, Let $N^* = \{n \in N | g^{-1}(n) \neq \varphi\}$ and $N^{**} = N \backslash N^*$. Then N is the disjoint union of N and N^* and N^{**}.If $n \in N^{**}$, then $g(K)(h) = 0$ and $g(\Phi)(h) = 0$ which imply that

$$\sum_{n\in N^{**}} \left(g(K)(n)^+ - g(\Phi)(n)\right) = 0 \; and \; \sum_{n\in N^{**}} (g(\Phi)(n) - g(K)(n)^-) = 0.$$

Since g is 1-1, $g^{-1}(n)$ is a singleton set for all $n \in N^*$ and $\{g^{-1}(n) | n \in N^*\} = H$.

Therefore $\sum_{h\in H} L(\Xi_C)(h) = \sum_{n\in N}(g(\Phi)(n) - g(K)(n)^-)$

$$= \sum_{n\in N^*} (g(\Phi)(n) - g(K)(n)^-) + \sum_{n\in N^{**}} (g(\Phi)(n) - g(K)(n)^-)$$

$$= \sum_{n\in N^*} (g(\Phi)(n) - g(K)(n)^-)$$

$$= \sum_{h\in H} (\Phi(h) - K(h)^-) \geq 0$$

And $\sum_{h\in H} R(\Xi_{g(C)})(h) = \sum_{n\in N}(g(K)(n)^+ - g(\Phi)(n))$

$$= \sum_{n\in N^*} (g(K)(n)^+ - g(\Phi)(n)) + \sum_{n\in N^{**}} (g(K)(n)^+ - g(\Phi)(n))$$

$$= \sum_{n\in N^*} (g(K)(n)^+ - g(\Phi)(n))$$

$$= \sum_{h\in H} (K(h)^+ - \Phi(h)) \geq 0$$

Hence $g(C) = \langle g(K), g(\Phi)\rangle$is an almost stable Plithogenic Neutrosophic cubic set in N.

Theorem 3.15 Let $g : H \to N$ be an 1-1 mapping. If $C =< K, \Phi >$is an almost stable Plithogenic Neutrosophic cubic set in H, then the negation of $g(C) = \langle g(K), g(\Phi)\rangle$is an almost stable Plithogenic Neutrosophic cubic set in N.

Theorem 3.16 Let $g : H \to N$ be a mapping. If $D =< M, \Omega >$is a stable Plithogenic Neutrosophic cubic set in N, then $g^{-1}(D) = \langle g^{-1}(M), g^{-1}(\Omega)\rangle$ is a stable Plithogenic Neutrosophic cubic set in H.

Proof: Let $D =< M, \Omega >$be a stable Plithogenic Neutrosophic cubic set in N. Then

$S_D = \{n \in N | L(\Xi_D(n)) \geq 0,\ R(\Xi_D(n)) \geq 0\} = N.$

Therefore $\Omega(n) - M(n)^- \geq 0$ and $M(n)^+ - \Omega(n) \geq 0$ for all $n \in N$.

It follows that

$$L\left(\Xi_{g^{-1}(D)}(h)\right) = g^{-1}(\Omega)(h) - g^{-1}(M)(h)^{-} = \Omega\left(g(h)\right) - M\left(g(h)\right)^{-} \geq 0$$

And

$$R\left(\Xi_{g^{-1}(D)}(h)\right) = g^{-1}(M)(h)^{+} - g^{-1}(\Omega)(h) = M\left(g(h)\right)^{+} - M\left(\Omega(h)\right) \geq 0 \text{ for all } h \in H.$$

Thus

$$S_{g^{-1}(D)} = \left\{n \in N | L\left(\Xi_{g^{-1}(D)}(n)\right) \geq 0,\ R\left(\Xi_{g^{-1}(D)}(n)\right) \geq 0\right\} = N.$$

Which implies $g^{-1}(D) = \langle g^{-1}(M), g^{-1}(\Omega)\rangle$ is a stable Plithogenic Neutrosophic cubic set in H.

Corollary 3.17 Let $g : H \to N$ be a mapping. If $C =< K, \Phi >$is a stable Plithogenic Neutrosophic cubic set in N, then $g^{-1}(D)^{c}$ is a stable Plithogenic Neutrosophic cubic set in H.

Corollary 3.18 Let $g : H \to N$ be a mapping. If $C =< K, \Phi >$and $D =< M, \Omega >$ are stable Plithogenic Neutrosophic cubic set in N, then the P-union and the P-intersection of $g^{-1}(C)$and $g^{-1}(D)$ are stable Plithogenic Neutrosophic cubic set in H

Corollary 3.19 Let $g : H \to N$ be a mapping. If $C =< K, \Phi >$and $D =< M, \Omega >$ are stable Plithogenic Neutrosophic cubic set in N, then $g^{-1}(C \cup D)$ and $g^{-1}(C \cap D)$ are stable Plithogenic Neutrosophic cubic set in H.

Corollary 3.20 Let $g : H \to N$ be a mapping. If $C =< K, \Phi >$is a stable Plithogenic Neutrosophic cubic set in H, then $g^{-1}\left(g(C)\right)$ is a stable Plithogenic Neutrosophic cubic set in H.

Corollary 3.21 Let $g : H \to N$ be a mapping. If $D =< M, \Omega >$is a stable Plithogenic Neutrosophic cubic set in H, then $g^{-1}\left(g(D)\right)$ is a stable Plithogenic Neutrosophic cubic set in H.

4. Application

Let T represents a set of environmental factors related to climate change, such as temperature (t_1), precipitation (t_2), humidity (t_3), cloudiness(t_4) and carbon emission(t_5).

Define a mapping $g : H \to N$.If $C =< K, \Phi >$is a stable Plithogenic Neutrosophic cubic set in H. Here K(h) denotes the interval valued neutrosophic set and $\Phi(h)$ denotes the neutrosophic set.

Measures of contradiction	0	0.3	0.5	0.7	1
Value of traits	temperature	precipitation	humidity	cloudiness	carbon emission
Appurtenance Measure K(h)	([0.6, 0.7], [0.3, 0.7], [0.1, 0.5])	([0.4, 0.6], [0.1, 0.4], [0.2, 0.5])	([0.3, 0.5], [0.2, 0.3], [0.7, 0.8])	([0.4, 0.7], [0.1,0.3], [0.6, 0.9])	([0.1, 0.5], [0.4,0.6], [0.7, 0.9])
$\Phi(h)$	([0.1,0.3, 0.4])	([0.3,0.2, 0.3])	([0.4,0.2,0.2])	([0.2,0.1,0.3])	([0.2,0.5,0.8])

The image of inverse of plithogenic cubic mapping is crucial for reversing cause-effect relationships, setting thresholds validating models which is calculated below. In this application, the analytical tool transforms complex relationships into actionable in sights, driving more effective interventions and strategies

Measures of contradiction	0	0.3	0.5	0.7	1
Value of traits	temperature	precipitation	humidity	cloudiness	carbon emission
Appurtenance Measure $g^{-1}(K(h))$	([0.3, 0.5], [0.2, 0.3], [0.7, 0.8])	([0.4, 0.7], [0.1,0.3], [0.6, 0.9])	([0.1, 0.5], [0.4,0.6], [0.7, 0.9])	([0.6, 0.7], [0.3, 0.7], [0.1, 0.5])	([0.4, 0.6], [0.1, 0.4], [0.2, 0.5])
$g^{-1}(\Phi(h))$	([0.4,0.2,0.2])	([0.2,0.1,0.3])	([0.2,0.5,0.8])	([0.1,0.3, 0.4])	([0.3,0.2, 0.3])

Inferences:

(i) Attributes like temperature and cloudiness have higher truth memberships in their original mapping, indicating stronger, more direct effects on the environmental system.

(ii) High Measures of contradictions in humidity and carbon emissions suggest significant uncertainty or variability in their impact.

(iii) The inverse mapping shows decreased truth memberships for most attributes, indicating weaker backward contributions.

(iv) The increase in Measures of contradictions in attributes like temperature and humidity reflects more uncertainty when tracing causes from observed effects.

5. Conclusion

In this article, we have examined the process of determining the inverse image for plithogenic cubic sets. We have also presented the essential features of stable and almost stable plithogenic cubic maps. Our study has shown that plithogenic cubic sets and their mappings can be used to model and analyze complex systems in a more general and flexible way.

6.Future Research

By exploring these areas of future research, we hope to further advance the state of the art in plithogenic cubic sets and their mappings and contribute to the growth of novel mathematical tools for dealing with complex and uncertain classifications.

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Application of the Plithogenic-TODIM Approach for Teaching Quality Evaluation in Vocational Colleges under the Background of New Era Education

Qiongyao Liu[1], Wenqiang Dai[2], Wei Wang[3], Yuemin Gao[4*]
[1]Basic Teaching Department, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
[2]Department of Architecture and Design, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
[3]Art Department, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
[4]Basic Teaching Department, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
*Corresponding author E-mail: gaoyuemin_2004@163.com

Abstract

The goal of vocational education reform is to improve quality, and one of its key tasks is to create a system for evaluating quality that is tailored to the requirements of contemporary vocational education. International research on models for evaluating the quality of vocational education has not yet been thoroughly examined, nevertheless. This study proposes a multi-criteria decision making (MCDM) methodology to evaluate the teaching quality in vocational colleges. We use the TODIM methodology under the plithogenic sets to deal with uncertainty information. The TODIM methodology is used to rank the alternatives. Ten criteria and ten alternatives are used in this study. Three experts evaluated criteria and alternatives using the plithogenic numbers. Then we used the plithogenic operator to combine these numbers. The results show the Curriculum Relevance and Industry Alignment criterion has the highest importance and Institutional Reputation and Partnerships criterion has the lowest weights. The sensitivity analysis is performed to show the different ranks of alternatives and robustness of the proposed method. The results show the proposed method is strong under different criteria weights.

Keywords: Teaching quality Evaluation; Vocational Colleges; Uncertainty; TODIM Method; MCDM.

1. Introduction

Improving quality is a major global concern for vocational education. Another crucial strategic issue for China's vocational education reform is creating a quality evaluation model and system to satisfy the development requirements of the country's contemporary system. Vocational education is becoming more and more popular worldwide, though the amounts and rates of growth vary by nation and region based on factors like economic development, labor supply and demand fluctuations, and the level of Information[1], [2].

The global vocational education market is projected to grow to 2026 under the macro backdrop of positive global vocational education development, surpassing USD 800 billion, despite the emergence of the global new crown pneumonia epidemic, political unrest in different nations, and severe natural disasters in recent years that have brought certain adverse effects on the world's economy, according to the "Global Vocational Education Industry Development Report 2022" released by Ariadne Consulting in July 2022. China's vocational education system produced 61 million highly qualified workers and technically proficient professionals for a variety of businesses between 2012 and 2022[3], [4].

Due to enrollment pressure and the negative social perception brought about by the Chinese public's lack of trust in vocational education, China's vocational education system needs to be modernized and redesigned with the goal of requiring quality improvement[5], [6]. In general, the growth of high-quality education is linked to the curriculum, professional environments, and the development of abilities. The skills developed via vocational education are given the importance of the times to contribute to industrial output and economic development as China's industrial transformation picks up speed. In contrast to conventional higher education, the quality of vocational education places more emphasis on skill competency and problem-solving abilities than it does on mastery of advanced information[7], [8].

This study integrates the plithogenic set, which was first presented by Samarandache, to solve the limitations of the conventional TODIM technique. A generalization of classical, fuzzy, and intuitionistic fuzzy sets (IFS) is the neutrophilic set. To describe the ambiguous and inconsistent human information, a neutrosophic set is defined by degrees of truth, indeterminacy, and falsity. The neutrosophic theory's degrees of truth, indeterminacy, and falsity are unrelated to one another. While the traditional fuzzy set and IFS can only handle partial or incomplete information, the neutrosophic set can handle inconsistent, indeterminate, and incomplete information[9], [10].

This study ranks the alternatives using the TODIM method. Additionally, this study uses the plithogenic set aggregation operator, which considers the equidistant contradiction degree value between each criterion and the most significant criterion (dominant one), in an effort to improve aggregation accuracy[11], [12]. In 2017, Smarandache created the concept of a plithogenic set, which is derived from plithogeny and is a generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets. An application is chosen to implement the suggested plithogenic based TODIM framework[13], [14].

The following are this study's primary contributions:

- In the context of developing teaching quality, this study adds to the body of knowledge on evaluating of teaching quality by using ten criteria and alternatives to compute the criteria and ranking the alternatives.
- We applied the proposed methodology on application to validate it.
- This study uses a TODIM framework based on plithogenic to calculate the weight criteria and ranking the alternatives.
- The alternatives are ranked in order of significance in the current investigation based on the weight achieved.

2. Research Method

This section shows the plithogenic set and the proposed method under the plithogenic sets.

2.1 Plithogenic set

The creation, conception, growth, and formation of new things from fusions of contradictory and non-contradictory numerous old entities is known as lithogenic set, as defined by Smarandache. Plithogenic sets P, A, V, d, c, and p are made up of several components known as multi-dimensional characteristics.

The value of each attribute (V) has a degree of appurtenance d x; according to certain conditions, (v) of the element x, to the set P. The relationship between each attribute and the dominant (most significant) attribute value is presented by the contradiction (dissimilarity) degree function c (v, D). The contradiction degree function is the key component that improves accuracy for the plithogenic aggregation operators (union, intersection, complement, inclusion, equality)[15], [16].

2.2 TODIM Method

This study shows the TODIM methodology steps[17] .

Build the decision matrix information.

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \quad (1)$$

Normalize the decision matrix

The decision matrix is normalized based on the beneficial and non-beneficial criteria such as:

$$t_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}}; i = 1, \ldots, m; j = 1, \ldots, n \quad (2)$$

$$t_{ij} = \frac{1/x_{ij}}{\sum_{i=1}^{m} 1/x_{ij}}; i = 1, \ldots, m; j = 1, \ldots, n \quad (3)$$

Relative weight

The relative weight is computed as:

$$w_j^- = \frac{w_j}{\max w} \quad (4)$$

The dominance degree

The dominance degree between one alternative and another is computed such as:

$$y_i = y(A_i, A_{i'}) = \sum_{j=1}^{n} r_i(A_i, A_{i'}) \quad (5)$$

Where the r can be computed such as:

$$r_i(A_i, A_{i'}) = \begin{cases} -\frac{1}{u}\sqrt{\frac{\left(\sum_{j=1}^{n} w_j^-\right)|t_{ij} - t_{ji}|}{w_j^-}} & if\ (t_{ij} - t_{ji}) < 0 \\ 0 & if\ (t_{ij} - t_{ji}) = 0 \\ \sqrt{\frac{\left(w_j^-\right)|t_{ij} - t_{ji}|}{\sum_{j=1}^{n} w_j^-}} & if\ (t_{ij} - t_{ji}) > 0 \end{cases} \quad (6)$$

Where u refers to the attenuation factor for the losses.

The overall dominance degree

The overall dominance degree is computed such as:

$$p_i = \frac{r_i - \min r_i}{\max r_i - \min r_i} \quad (7)$$

Rank the alternatives.

The alternatives are ranked based on the highest value in p_i.

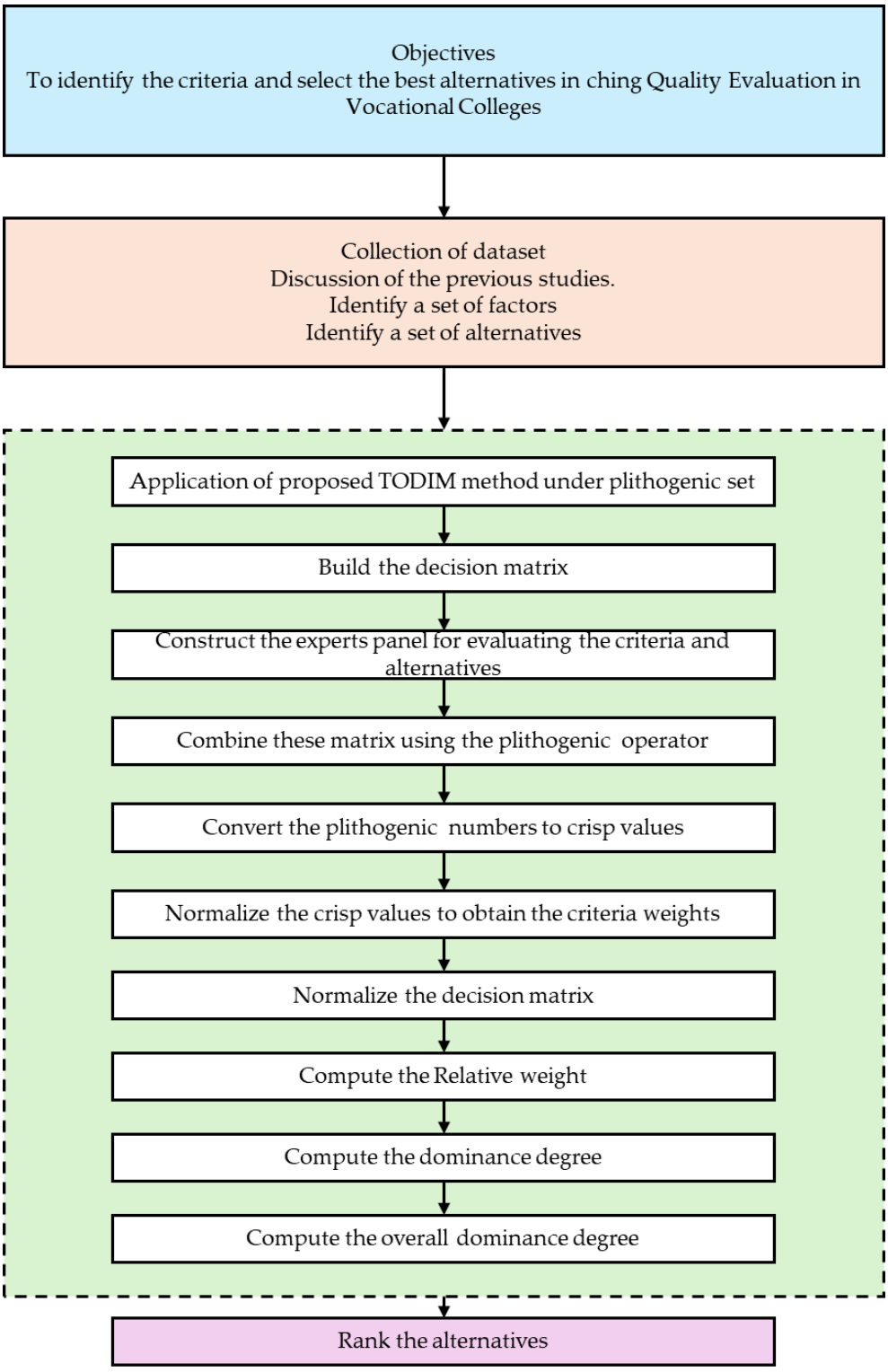
Objectives
To identify the criteria and select the best alternatives in ching Quality Evaluation in Vocational Colleges
Collection of dataset
Discussion of the previous studies.
Identify a set of factors
Identify a set of alternatives
Application of proposed TODIM method under plithogenic set
Build the decision matrix
Construct the experts panel for evaluating the criteria and alternatives
Combine these matrix using the plithogenic operator
Convert the plithogenic numbers to crisp values
Normalize the crisp values to obtain the criteria weights
Normalize the decision matrix
Compute the Relative weight
Compute the dominance degree
Compute the overall dominance degree
Rank the alternatives

Figure 1. The roadmap of the research.

2.3 Research framework

This research can analyze the factors for evaluating the Teaching Quality Evaluation in Vocational Colleges. Figure 1 presents the brief roadmap of the present work. The step by step is developed to obtain the goals of the research work is discussed below:

- Identify the factors of Teaching Quality Evaluation in Vocational Colleges thorough set of previous studies.
- Select the best alternative based on the decision-making panel to apply the proposed TODIM methodology under uncertainty.
- Discuss the factors of the Teaching Quality Evaluation in Vocational Colleges with the decision makers and experts to obtain the highest importance.
- Conducted a decision matrix between the criteria and alternatives based on opinions of a set of experts.
- Combine the opinions of the expert's matrix using the plithogenic aggregation operator to obtain a single matrix.
- Compute the crisp values and then compute the criteria weights using the normalization of crisp values.
- Then apply the TODIM method under the plithogenic sets to rank the alternatives and select the best one.

3. Results and Discussion

The proposed plithogenic-TODIM method is applied to case study to rank a set of alternatives.

3.1 Application of plithogenic based TODIM framework

In this part, the proposed methodology is applied to compute the criteria weights and rank the alternatives. the application and the results obtained are as follows:

Stage 1: Validating of criteria and alternatives.

In this stage, the set of criteria and alternatives are identified based on previous studies analysis and opinions of three experts and decision makers. The decision makers panel was also authorized.

Then with the assistance of the decision makers panel, the ten criteria and ten alternatives are collected in this study. Then the relationship between the criteria and alternatives is built to show the importance of the criteria by the decision makers.

Stage 2: Computing the criteria weights using the plithogenic sets.

In this stage, the decision makers panel provides an evaluation of the criteria and alternatives in the decision matrix. Table 1 shows the information about the decision maker's panel.

Table 1. Demographic profile of expert's panel.

	Positing in firms	*Academic degree*	*Experience*
DM1	Senior	M.Sc.	20
DM2	Manger	Ph.D.	25
DM3	Manger	Ph.D.	30

Ten criteria and alternatives are used in this work. Tables 2-4 show the plithogenic numbers in the decision matrix. Then we used the plithogenic operator to aggregate these decision matrices. Then we convert these numbers into crips values. Then we normalize these values to compute the criteria weights. Table 5 shows the criteria weights.

Table 2. The decision matrix 1

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A_2	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_3	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A_4	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_5	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_8	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_9	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)
A_{10}	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)

Table 3. The decision matrix 2

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)
A_2	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)
A_3	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)
A_4	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_5	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)
A_6	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_7	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)
A_8	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A_9	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)
A_{10}	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)

Table 4. The decision matrix 3

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_2	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)
A_3	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_5	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)
A_6	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)
A_7	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_8	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_9	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_{10}	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)

Table 5. Criteria weights.

	Criteria	*Weights*
C_1	Teaching Effectiveness	0.104738
C_2	Resource Availability and Infrastructure	0.099943
C_3	Teacher Qualifications and Professional Development	0.102379
C_4	Student Engagement and Satisfaction	0.094189
C_5	Integration of Technology in Teaching	0.099986
C_6	Support Services and Career Guidance	0.09307
C_7	Institutional Reputation and Partnerships	0.092526
C_8	Student Learning Outcomes	0.100843
C_9	Practical Training and Skill Development	0.104518
C_{10}	Curriculum Relevance and Industry Alignment	0.107808

Stage 3: Ranking the alternatives using the TODIM method under the plithogenic numbers. We starting with a combined decision matrix to normalize the decision matrix between the criteria and alternatives as shown in Table 6. All criteria are beneficial. Then we compute the relative by dividing the criteria weights by the maximum value in criteria weights. Then we compute the dominance degree. Then we compute the overall dominance degree. Then we ranked the alternatives as shown in Figure 2.

Table 6. The normalization matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	0.105892	0.080366	0.088016	0.108566	0.125262	0.159334	0.086808	0.089356	0.082965	0.057619
A_2	0.105892	0.08282	0.139967	0.109669	0.087904	0.084197	0.138315	0.042588	0.086214	0.119232
A_3	0.106054	0.122713	0.084698	0.095669	0.136423	0.056777	0.08887	0.100739	0.070835	0.083584
A_4	0.070687	0.105585	0.088016	0.06595	0.112303	0.059036	0.114048	0.11335	0.097197	0.042061
A_5	0.031266	0.112352	0.088328	0.135872	0.071938	0.153382	0.141203	0.111349	0.122444	0.112393
A_6	0.086033	0.107249	0.103072	0.159614	0.135993	0.077284	0.105929	0.08154	0.070835	0.13113
A_7	0.105148	0.115031	0.106697	0.087301	0.150361	0.138113	0.138925	0.08154	0.133149	0.057619

A_8	0.098509	0.090162	0.112293	0.085275	0.045351	0.146099	0.059383	0.120156	0.107433	0.129087
A_9	0.141059	0.103356	0.088016	0.076365	0.072338	0.059036	0.059383	0.120156	0.098831	0.137042
A_{10}	0.149462	0.080366	0.100896	0.075718	0.062127	0.066743	0.067136	0.139226	0.130096	0.130232

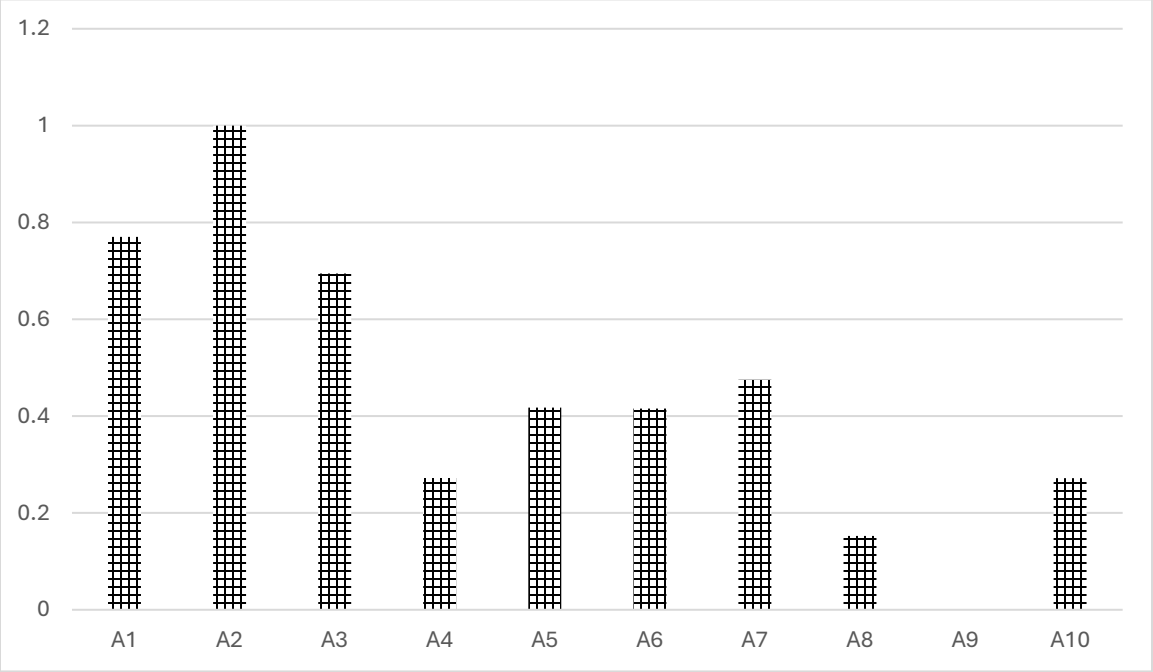

Figure 2. The rank of alternatives.

4. Analysis

In this work, sensitivity analysis of the criteria weights and ranking of alternatives has been performed. There are two reasons to perform this analysis. In the first reason, to show the robust of the ranking of this study. In the second reason, to show the ranking of alternative is sensitive to the criteria weights. The sensitivity analysis is performed by two ways. In the first way, we can change the weights of decision makers and experts. In the second way, we can change the criteria weights.

In this work, we change the criteria weights to show the sensitivity analysis tasks. We change the criteria weights by increasing 5% and decreasing 5% as shown in Table 7. Then we apply the proposed methodology under these changes. We show the rank of alternatives is robust as shown in Table 8.

Table 7. Change in the weights to show ranks.

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*	*Case 6*	*Case 7*	*Case 8*	*Case 9*	*Case 10*
C_1	0.154738	0.104738	0.104738	0.104738	0.104738	0.104738	0.104738	0.104738	0.104738	0.104738
C_2	0.049943	0.149943	0.099943	0.099943	0.099943	0.099943	0.099943	0.099943	0.099943	0.099943
C_3	0.102379	0.052379	0.152379	0.102379	0.102379	0.102379	0.102379	0.102379	0.102379	0.102379
C_4	0.094189	0.094189	0.044189	0.144189	0.094189	0.094189	0.094189	0.094189	0.094189	0.094189
C_5	0.099986	0.099986	0.099986	0.049986	0.149986	0.099986	0.099986	0.099986	0.099986	0.099986
C_6	0.09307	0.09307	0.09307	0.09307	0.04307	0.14307	0.09307	0.09307	0.09307	0.09307
C_7	0.092526	0.092526	0.092526	0.092526	0.092526	0.042526	0.142526	0.092526	0.092526	0.092526
C_8	0.100843	0.100843	0.100843	0.100843	0.100843	0.100843	0.050843	0.150843	0.100843	0.100843
C_9	0.104518	0.104518	0.104518	0.104518	0.104518	0.104518	0.104518	0.054518	0.154518	0.054518
C_{10}	0.107808	0.107808	0.107808	0.107808	0.107808	0.107808	0.107808	0.107808	0.057808	0.157808

Table 8. Rank of alternatives from increasing weights by 5%.

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*	*Case 6*	*Case 7*	*Case 8*	*Case 9*	*Case 10*
A_1	9	9	9	9	9	9	9	9	9	9
A_2	10	10	10	10	10	10	10	10	10	10
A_3	8	8	8	8	8	8	8	8	8	8
A_4	4	3	3	4	4	4	4	3	3	4
A_5	6	6	6	7	5	6	6	6	5	6
A_6	5	5	5	5	6	5	5	5	7	5
A_7	7	7	7	6	7	7	7	7	6	7
A_8	2	2	2	2	2	2	2	2	2	2
A_9	1	1	1	1	1	1	1	1	1	1
A_{10}	3	4	4	3	3	3	3	4	4	3

5. Conclusions

This study proposed a MCDM methodology to rank the alternatives of teaching quality of vocational colleges. The plithogenic sets are used in this study to dela with the uncertainty data. Three experts evaluated the criteria and then we use the plithogenic numbers to evaluate the criteria and alternatives. Then we used the plithogenic operators to combine these numbers. Then we obtained the crisp values. We validated the proposed methodology through an application with ten criteria and ten alternatives. The criteria weights are computed using the normalized values. Then we applied the TODIM methodology to rank the alternatives. The results show alternative 2 has the highest rank and alternative 9 has the lowest rank. We performed a sensitivity analysis to show the different rank of alternatives. The results show our methodology has a stable different rank of alternatives under ten cases in change of criteria weights.

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Weighted Statistical Convergence in Neutrosophic Normed Linear Spaces

P. Jenifer[1], M. Jeyaraman[2,*], and M. Mursaleen[3,4]

[1]Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.

E-mail: jenifer87.math@g-mail.com, ORCID: https://orcid.org/0009-0002-7569-0125

[2]P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.

E-mail: jeya.math@gmail.com, ORCID: https://orcid.org/0000-0002-0364-1845

[3]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan.

[4]Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.

E-mail: mursaleenm@gmail.com

*Correspondence: jeya.math@gmail.com

Abstract:. In this research, the authors provide a definition among the Neutrosophic norm(NN)$(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ in terms of the summability property $(\bar{N}, \mathfrak{q}_n)$. We generalize the idea among Statistical Convergence $(\mathfrak{SC})$ in terms of NN by introducing the notion of $(\bar{N}, \mathfrak{q}_n)$-summability with respect to NN(μ, ν, ω). Here, the novel strategy is referred to by the term neutrosophic weighted statistical convergence. We also investigated its dependence on the NN$(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ in terms of statistical convergence and $(\bar{N}, \mathfrak{q}_n)$-summability.

Keywords: Neutrosophic normed linear space, weighted statistical convergence, statistical convergence, strong summability.

1. Introduction

Zadeh [1] first developed the concept among a set in 1965. Since then, scholars have extended the idea among sets to many other areas, including computing [2], quantum mechanics [6], demographics [5], systems of non-linear dynamic [4] as well as chaos controls [3]. Generalized the sets by introducing the idea of neutrosophic sets by Atanassov [7]. The neutrosophic metric spaces were first established by Park [8] using the concept of Neutrosophic sets, while $\mathfrak{NMLS}$ were first presented by Saadati and Park [9].

To improve the convergence sequence, Fast [11] described the idea of $\mathfrak{SC}$ along with Steinhaus [10]. Many scholars have examined $\mathfrak{SC}$ within the theory of ergodic theory, number theory, and Fourier analysis over the years. In addition, research has been conducted on the theory of summation of $\mathfrak{SC}$. Schoenberg [12] established a link between computability theory in addition $\mathfrak{SC}$. Salat, Connor, Fridy, along with Miller ([13], [14], [15], and [16]) investigated $\mathfrak{SC}$ as a summary technique.

Kirisci and Simsek [17] define $\mathfrak{NNLS}$ and $\mathfrak{SC}$, examining statistically Cauchy sequences and the concept of statistically completeness in this specialized mathematical context. Various investigations [18] have been carried out, expanding this idea. Sharma et. al., [19,28] discussed various types of summability in neutrosophic normed space.

Moricz along with Orhan [21] introduced statistical summability $(\bar{N}, \mathfrak{q}_n)$ as follows:
Let $\mathfrak{q} = (\mathfrak{q}_\iota)_{\iota=0}^{\infty}$ represent a sequence of non-negative numbers in which $\mathfrak{q}_0 > 0$ as well as $\mathfrak{Q}_n = \sum_{\iota=0}^{n} \mathfrak{q}_\iota \to \infty$ as $n \to \infty$. Demonstrate that $\varpi_n = \frac{1}{\mathfrak{Q}_n} \sum_{\iota=0}^{n} \mathfrak{q}_\iota x_\iota, n = 0, 1, 2, ...$
$\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is statistically summable towards L using the weighted mean technique based on the sequence $(\mathfrak{q}_\iota)$ or statistically summable $(\bar{N}, \mathfrak{q}_n)$ in a brief statistical manner if $S - \lim_{n\to\infty} \varpi_n = L$.

Thus, we put $\bar{N}(S) - \lim \hat{\mathfrak{u}} = L$. $\bar{N}(S)$ denotes the set among all statistically summable $(\bar{N}, \mathfrak{q}_n)$ sequences. Conversely, if $\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \mathfrak{q}_\iota |\hat{\mathfrak{u}}_\iota - L| = 0$ then the sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is called as strongly $(\bar{N}, \mathfrak{q}_n)$ - summable to L. Here we write $|\bar{N}, \mathfrak{q}_n| - \lim \hat{\mathfrak{u}} = L$.

Karkaya together with Chishti [22] applied $(\bar{N}, \mathfrak{q}_n)$-compatibility to apply the concept about $\mathfrak{SC}$ generically and it also known as new approach $\mathfrak{WSC}$. Mursaleen together with others [23] modified the definition of $\mathfrak{WSC}$ and alos $(\bar{N}, \mathfrak{q}_n)$ - noted how they relate to the idea of computability. Additionally, it was shown why the definition given below should be applied:

If the limit exists, then $\delta_{\bar{N}}(K) = \lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} |K_{\mathfrak{Q}_n}|$ defines the weighted density on $K \subseteq N$. A sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ represent weighted statistically convergent (or SN-convergent) towards $\tilde{\mathfrak{P}}$ when, for each $\acute{\varepsilon} > 0$,

$$\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \mathfrak{q}_\iota |\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}| \geq \acute{\varepsilon}\}) = 0, \text{ or similarly } \lim_{n\to\infty} \frac{1}{\mathfrak{Q}} |\{\iota \leq \mathfrak{Q}_n : \mathfrak{q}_\iota |\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}| \geq \acute{\varepsilon}\}| = 0.$$

Thus, we put $S_{\bar{N}} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. Here, we present a new concept among $\mathfrak{SC}$ it is defined as $\mathfrak{WSC}$ within $\mathfrak{NNLS}$. Few links between this concept and $(\bar{N}, q_n)$- summability within $\mathfrak{NNLS}$ are shown.

2. Basic definitions

Definition 2.1. The 7-tuple $(\hat{\Im}, \dot{\mathcal{J}}, \ddot{\mathcal{G}}, \dddot{\mathcal{H}}, \check{*}, \check{\odot}, \check{\Diamond})$ be referred as $\mathcal{NNS}$ when $\hat{\Im}$ represent a linear space, $\check{*}$ denote a continuous ϖ-norm, $\check{\odot}$ as well as $\check{\Diamond}$ indicate continuous ϖ-co-norm, $\dot{\mathcal{J}}, \ddot{\mathcal{G}}$ and $\dddot{\mathcal{H}}$ represent fuzzy sets on $\hat{\Im} \times (0, \infty)$ fulfills the below conditions:
For each $\lceil, \dagger \in \hat{\Im}$ and $\int, \varpi > 0$;

(a) $0 \leq \dot{\mathcal{J}}(\lceil, \varpi) \leq 1; 0 \leq \ddot{\mathcal{G}}(\lceil, \varpi) \leq 1; 0 \leq \dddot{\mathcal{H}}(\lceil, \varpi) \leq 1$,
(b) $\dot{\mathcal{J}}(\lceil, \varpi) + \ddot{\mathcal{G}}(\lceil, \varpi) + \dddot{\mathcal{H}}(\lceil, \varpi) \leq 3$,
(c) $\dot{\mathcal{J}}(\lceil, \varpi) > 0$,
(d) $\dot{\mathcal{J}}(\lceil, \varpi) = 1$ if and only if $\lceil = 0$,
(e) $\dot{\mathcal{J}}(\acute{\alpha}\lceil, \varpi) = \dot{\mathcal{J}}\left(\lceil, \frac{\varpi}{|\acute{\alpha}|}\right)$ for each $\acute{\alpha} \neq 0$,
(f) $\dot{\mathcal{J}}(\lceil, \varpi) \check{*} \dot{\mathcal{J}}(\dagger, \int) \leq \dot{\mathcal{J}}(\lceil + \dagger, \varpi + \int)$,
(g) $\dot{\mathcal{J}}(\lceil, \varpi) : (0, \infty) \to [0, 1]$ is continuous,
(h) $\lim_{\varpi \to \infty} \dot{\mathcal{J}}(\lceil, \varpi) = 1$ and $\lim_{\varpi \to 0} \dot{\mathcal{J}}(\lceil, \varpi) = 0$,
(i) $\ddot{\mathcal{G}}(\lceil, \varpi) < 1$,
(j) $\ddot{\mathcal{G}}(\lceil, \varpi) = 0$ if and only if $\lceil = 0$,
(k) $\ddot{\mathcal{G}}(\acute{\alpha}\lceil, \varpi) = \ddot{\mathcal{G}}\left(\lceil, \frac{\varpi}{|\acute{\alpha}|}\right)$ for each $\acute{\alpha} \neq 0$,
(l) $\ddot{\mathcal{G}}(\lceil, \varpi) \check{\odot} \ddot{\mathcal{G}}(\dagger, \int) \geq \ddot{\mathcal{G}}(\lceil + \dagger, \varpi + \int)$,
(m) $\ddot{\mathcal{G}}(\lceil, \varpi) : (0, \infty) \to [0, 1]$ is continuous,
(n) $\lim_{\varpi \to \infty} \ddot{\mathcal{G}}(\lceil, \varpi) = 0$ and $\lim_{\varpi \to 0} \ddot{\mathcal{G}}(\lceil, \varpi) = 1$,
(o) $\dddot{\mathcal{H}}(\lceil, \varpi) < 1$,
(p) $\dddot{\mathcal{H}}(\lceil, \varpi) = 0$ if and only if $\lceil = 0$,
(q) $\dddot{\mathcal{H}}(\acute{\alpha}\lceil, \varpi) = \dddot{\mathcal{H}}\left(\lceil, \frac{\varpi}{|\acute{\alpha}|}\right)$ for each $\acute{\alpha} \neq 0$,
(r) $\dddot{\mathcal{H}}(\lceil, \varpi) \check{\Diamond} \ddot{\mathcal{G}}(\dagger, \int) \geq \dddot{\mathcal{H}}(\lceil + \dagger, \varpi + \int)$
(s) $\dddot{\mathcal{H}}(\lceil, \varpi) : (0, \infty) \to [0, 1]$ is continuous,
(t) $\lim_{\varpi \to \infty} \dddot{\mathcal{H}}(\lceil, \varpi) = 0$ and $\lim_{\varpi \to 0} \dddot{\mathcal{H}}(\lceil, \varpi) = 1$.

Example 2.2. [25] Let $(\hat{\mathfrak{A}}, |\cdot|)$ denote a normed space, in addition let $\acute{\alpha} \check{*} \acute{\gamma} = \acute{\alpha}\gamma$ as well as $\acute{\alpha} \check{\odot} \acute{\gamma} = \min\{\acute{\alpha} + \gamma, 1\}$ for every $\acute{\alpha}, \acute{\gamma} \in [0, 1]$. For all $\hat{\mathfrak{u}} \in \hat{\mathfrak{A}}$ and every $\hat{\varphi} > 0$, take $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\varpi}{\varpi + \|\hat{\mathfrak{u}}\|}, \ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\|\hat{\mathfrak{u}}\|}{\varpi + \|\hat{\mathfrak{u}}\|}$ as well as $\dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\|\hat{\mathfrak{u}}\|}{\|\varpi + \hat{\mathfrak{u}}\|}$. Then $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \check{*}, \check{\odot}, \check{\Diamond})$ is an $\mathfrak{NMLS}$.

Definition 2.3. [25] Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \check{*}, \check{\odot}, \check{\Diamond})$ represent an $\mathfrak{NMLS}$. A $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ sequence within $\hat{\mathfrak{A}}$ is convergent towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ when, for any $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, that $\iota_0 \in \mathbb{N}$ exists which means $\dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) > 1 - \acute{\varepsilon}, \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) < \acute{\varepsilon}$ as well as $\dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) < \acute{\varepsilon}$ for everyone $\iota \geq \iota_0$ where $\iota \in \mathbb{N}$. It's indicated by $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho}) - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

Theorem 2.4. *[23] Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \bar{*}, \breve{\odot}, \breve{\Diamond})$ represent a $\mathfrak{NMLS}$. Then, $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ sequence within $\hat{\mathfrak{A}}$ which is convergent towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ if and only if $\lim_{\iota\to\infty} \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) = 1$, $\lim_{\iota\to\infty} \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) = 0$ as well as $\lim_{\iota\to\infty} \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) = 0$.*

3. Weighted statistical convergence in $\mathfrak{NMLS}$

Weighted Statistical Convergent ($\mathfrak{WSC}$) related the new idea NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ we're going to developed in the below subsection. Here, in addition examine few connections between this description and $\mathfrak{SC}$ as well as $(\bar{N}, \mathfrak{q}_n)$- summability within $\mathfrak{NMLS}$.

Definition 3.1. Let a sequence $\mathfrak{q} = (\mathfrak{q}_\iota)_{\iota=0}^{\infty}$ among nonnegative numbers where $\mathfrak{q}_0 > 0$ along with $\mathfrak{Q}_n = \sum_{\iota=0}^{n} \mathfrak{q}_\iota \to \infty$ as $n \to \infty$ in addition $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \bar{*}, \breve{\odot}, \breve{\Diamond})$ be an $\mathfrak{NMLS}$. A sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ in $\hat{\mathfrak{A}}$ is known as weighted statistically convergent towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ with respect to the NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ (or $S_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$ - convergent to $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$) when, for any $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$,

$$\delta_{\bar{N}} \left(\left\{ \begin{array}{c} \iota \in \mathbb{N} : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \right) = 0, \tag{1}$$

or similarly

$$\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} \left| \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \right| = 0.$$

Thus, we put $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. Weighted statistically convergence can be simplified to statistically convergence in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ described by Karakus et al. [17], if we assume that $\mathfrak{q}_\iota = 1$ for all $\iota \in \mathbb{N}$ and thus, we put $\mathfrak{S}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

Using equality (1) and the weighted density's properties, we may rapidly arrive at the following lemma.

Lemma 3.2. *Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \bar{*}, \breve{\odot}, \breve{\Diamond})$ being a $\mathfrak{NMLS}$ along with $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ within $\hat{\mathfrak{A}}$. After that for all $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, the given statements are identical:*

(i) $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

(ii) $\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dot{\xi}(q_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \ddot{\zeta}(q_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dddot{\varrho}(q_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}\}) = 0$.

(iii) $\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon}, \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon} \text{ and } \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\}) = 1$.

(iv) $\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\}) = 1$.

Definition 3.3. Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \bar{*}, \breve{\odot}, \breve{\Diamond})$ being a $\mathfrak{NMLS}$. A sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ within $\hat{\mathfrak{A}}$ is referred to as $(\bar{N}, \mathfrak{q}_n)$ - summable towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ (or $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$-

summable towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$) when, for all $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, that $n_0 \in \mathbb{N}$ exists which means

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon},$$

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon} \text{and}$$

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}.$$

for any a$n \geq n_0$. Therefore, we have, $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. For any $\iota \in \mathbb{N}$, if we assume that $\mathfrak{q}_\iota = 1$, then $(\bar{N}, \mathfrak{q}_n)$- summability in terms of the NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ is simplified towards $(C, 1)$-summability in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ and thus we put $(C, 1)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

The first theorem establishes the link between $\mathfrak{SC}$ and $\mathfrak{WSC}$ in $\mathfrak{NNLS}$.

Theorem 3.4. *Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \ast, \breve{\odot})$ being a $\mathfrak{NNLS}$ and in addition $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ represent a sequence in $\hat{\mathfrak{A}}$. The following conditions are true:*

(i) *If $\mathfrak{q}_\iota \geq 1$ for each $\iota \in \mathbb{N}$, $\lim\limits_{n\to\infty}\sup\dfrac{\mathfrak{Q}_n}{n} < \infty$ along with $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$, then $\mathfrak{S}(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho}) - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.*

(ii) *If $\mathfrak{q}_\iota \leq 1$ for all $\iota \in \mathbb{N}$, $\lim\limits_{n\to\infty}\inf\dfrac{\mathfrak{Q}_n}{n} > 0$ and in addition $\mathfrak{S}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$, after that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.*

Proof. (i) Assume that $\mathfrak{q}_\iota \geq 1$ for each $\iota \in \mathbb{N}$, $\lim\limits_{n\to\infty}\sup\dfrac{\tilde{\mathfrak{P}}_n}{n} < \infty$. Given, that positive constant $\mathfrak{K}$ exists which means $1 \leq \frac{\tilde{\mathfrak{P}}_n}{n} \leq \mathfrak{K}$. Furthermore, $\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$, $\ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$ and $\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$ given that $\mathfrak{q}_\iota \geq 1$ for each $\iota \in \mathbb{N}$ along with $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi})$ indicate increasing function of $\hat{\varphi} \in \mathbb{R}^+$, $\ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}), \dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi})$ represent decreasing function of $\hat{\varphi} \in \mathbb{R}^+$ based on the Definition (2.1) provided in [**?**]. After that, we obtain for each $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$

$$\frac{1}{n}\left|\left\{\begin{array}{c}\iota \leq n : \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon}, \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon}\end{array}\right\}\right| \leq \frac{1}{n}\left|\left\{\begin{array}{c}\iota \leq n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}\end{array}\right\}\right|$$

$$\leq \frac{\mathfrak{K}}{\mathfrak{Q}_n}\left|\left\{\begin{array}{c}\iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}\end{array}\right\}\right|.$$

Given that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$, as a result $\mathfrak{S}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

(ii) Assume that $\mathfrak{q}_\iota \leq 1$ for each $\iota \in \mathbb{N}$ together with $\lim\limits_{n\to\infty}\inf\dfrac{\mathfrak{Q}_n}{n} > 0$. Hence, that $\delta > 0$ exists where $\delta \leq \frac{\mathfrak{Q}_n}{n} \leq 1$. Furthermore,

$\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}), \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$ and $\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$

because $\mathfrak{q}_\iota \leq 1$ for any $\iota \in \mathbb{N}$ along with $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi})$ is an increasing function among $\hat{\varphi} \in \mathbb{R}^+, \ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}), \dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi})$ indicate decreasing function among $\hat{\varphi} \in \mathbb{R}^+$. Consequently, we have

$$\frac{1}{n}\left|\left\{\begin{array}{c} \iota \leq n : \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right| \geq \frac{1}{n}\left|\left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right|$$

$$\geq \frac{\delta}{\mathfrak{Q}_n}\left|\left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_k(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right|.$$

The concept implies that $\mathfrak{S}_{\tilde{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. □

The below conditions demonstrate that the converse among theorem (3.4) (ii) denote not typically true.

Example 3.5. The symbol $(\mathbb{R}, |\cdot|)$ represents the space of real number with the ordinary norm along with for each $\acute{\alpha}, \acute{\beta} \in [0, 1]$, let $\acute{\alpha} \acute{*} \acute{\beta} = \acute{\alpha}\acute{\beta}$ as well as $\acute{\alpha} \acute{\odot} \acute{\beta} = \min\{\acute{\alpha} + \acute{\beta}, 1\}$. All $x \in \mathbb{R}$ in addition any of the $\hat{\varphi} > 0$, take that $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\hat{\varphi}}{\hat{\varphi} + |\hat{\mathfrak{u}}|}, \ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{|\hat{\mathfrak{u}}|}{\hat{\varphi} + |\hat{\mathfrak{u}}|}$ along with $\dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{|\hat{\mathfrak{u}}|}{|\hat{\varphi} + \hat{\mathfrak{u}}|}$.
Since, $(\mathbb{R}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \acute{*}, \acute{\odot}, \acute{\Diamond})$ represent an $\mathfrak{NMLGS}$. Let describe the sequences $\tilde{\mathfrak{P}}_\iota = 3^{\iota - 1}$ for any $\iota \in \mathbb{N}$ alongwith $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ which means $m \in \mathbb{N}$ as well as $\hat{\mathfrak{u}}_\iota = \begin{cases} 1, & \text{first } \left[\sqrt{3}^{m-1}\right] \text{ integers in} (\mathfrak{Q}_{m-1}, \mathfrak{Q}_m], \\ 0, & \text{otherwise.} \end{cases}$ We have, for every $0 < \acute{\varepsilon} < 1$ as well as $\hat{\varphi} > 0$, notify

$$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) := \left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}_0(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \leq 1 - \acute{\varepsilon}, \\ \ddot{\zeta}_0(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon}, \text{and} \\ \dddot{\varrho}_0(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}.$$

Given that

$$\begin{aligned} \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) &= \left\{\iota \leq \mathfrak{Q}_n : \frac{\hat{\varphi}}{\hat{\varphi} + 3^{\iota-1}|\hat{\mathfrak{u}}_\iota|} \leq 1 - \acute{\varepsilon} yada \frac{3^{\iota-1}|\hat{\mathfrak{u}}_\iota|}{\hat{\varphi} + 3^{\iota-1}|\hat{\mathfrak{u}}_\iota|} \geq \acute{\varepsilon}\right\} \\ &= \left\{\iota \leq \mathfrak{Q}_n : |\hat{\mathfrak{u}}_\iota| \geq \frac{\acute{\varepsilon}\hat{\varphi}}{3^{\iota-1}(1 - \acute{\varepsilon})}\right\} \\ &= \{\iota \leq \mathfrak{Q}_n : \hat{\mathfrak{u}}_\iota = 1\}, \end{aligned}$$

we get

$$\begin{aligned}\frac{1}{\mathfrak{Q}_n}|\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})| &= \frac{2}{3^n-1}\left|\left\{\iota \leq \frac{3^n-1}{2} : \hat{\mathfrak{u}}_\iota = 1\right\}\right| \\ &= \frac{2}{3^n-1}\left(1+\left[\sqrt{3}\right]+\left[\sqrt{3}^2\right]+\cdots+\left[\sqrt{3}^{n-1}\right]\right) \\ &\leq \frac{2}{3^n-1}\left(1+(1.8)+(1.8)^2+\cdots+(1.8)^{n-1}\right) \\ &\leq \frac{5}{2}\left(\frac{1.8}{3}\right)^n \frac{\left(1-\left(\frac{1}{1.8}\right)^n\right)}{\left(1-\left(\frac{1}{3}\right)^n\right)},\end{aligned}$$

it attains $\lim_{n\to\infty}\frac{1}{\mathfrak{Q}_n}|\mathfrak{K}\mathfrak{Q}_n(\acute{\varepsilon})| = 0$.
A positive integer m like that $\mathfrak{Q}_{m-1} < n \leq \mathfrak{Q}_m$ exist given a large enough number n. After that, for every $0 < \acute{\varepsilon} < 1$ along with $\hat{\varphi} > 0$, let
$\mathfrak{K}_n(\acute{\varepsilon}) := \left\{\iota \leq n : \dot{\xi}_0(\hat{\mathfrak{u}}_\iota, \hat{\varphi}) \leq 1-\acute{\varepsilon}\ddot{\zeta}_0(\hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon}, \dddot{\varrho}_0(\hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon}\right\}$
Given that

$$\begin{aligned}\mathfrak{K}_n(\acute{\varepsilon}) &= \left\{\iota \leq n : \frac{\hat{\varphi}}{\hat{\varphi}+|\hat{\mathfrak{u}}_\iota|} \leq 1-\acute{\varepsilon} yada \frac{|\hat{\mathfrak{u}}_\iota|}{\hat{\varphi}+|\hat{\mathfrak{u}}_\iota|} \geq \acute{\varepsilon}\right\} \\ &= \left\{\iota \leq n : |\hat{\mathfrak{u}}_\iota| \geq \frac{\acute{\varepsilon}\hat{\varphi}}{(1-\acute{\varepsilon})}\right\} \\ &= \left\{\iota \leq n : \hat{\mathfrak{u}}_\iota = 1\right\},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{|\mathfrak{K}_n(\acute{\varepsilon})|} &= \frac{1}{n}\left|\{\iota \leq n : \hat{\mathfrak{u}}_\iota = 1\}\right| \\ &> \frac{1}{n}\left\{1+(1.2)+(1.2)^2+\cdots+(1.2)^{m-1}\right\} \\ &> \frac{1}{n}(1.2)^m - \frac{1}{n} \\ &> \frac{1}{n}(1.2)^{\frac{n}{\log 2}} - \frac{1}{n} \text{ since } m > \frac{n}{\log 2},\end{aligned}$$

in which $(\hat{\mathfrak{u}}_n)$ is $\mathfrak{WSC}$ to 0 with respect towards the NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$; however, $\mathfrak{SC}$ not with respect towards the NN.

In the following theorems, we prove the link between $\mathfrak{WSC}$ and $(\bar{N}, \mathfrak{q}_n)$ - summability within $\mathfrak{NMLS}$.

Theorem 3.6. *Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \bigstar, \breve{\odot}, \breve{\Diamond})$ be an $\mathfrak{NMLS}$ and in addition $\frac{\mathfrak{Q}_n}{n} \geq 1$ for each $n \in \mathbb{N}$. If $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ within $\hat{\mathfrak{A}}$ is $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$- summable towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$, after that $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$ - convergent to $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$.*

Proof. Assume that $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$ - summable to $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$. For all $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, consider

$$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \text{ and}$$

$$\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon} \text{ and} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon} \end{array} \right\}$$

After that,

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) + \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi})$$

$$\geq \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > \frac{1}{\mathfrak{Q}_n} \left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right| (1 - \acute{\varepsilon}). \tag{2}$$

Resulting from inequity (2), we get $\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} \left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right| = 1$. Similarly, for every $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$,

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) + \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi})$$

$$\leq \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \frac{1}{\mathfrak{Q}_n} \left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right| \acute{\varepsilon}. \tag{3}$$

By inequality (3), we have $\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} |\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})| = 0$. Hence proved. □

Theorem (3.4) is not often true, as demonstrated by the case that follows.

Example 3.7. Let $(\mathbb{R}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \bar{*}, \breve{\odot}, \breve{\Diamond})$ be as shown in Example (3.5). Take that $\mathfrak{q}_\iota = \frac{1}{\iota+1}$ for each $\iota \in \mathbb{N}$ and create a sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ which is defined by

$$\hat{\mathfrak{u}}_\iota = \begin{cases} \iota, & \text{if } \iota = m^2 (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $0 < \acute{\varepsilon} < 1$ along with each $\hat{\varphi} > 0$, consider

$$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\}.$$

Given that

$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \iota \leq \mathfrak{Q}_n : \frac{\hat{\varphi}}{\hat{\varphi} + |\hat{\mathfrak{u}}_\iota|} \leq 1 - \acute{\varepsilon} \text{ or } \frac{\frac{|\hat{\mathfrak{u}}_\iota|}{\iota+1}}{\hat{\varphi} + \frac{|\hat{\mathfrak{u}}_\iota|}{\iota+1}} \geq \acute{\varepsilon} \right\} = \left\{ \iota \leq \mathfrak{Q}_n : |\hat{\mathfrak{u}}_\iota| \geq \frac{(\iota+1)\acute{\varepsilon}\hat{\varphi}}{(1-\acute{\varepsilon})} \right\}$

$\subseteq \{\iota \leq \mathfrak{Q}_n : \hat{\mathfrak{u}}_\iota = \iota\}$, we get $\frac{1}{\mathfrak{Q}_n} |\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})| \leq \frac{\sqrt{1 + \frac{1}{2} + \cdots + \frac{1}{n}}}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}$

it yields that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = 0$.

Therefore,

$$\begin{aligned}
&\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})\\
&=\frac{1}{\mathfrak{Q}_{m^2+s}}\sum_{\iota=1}^{m^2+s}\dot{\xi}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})\\
&=\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\left(\begin{array}{c}\frac{\hat{\varphi}}{\hat{\varphi}+1}\underbrace{1+1}_{2}+\frac{4\hat{\varphi}}{4\hat{\varphi}+4}+\underbrace{1+1+1+1}_{4}\\ +\frac{9\hat{\varphi}}{9\hat{\varphi}+9}+\cdots+\underbrace{1+1+1+1}_{2(m-1)}+\frac{m^2\hat{\varphi}}{m^2\hat{\varphi}+m^2}+\underbrace{1+1+1+1}_{2m}\end{array}\right)\\
&=\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\left(\frac{mt}{t+1}+m^2+m\right)\text{ and}\\
&\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\ddot{\zeta}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})=\frac{1}{\mathfrak{Q}_{m^2+s}}\sum_{\iota=1}^{m^2+s}\ddot{\zeta}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})\\
&=\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\left(\frac{\hat{\varphi}}{\hat{\varphi}+1}+\frac{4\hat{\varphi}}{4\hat{\varphi}+4}+\frac{9\hat{\varphi}}{9\hat{\varphi}+9}+\cdots+\frac{m^2\hat{\varphi}}{m^2\hat{\varphi}+m^2}\right)\\
&=\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\frac{m\hat{\varphi}}{\hat{\varphi}+1}
\end{aligned}$$

about $m,s\in\mathbb{N}$. Therefore, $\hat{\mathfrak{u}}=(\hat{\mathfrak{u}}_\iota)$ is not $(\bar{N},\mathfrak{q}_n)$ - summable towards 0 in terms of NN $(\dot{\xi},\ddot{\zeta},\dddot{\varrho})$, because to the statement that

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})\to\infty,\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\ddot{\zeta}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})\to\infty\text{ and }\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dddot{\varrho}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi})\to\infty$$

as $n\to\infty$.

Theorem 3.8. *Let $(\hat{\mathfrak{A}},\dot{\xi},\ddot{\zeta},\dddot{\varrho},\divideontimes,\odot,\lozenge)$ being a $\mathfrak{MMLGS},\hat{\mathfrak{u}}=(\hat{\mathfrak{u}}_\iota)$ in $\hat{\mathfrak{A}}$ and $\frac{\mathfrak{Q}_n}{n}\geq 1$ for all $n\in\mathbb{N}$. When $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})}-\lim\hat{\mathfrak{u}}=\tilde{\mathfrak{P}},\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota-\tilde{\mathfrak{P}}),\hat{\varphi})\geq 1-M,\ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota-\tilde{\mathfrak{P}}),\hat{\varphi})\leq M$ and in addition $\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota-\tilde{\mathfrak{P}}),\hat{\varphi})\leq M$ regards to $M\in(0,1)$ and $\iota\in\mathbb{N}$, after that $(\bar{N},\mathfrak{q}_n)^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})}-\lim\hat{\mathfrak{u}}=\tilde{\mathfrak{P}}$.*

Proof. Assume that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})}-\lim\hat{\mathfrak{u}}=\tilde{\mathfrak{P}}$. After that, for each $\acute{\varepsilon}>0$ as well as $\hat{\varphi}>0$,

$$\begin{aligned}
\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota(\iota-\tilde{\mathfrak{P}}),\hat{\varphi})&=\frac{1}{\mathfrak{Q}_n}\sum_{\substack{\iota=1\\ \iota\in\hat{\mathfrak{K}}_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota-\tilde{\mathfrak{P}}),\hat{\varphi})+\frac{1}{\mathfrak{Q}_n}\sum_{\substack{\iota=1\\ \iota\in\hat{\mathfrak{K}}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota-\tilde{\mathfrak{P}}),\hat{\varphi})\\
&=\mathfrak{S}_1(n)+\mathfrak{S}_2(n)
\end{aligned}$$

in which

$$\mathfrak{S}_1(n)=\frac{1}{\mathfrak{Q}_n}\sum_{\substack{\iota=1\\ \iota\in\hat{\mathfrak{K}}_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota-\tilde{\mathfrak{P}}),\hat{\varphi})\tag{4}$$

as well as

$$\mathfrak{S}_2(n) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1 \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \tag{5}$$

When $\iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})$, after that

$$\mathfrak{S}_1(n) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1 \\ \iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \frac{|\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})|}{\mathfrak{Q}_n}(1 - M). \tag{6}$$

Given that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$,

$$\lim_{n \to \infty} \mathfrak{S}_1(n) \geq 0. \tag{7}$$

If $\iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})$, then we obtain

$$\mathfrak{S}_2(n) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1 \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > \frac{\left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right|}{\mathfrak{Q}_n}(1 - \acute{\varepsilon}) \tag{8}$$

which gives that

$$\lim_{n \to \infty} \mathfrak{S}_2(n) > (1 - \acute{\varepsilon}). \tag{9}$$

Using equality (4)-(5) and inequalities (6)-(9), we obtain

$$\lim_{n \to \infty} \frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) = 1. \tag{10}$$

Similarly, we obtain

$$\lim_{n \to \infty} \frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) = 0. \tag{11}$$

As a result from equality (10) and (11), we get

$$(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}. \tag{12}$$

□

4. **Conclusion**

We present a new method among summability within $\mathfrak{MNLS}$ in this paper, represented by the notation $(\bar{N}, \mathfrak{q}_n)$ -summability, as well as apply that summability to develope a new kind of $\mathfrak{SC}$ in $\mathfrak{MNLS}$, denoted by the notation $\mathfrak{WSC}$. The connections among these ideas are also explored. The derived findings were greater in scope compared to the associated findings for normed linear spaces because any crispness norm may generate a NN. While this paper's findings do overlap with those of previous studies on the subject, the arguments presented here were proven using an alternative methodology for the largest part. The authors of this research want to allow space for follow-up studies.

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Multiple-Criteria Decision-Making and Plithogenic Sets-Based TOPSIS Framework for 'Dual-Qualified' Teachers Quality Evaluation in the Context of High-Quality Development of Vocational Education

Wenjing Liu[1],Yuemin Gao[2] ,Qiongyao Liu[3] ,Wenqiang Dai[4*]
[1]Art Department, Qinhuangdao Vocational and Technical College, Qinhuangdao, 066100, Hebei, China
[2]Basic Teaching Department, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
[3]Basic Teaching Department, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
[4]Department of Architecture and Design, Qinhuangdao Vocational and Technical College, Qinhuangdao 066100, Hebei, China
*Corresponding author, E-mail: dwq@qvc.edu.cn

Abstract: This study proposed a multi-criteria decision making (MCDM) methodology for evaluating the Dual-Qualified" Teachers Quality. This evaluation has various criteria and factors. So, the MCDM can deal with these factors. This study combines the MCDM methodology with the plithogenic set to deal with uncertainty and vague information. Three experts are invited to evaluate the criteria and alternatives. Then we replace their opinions by using plithogenic numbers. We used plithogenic operators to combine these numbers. Eight criteria and eleven alternatives are used in this study. We used the TOPSIS method under the plithogenic numbers to rank the alternatives. the results show the Professional Knowledge and Industry Expertise criterion has the highest weights and Research and Innovation in Teaching criterion has the lowest weights. We validated the proposed methodology by an application. The results show the proposed methodology is effective.

Keywords: Multiple-Criteria Decision-Making; Dual-Qualified; Vocational Education; TOPSIS Method.

1. Introduction

Higher vocational education's goal, as we all know, is to prepare highly qualified individuals and talented individuals to fill front-line production, construction, management, and service positions in industry businesses. Higher vocational colleges also propose the requirement of "double qualification" for teachers, based on the unique

orientation of running a school and the purpose of talent training. Double qualification is the term used to describe the Chinese Ministry of Education's requirement that teachers complete ongoing training through additional coursework and exposure to the industry[1], [2].

This is a crucial component of the talent team in higher vocational colleges; "double-qualified" teachers are the main source of assistance in showcasing the operational features of the school and bolstering the development of talented individuals. The State has released the Execution Plan for Accelerating School Modernization in successive years[3], [4].

Higher vocational education plays a crucial role in China's educational process, and higher standards are proposed for its development in various documents, including "the Implementation Plan for Improving the Quality of Teachers in Vocational Colleges," "the Opinions of the CPC Central Committee and the State Council on Comprehensively Deepening the Reform of the Building of Teachers in the New Era," and Modernization of China's Education[5], [6].

Teachers are crucial to the integration of production, study, research, invention, and application in the modern period. The development of a "Double-qualified" teaching staff is a crucial step in advancing higher vocational education reform and achieving modernization. Vocational colleges must produce competent applied skills in the context of China's social development and industrial upgrading in the new era, and the main assurance is the training of "Double-qualified" teachers[7], [8].

To greatly enhance the quality of talent training and the degree of modernization of China's higher vocational education, support high-tech skilled talent, encourage the social economy's sustainable growth, boost national competitiveness, and assemble a group of "Double-qualified" educators with high moral standards[1], [9].

One of the main issues that decision-makers may encounter in the evaluation process is vagueness and uncertainty. There are several shortcomings in recent studies assessing teacher quality, namely in taking uncertain elements into account that hinder desirable decision-making. Inaccurate conclusions are also produced because the decision maker's priorities and the degree of conflict between factors are not considered[10], [11].

Therefore, the suggested method in this study increased the degree of uncertainty consideration and offered a high degree of accuracy in decision-making situations. By considering the truth-membership function, falsity-membership function, and

indeterminacy-membership function, the plithogenic set—a generalization of the neutrosophic set—is more effective in managing ambiguous judgments[12], [13].

The TOPSIS method based integrated plithogenic approach strengthens the consideration of uncertainty and offers a highly reliable and accurate indicator of teacher quality. To address the uncertainty environment, we employ the plithogenic TOPSIS method, which has no other application in the field of teacher quality. This study contributes to the development of a framework evaluation of teacher quality. The suggested method is used to assess the teacher quality[14], [15], [16].

1.1 Objectives

- This paper's primary goal is to assess teacher quality using plithogenic set theory to identify the criteria and alternatives.
- To measure the teacher quality variables, we suggest a combined plithogenic technique based on the TOPSIS method.
- The accuracy of assessments based on decision makers' evaluations is increased by utilizing plithogenic set operations' properties.

1.2 Organization

The rest of the document is structured as follows: Section 2 provides information and definitions regarding the techniques and guiding principles that were employed. Section 3 provides a detailed description of the suggested methodology, including its features and procedures. The use of the suggested method is shown in Section 4, along with a discussion of its outcomes. Section 5 shows the results and discussion. In Section 6, the work's summary is finally provided.

2. Methods

2.1 Plithogenic set

A set that consists of different items defined by several qualities is called a Plithogenic set (P, A, V, d, c). Each attribute has a value V = {v1, v2, …, vn}, with n ≥ 1. A = {α1, α2, …, αm}, m ≥ 1. The two primary characteristics that set apart plithogenic sets are the degree of contradiction and the degree of appurtenance. D(x,v) is the appurtenance degree function of element x with regard to the set of specified criteria. Each attribute value is distinguished from the dominant attribute value using the contradiction (dissimilarity) degree function c(v,D). Intersection, union, complement, inclusion, and equality are examples of plithogenic set operations[9], [17].

Let A be a non-empty set of uni-dimensional attributes A = {α1, α2, …, αm}, m ≥ 1, and let $\alpha \in$ A be an attribute whose value spectrum is the set S. S can be defined as an infinitely countable set S = {s1, s2, …, s∞}, or as a finite discrete set S = {s1, s2, …, sl}, 1 ≤ l <∞, or as an infinitely uncountable (continuum) set S =]a,b [, a < b, with the range of all attribute values defined by experts depending on the application, V = {v1, v2, …, vn} for n ≥ 1. The decision-maker determines the dominant attribute value in set V, which stands for the most significant value, based on the evaluation and the nature of the issue[18], [19].

2.2 Technique in Order of Preference by similarity to ideal solution (TOPSIS)

TOPSIS is an excellent mathematical MCDM technique that has shown remarkable results in numerous studies and is used to tackle decision-making problems in a variety of fields. This method's primary objective is to determine the best option based on a set of predetermined standards by calculating how far each alternative is from a positive and negative ideal solution. The following is a clear definition of the TOPSIS steps:

1. Step 1: Identify the issue for which you need to determine the best solution given a range of options and the standards by which those options will be evaluated. Next, use the decision matrix to assess each option according to predetermined standards.
2. Step 2: Apply the TOPSIS vector normalization formula to normalize the decision matrix.

$$p_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$$

Where x_{ij} refers to the value in decision matrix.

3. Step 3: Construct the weighted decision matrix as:

$$q_{ij} = w_j \times p_{ij}$$

4. Step 4: Identify the positive and negative ideal solution

$$B^+ = \{q_1^+, \dots, q_n^+\}$$
$$B^+ = \left\{\left(\max q_{ij} \,|j \in J_b\right), \left(\min q_{ij} \,|j \in J_b\right)| \ \in [1 \dots m] \right\}$$
$$B^- = \{q_1^-, \dots, q_n^-\}$$
$$B^- = \left\{\left(\min q_{ij} \,|j \in J_b\right), \left(\max q_{ij} \,|j \in J_b\right)| \ \in [1 \dots m] \right\}$$

5. Step 5: Compute the distance of every alternative using the Euclidean distance as:

$$D_i^+ = \left[\sum_{j=1}^{m}\left(B_i - B_j^+\right)^2\right]^{\frac{1}{2}}$$

$$D_i^- = \left[\sum_{j=1}^{m}\left(B_i - B_j^-\right)^2\right]^{\frac{1}{2}}$$

6. Step 6: Compute the closeness coefficient of every alternative.

$$R_i = \frac{D_i^-}{D_i^+ - D_i^-}$$

7. Step 7: Rank the alternatives

3. Proposed Approach

To assess sustainable Teachers Quality, we present a plithogenic TOPSIS technique in this study. This approach's significance stems from its high degree of uncertainty consideration and enhanced evaluation process accuracy. Features of plithogenic set aggregation, like the contradiction degree function, guarantee more precise conclusions that surpass those of other researchers at the same time. To choose the optimum course of action, TOPSIS primarily focuses on evaluating the set of criteria by contrasting the options with the best and worst solutions. Based on the criteria's conflict nature and degree of difference, we determine the objective weight. This method creates a robust model by combining the benefits of the TOPSIS, and plithogenic aggregation operator. Figure 1 shows the steps of the proposed methodology.

- Step 1: Determine committee of experts to help in the evaluation process in the MCDM methodology. Set of experts as $D = \{D_1, \dots, D_k\}$, criteria $C = \{C_1, \dots, C_n\}$, and alternatives $A = \{A_1, \dots, A_m\}$.
- Step 2. Establish the language scale that DMs will use to assess the criteria and options. This method defines the plithogenic numbers. DMs assess the specified set of criteria and options using this scale.
- Step 3. Build the combination of decision matrix. Then compute the crisp values.
- Step 4. Compute the criteria weights by normalizing the crisp values.
- Step 5. Apply the steps of the TOPSIS method under the plithogenic.
- Step 6. Normalize the decision matrix.
- Step 7. Build the weighted normalized decision matrix.

- Step 8. Find the distance of every alternative from positive and negative ideal solution.
- Step 9. Compute the closeness coefficient.
- Rank the alternatives.

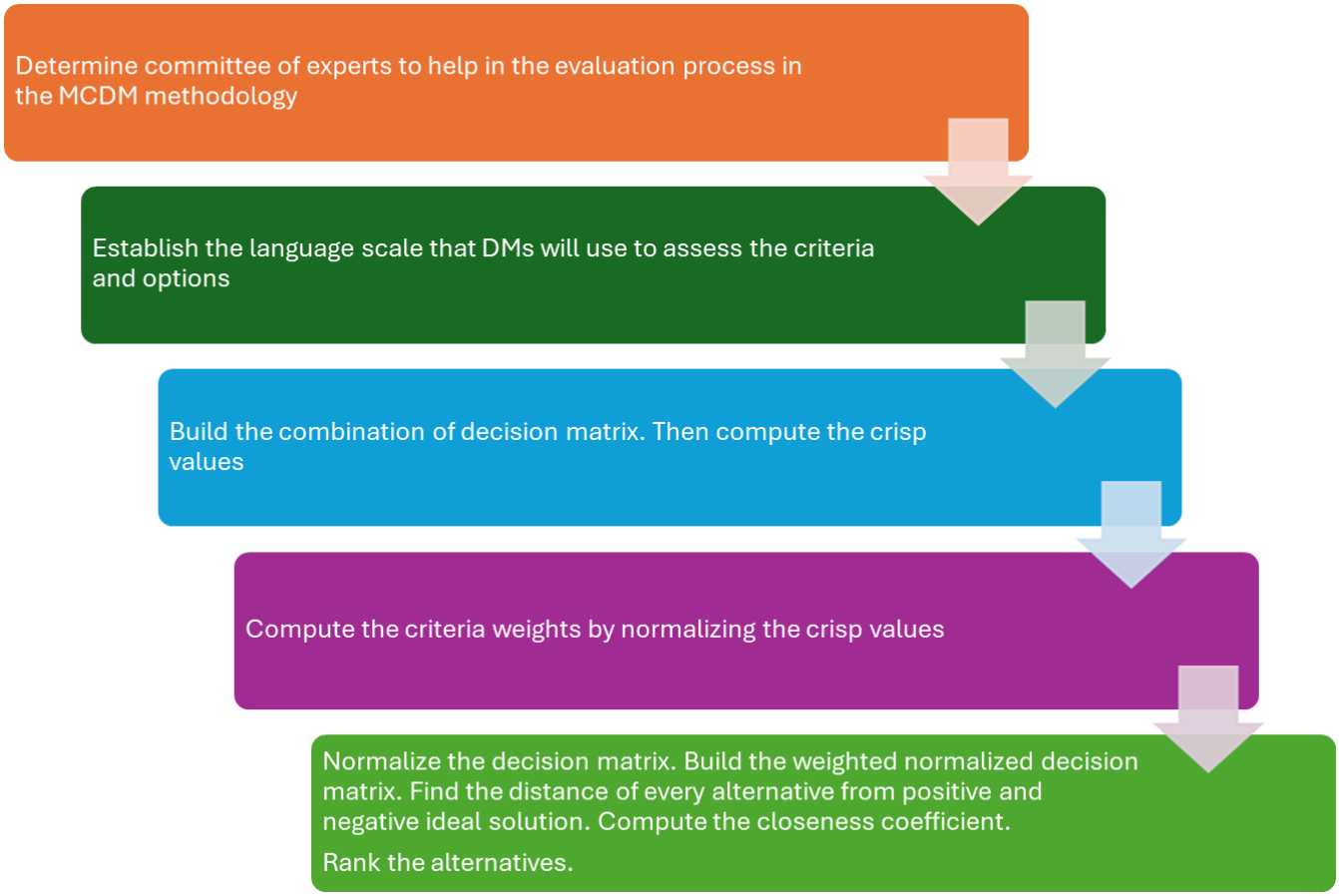
Determine committee of experts to help in the evaluation process in the MCDM methodology
Establish the language scale that DMs will use to assess the criteria and options
Build the combination of decision matrix. Then compute the crisp values
Compute the criteria weights by normalizing the crisp values
Normalize the decision matrix. Build the weighted normalized decision matrix. Find the distance of every alternative from positive and negative ideal solution. Compute the closeness coefficient.
Rank the alternatives.

Figure 1. Steps of the proposed model.

4. Application of proposed approach and discussion

This section shows the criteria and alternatives used in this study. Three experts collected eight criteria and eleven alternatives as shown in Figure 2. Three experts have experience in field of teacher quality.

Collaboration with Industry
Teaching Skills and Pedagogical Ability
Research and Innovation in Teaching
Professional Knowledge and Industry Expertise
Evaluation of teacher quality
Student Mentorship and Career Guidance
Curriculum Design and Development
Practical Training and Technical Skills
Continuous Professional Development

Figure 2. Aspects of evaluation of this study.

Table 1. Judgment of experts.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)
A_2	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_3	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_4	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_5	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_6	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_7	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_8	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)
A_9	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_{10}	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)

Table 2. Judgment of experts.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_2	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)
A_3	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A_4	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A_5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_6	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A_7	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)
A_8	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_9	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)

A10	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A11	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)

Table 3. Judgment of experts

	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
A1	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A2	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A3	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A4	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A5	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A6	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)
A7	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A8	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A9	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A10	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)
A11	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)

5. Results and Discussion

Applying the proposed model on application of eight criteria eleven alternatives in this study as following:

- The objective of this study is determining criteria and alternatives.
- Three decision makers evaluate the criteria and alternatives as shown in Tables 1-3.
- We combine the decision matrix with the plithogenic operators.
- We convert the plithogenic numbers into crisp values.
- Then we normalize the crips values to compute the criteria weights as shown in Figure 3.

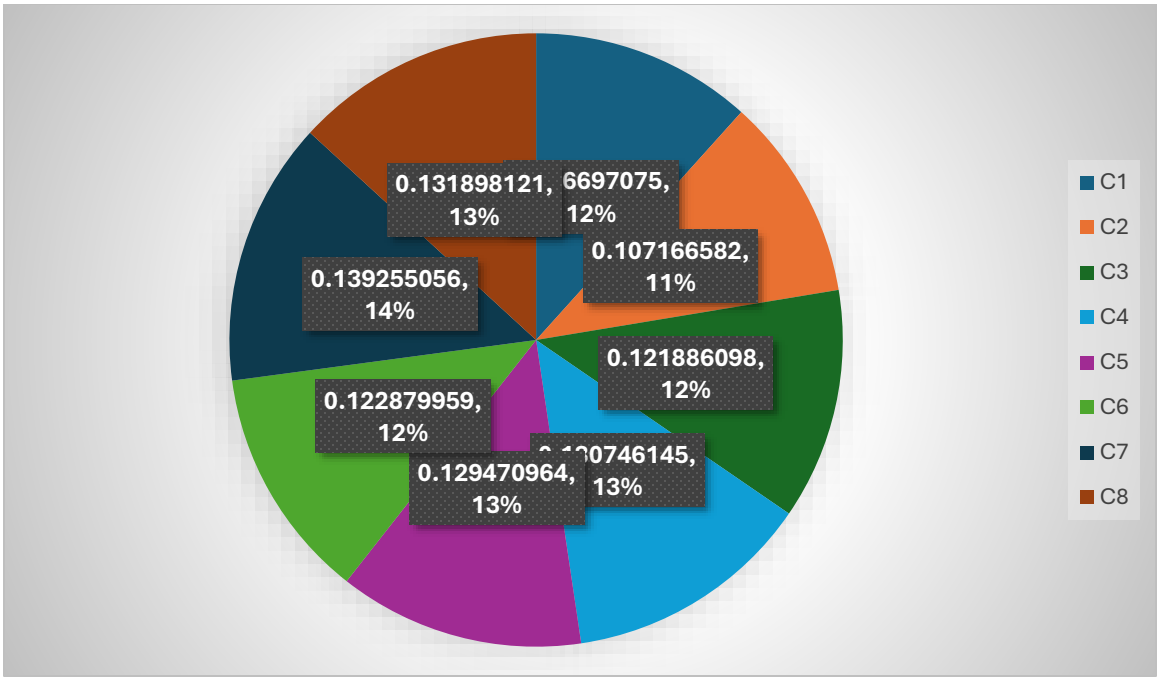

Figure 3. The criteria weights.

- ❖ We applied the steps of the TOPSIS method to rank the alternatives. We normalize the decision matrix as shown in Table 4.
- ❖ We compute the weighted normalized decision matrix as shown in Table 5.
- ❖ Then we compute the distance of every alternative from positive and negative ideal solution.
- ❖ Then we compute the closeness values as shown in Table 6.
- ❖ Then we rank the alternatives.

Table 4. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.289932624	0.128771	0.283297	0.226486	0.255382	0.299311	0.297757	0.34313
A_2	0.371820062	0.248181	0.268702	0.35278	0.394396	0.27096	0.337874	0.28229
A_3	0.344631861	0.215666	0.328476	0.322738	0.164622	0.406347	0.213534	0.282466
A_4	0.265621852	0.456793	0.383795	0.317615	0.312282	0.210528	0.315208	0.162682
A_5	0.275419149	0.336469	0.211403	0.296249	0.164622	0.413248	0.279118	0.359101
A_6	0.364830013	0.283421	0.344275	0.340964	0.363384	0.120945	0.369877	0.237784
A_7	0.359622568	0.407699	0.276341	0.211889	0.24062	0.214231	0.297757	0.25633
A_8	0.254112133	0.336469	0.317011	0.283789	0.259388	0.275517	0.272618	0.221097
A_9	0.286339346	0.363307	0.378631	0.311437	0.326421	0.293742	0.276498	0.304027
A_{10}	0.31553473	0.191284	0.324529	0.352471	0.377485	0.341345	0.3122	0.389747
A_{11}	0.06286833	0.15844	0.059921	0.261779	0.34819	0.34285	0.317197	0.389747

Table 5. The weighted normalized matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.033834289	0.0138	0.03453	0.029612	0.033065	0.036779	0.041464	0.045258
A_2	0.043390314	0.026597	0.032751	0.046125	0.051063	0.033296	0.047051	0.037233
A_3	0.04021753	0.023112	0.040037	0.042197	0.021314	0.049932	0.029736	0.037257
A_4	0.030997293	0.048953	0.046779	0.041527	0.040431	0.02587	0.043894	0.021457
A_5	0.032140609	0.036058	0.025767	0.038733	0.021314	0.05078	0.038869	0.047365
A_6	0.042574595	0.030373	0.041962	0.04458	0.047048	0.014862	0.051507	0.031363
A_7	0.041966902	0.043692	0.033682	0.027704	0.031153	0.026325	0.041464	0.033809
A_8	0.029654143	0.036058	0.038639	0.037104	0.033583	0.033855	0.037963	0.029162
A_9	0.033414964	0.038934	0.04615	0.040719	0.042262	0.036095	0.038504	0.040101
A_{10}	0.03682198	0.020499	0.039556	0.046084	0.048873	0.041944	0.043475	0.051407
A_{11}	0.00733655	0.016979	0.007304	0.034227	0.04508	0.042129	0.044171	0.051407

Table 6. The rank of alternatives.

	Closeness values	**Rank**
A_1	0.049075257	10
A_2	0.034972031	3
A_3	0.047944297	9
A_4	0.043164864	6
A_5	0.043012262	5
A_6	0.04560096	8
A_7	0.044108413	7
A_8	0.042104851	4
A_9	0.02862716	1
A_{10}	0.032439267	2
A_{11}	0.064703217	11

6. Conclusions

This study proposed a MCDM methodology to evaluate the teacher quality. We used the TOPSIS method to rank the alternatives. The TOPSIS method is integrated with the plithogenic number to deal with vague and uncertainty information. Eight criteria and eleven alternatives are gathered from previous studies. These data are evaluated by the three experts. We compute the criteria weights and rank the alternatives. We used plithogenic operators to combine the plithogenic numbers into single matrix. We compute the closeness value to rank the alternatives. The alternatives are ranked based on the highest value in the closeness values. The results show alternative 11 is the best and alternative 9 is the worst.

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EDAS approach for Psychological Education Effectiveness Evaluation of College Students based on Plithogenic Sets

Kaiyun Luo*

Wuxi Vocational College of Science and Technology, Wuxi, 214028, Jiangsu, China

*Corresponding author, E-mail: luokaiyun0125@163.com

Abstract: This study proposes a framework of using the plithogenic sets and Evaluation based on Distance from Average Solution (EDAS) method for psychological education effectiveness evaluation. The uncertainty, vague and inconsistent information are overcoming by using plithogenic numbers and computing the contradiction degrees of attribute values. This study uses the plithogenic multi-criteria decision making (MCDM) methodology to deal with various criteria. The criteria weights are computed and the rank of colleges. This study uses ten criteria and 6 colleges to be ranked and select the optimal one. The results show the Mental Health Improvement criterion has the highest importance and the Program Accessibility and Inclusivity criterion has the lowest importance. We conduct a sensitivity analysis to show the different ranks of alternatives under different weights. The results show the optimal alternative is stable under different weights.

Keywords: Plithogenic Sets; Multi-Criteria Decision Making; EDAS Method; Psychological Education.

1. Introduction

The study of human learning and memory, particularly in educational environments like classrooms, is known as educational psychology. This covers learning processes that are social, emotional, and cognitive. Teaching, testing and assessment techniques, psychometrics, classroom or learning environments, learning, social and behavioral issues that could hinder learning, and the use of technology in education are some possible areas of attention. Graduates find jobs in government organizations, the testing business, educational systems, research institutes, and the private sector as academics, education specialists, learning analyzers, and program assessors[1], [2].

Classroom management strategies, programs, curricula, and lesson planning can all be impacted by educational psychology. Teachers can, for instance, apply ideas from education psychology to comprehend and deal with the ways that quickly evolving technology both support and undermine their students' learning. Furthermore, educational psychologists are crucial in informing administrators, parents, and teachers on the best ways to support students who have difficulty with traditional teaching approaches[3], [4].

To enhance a child's learning results, these educational psychologists, who are psychologists, frequently work one-on-one with kids as well as in tandem with parents or guardians and educators[5], [6]. However, educational psychologists can also work as teachers, researchers, and consultants in a range of settings, such as learning centers, government research facilities, community organizations, and schools[7], [8].

Many experts distinguish five primary schools of thought—behaviorism, cognitivism, constructivism, experientialism, and social contextual learning theories—despite the fact that educational psychology programs incorporate a wide variety of theories.

The MCDM methodology is suitable for evaluating the Psychological Education Effectiveness since the quality is measured using a variety of qualitative and quantifiable criteria. Numerous statistical techniques, such as Quality Function Deployment (QFD), VIKOR, TOPSIS, MABAC, and EDAS are available for evaluating the Psychological Education Effectiveness. Because of these methods, EDAS is a useful tool in MCDM, especially when experts are unsure of how to express preferences throughout the decision-making process[9], [10].

In 2015, Keshavarz Ghorabaee, Zavadskas, Olfat, and Turskis presented the EDAS approach. This approach, which determines the optimum option by calculating each alternative's distance from the ideal value, is highly useful in situations where the traits are contradictory. Decision-making difficulties are evaluated using the EDAS approach[11], [12].

This subject should be tackled in a vagueness and plithogenic framework because the insights into evaluating the Psychological Education Effectiveness are derived from the perspectives of different experts and from varied semantic expressions. As a collection of factors or elements with a wide range of qualities, Smarandache introduced the plithogenic set as a generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets[13], [14]. According to certain specified criteria, an attribute value v has a

corresponding (fuzzy, intuitionistic fuzzy, or neutrosophic) degree of appurtenance d(x, v) of the element x to the set P[15], [16].

Therefore, the main goal of this study is to propose a group plithogenic established EDAS technique to determine the significance weights of evaluating standards and to use the contradiction degree.

The rest of this study is organized as follows: section 2 shows the proposed methodology. Section 3 shows the results of the proposed methodology. Section 4 shows the discussions and sensitivity analysis. Section 5 shows the conclusions.

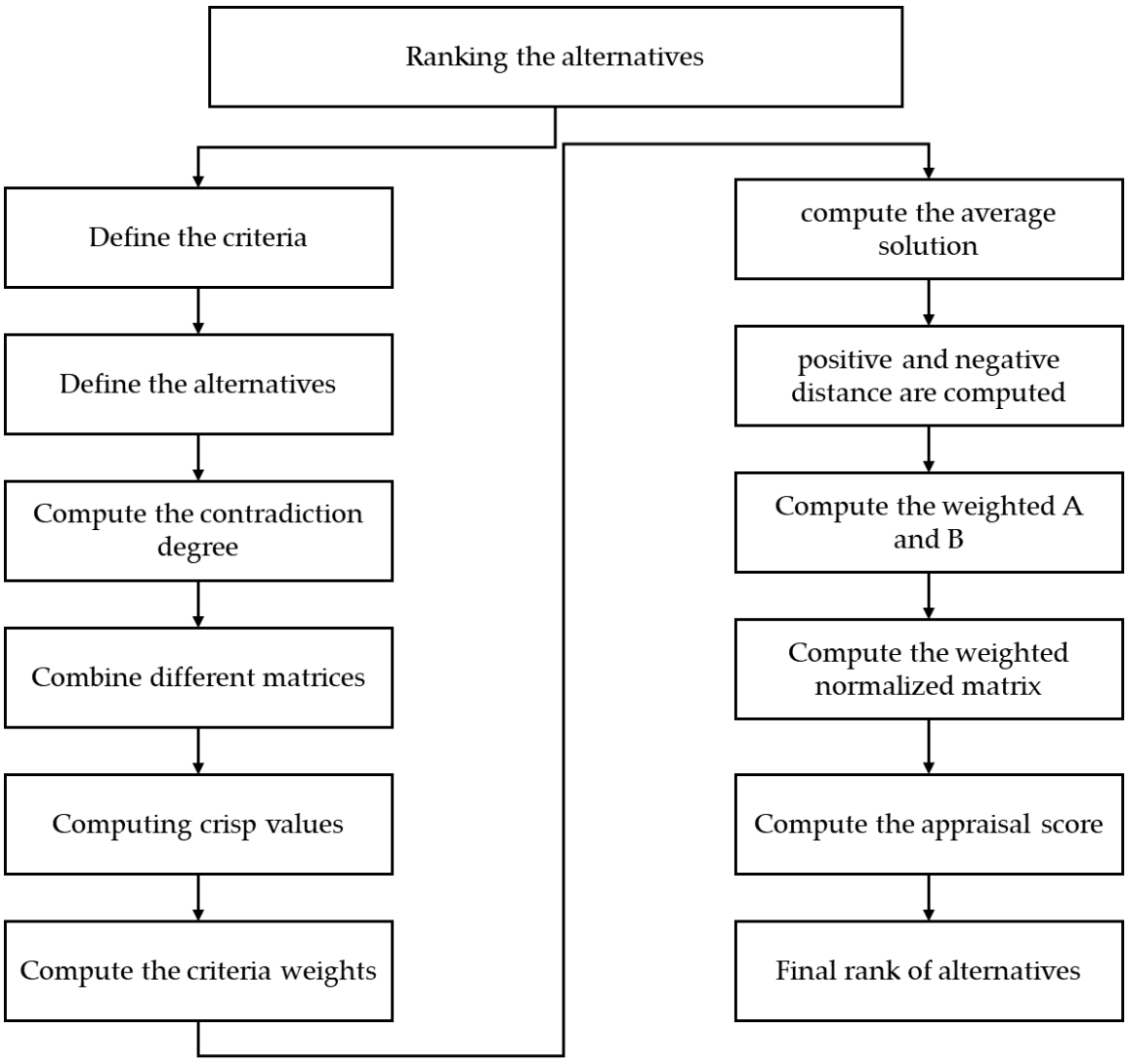
Ranking the alternatives
Define the criteria
Define the alternatives
Compute the contradiction degree
Combine different matrices
Computing crisp values
Compute the criteria weights
compute the average solution
positive and negative distance are computed
Compute the weighted A and B
Compute the weighted normalized matrix
Compute the appraisal score
Final rank of alternatives

Figure 1. The proposed methodology.

2. The Proposed Model

In this section, the producers of the suggested EDAS methods under plithogenic framework are discussed in detail for ranking the alternatives and computing the criteria weights[17], [18]. Figure 1 shows the framework of this study. The steps of the methodology include: compute the problem filed by opinions of experts and decision makers, determine the linguistic terms, define the criteria, define the alternatives, compute the plithogenic criteria weights, rank the alternatives. The whole plithogenic MCDM problem is shown as:

$$D = \begin{matrix} contradistiondegrees \\ d_1 \\ d_n \end{matrix} \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix}; i = 1, \dots, m; j = 1, \dots, n \quad (1)$$

Where $r_{ij} = (T, I, F)$ is a linguistic expression by plithogenic numbers, T,F average the truth and falsehood of plithogenic number and I is the indeterminacy value.

2.1 Determine suitable linguistic expression

This part defines the linguistic terms to evaluate the criteria and alternatives by a set of experts and decision makers. The linguistic terms are shown in Table 1.

Table 1. Linguistic terms.

Terms	*The plithogenic numbers*
Low Medium	(0.10, 0.70, 0.80)
Medium	(0.30, 0.40, 0.80)
High Medium	(0.50, 0.40, 0.60)
Very high Medium	(0.70, 0.30, 0.10)
Absolutely Medium	(0.90, 0.10, 0.10)

2.2 Determine the criteria for evaluating alternatives

This study defines ten criteria to evaluate the alternatives. These criteria are evaluated from the previous studies and opinions of experts and decision makers. Table 2 shows a set of criteria.

Table 2. Define of criteria.

Criteria	*Criteria names*
C_1	Participation and Engagement in Programs
C_2	Coping Mechanisms
C_3	Academic Performance and Focus
C_4	Emotional Regulation Skills
C_5	Mental Health Improvement
C_6	Social Interaction and Communication
C_7	Self-awareness and Reflection
C_8	Faculty Feedback

C_9	Program Accessibility and Inclusivity
C_{10}	Reduction in Negative Behaviors

2.3 Determine the plithogenic weights of criteria

A set of experts can evaluate the criteria in this study to compute the weights of criteria. This study uses the way of determining contradiction degree between all criteria. Then we compute the contradiction degree between all experts in combined matrix of every criteria value as:

$$W_1^e = (T^e, I^e, F^e) = \begin{pmatrix} (1-c_j)[d_A(R,V_T)\wedge_F d_B(R,V_T)] + [d_A(R,V_T)\vee_F d_B(R,V_T)], \\ \frac{1}{2}[d_A(R,V_I)\wedge_F d_B(R,V_I)] + [d_A(R,V_I)\vee_F d_B(R,V_I)], \\ (1-c_j)[d_A(R,V_F)\wedge_F d_B(R,V_F)] + [d_A(R,V_F)\vee_F d_B(R,V_F)] \end{pmatrix}$$

(2)

$$S(w) = \frac{2+T-I-F}{3} \tag{3}$$

2.4 Build the performance rating matrix

Experts are evaluate the criteria and alternatives by building the decision matrix by their opinions and using the linguistic terms. This study uses contradiction degree and computing the plithogenic performance value based on a set of experts. Then computing crisp values.

$$R_{ij}^e = \left(T_{ij}^e, F_{ij}^e, F_{ij}^e\right) \tag{4}$$

Then we compute the average solution

$$V_j = \frac{\sum_{i=1}^{m} r_{ij}}{m} \tag{5}$$

2.5 The positive and negative distance from average solution

The positive and negative distance are computed for the positive and negative criteria as:

$$A_{ij} = \frac{\max\left(0,(r_{ij}-V_j)\right)}{V_j} \tag{6}$$

$$B_{ij} = \frac{\max\left(0,(V_j-r_{ij})\right)}{V_j} \tag{7}$$

For cost criteria

$$A_{ij} = \frac{\max\left(0,(V_j-r_{ij})\right)}{V_j} \tag{8}$$

$$B_{ij} = \frac{\max\left(0,(r_{ij}-V_j)\right)}{V_j} \tag{9}$$

2.6 Compute the weighted A and B

Weights of criteria are multiplied by the A and B values such as:

$$SA_i = \sum_{j=1}^{n} A_{ij} w_j \tag{10}$$

$$SB_i = \sum_{j=1}^{n} B_{ij} w_j \tag{11}$$

2.7 Compute the weighted normalized matrix

The weighted normalized matrix is computed such as:

$$NSA_i = \frac{SA_i}{\max SA_i} \tag{12}$$

$$NSB_i = \frac{SB_i}{\max SB_i} \tag{13}$$

2.8 Compute the appraisal score

The appraisal score is computed as:

$$S_i = \frac{1}{2} * (NSA_i + NSB_i) \tag{14}$$

2.9 Final rank of alternatives

3. Results Statistical Analysis

This study applied the proposed methodology of 6 colleges. The main aim is to select the optimal college from different colleges. Four experts are invited to evaluate the criteria and alternatives in this study. These experts are evaluated by 6 colleges and ten criteria. The proposed methodology combines the plithogenic set with the EDAS method to deal with infirmary serving appraisal problem.

3.1 Computing the criteria weights

The criteria weights are evaluated by the set of experts. We compute the criteria weights by using the plithogenic MCDM methodology. Let four experts evaluate the criteria. Table 3 shows the opinions of experts for ten criteria.

Table 3. Computing weights of criteria by experts.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
E_1	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
E_2	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)

E_3	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
E_4	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)

Obtain the contradiction degree for every criterion which is computed by set of experts and decision makers. We used the Equi-distant in the contradiction degree and computing the dominant criteria value is 0.

Build the matrix between the four experts for all criteria using the linguistic terms and computing the contradiction degree.

Compute the contradiction degree for dominant criteria. Then we combine the matrix of opinions.

After we combined the values of matrix, we compute crip values. Then we normalize these values to compute the criteria weights as shown in Table 4.

Table 4. The criteria weights.

Criteria	*Crisp*	*Normalized weight*
C_1	0.313367	0.042526
C_2	0.5459	0.074082
C_3	1.002633	0.136063
C_4	1.319933	0.179123
C_5	1.581267	0.214587
C_6	0.660767	0.08967
C_7	1.121633	0.152212
C_8	0.196767	0.026702
C_9	0.196767	0.026702
C_{10}	0.429833	0.058331

We built the decision matrix between the criteria and alternatives by using the plithogenic numbers as shown in Tables 5-8. And then combine all matrices into single matrix.

Table 5. Ordering alternatives by expert1.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_8	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_{10}	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)

Table 6. Ordering alternatives by expert2.

	A_1	A_2	A_3	A_4	A_5	A_6

C_1	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)
C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_8	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_{10}	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)

Table 7. Ordering alternatives by expert$_3$.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_3	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_8	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_{10}	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)

Table 8. Ordering alternatives by expert$_4$.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_7	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_8	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_{10}	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)

Convert the combined values by all experts into crisp values. Then we compute the average solution. Then we compute the positive and negative distance from the average solution as shown in Tables 9-10 . All criteria in this study are positive.

Table 9. The positive distances.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0	0.681084	0.027055	0.32608	0.284786	0
C_2	0	0.585374	0	0.276501	0	0
C_3	0	0	0	0.11657	0.423476	0.280191
C_4	0.250417	0.062561	0	0	0.272237	0
C_5	0.559776	0	0	0	0.17511	0.559776
C_6	0	0	0	0.411481	0.32144	0.479442
C_7	0.245014	0	0.397824	0.308654	0	0.112924
C_8	0	0.563506	0.833068	0	0	0
C_9	0	0.432833	0.529264	0	0.367013	0
C_{10}	0	0.044472	0.224548	0.158207	0	0.23796

Table 10. The negative distances.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.581676	0.378726	0.027556	0	0	0.259382
C_2	0	0	0.470537	0	0.690892	0.779455
C_3	0	0.010976	0.322144	0.387928	0.362681	0.173526
C_4	0	0	0	0.075118	0.241089	0
C_5	0	0.13533	0	0	0	0
C_6	0.737329	0.336845	0	0.122169	0	0
C_7	0.581676	0.378726	0.027556	0	0	0.259382
C_8	0	0	0.470537	0	0.690892	0.779455
C_9	0	0.010976	0.322144	0.387928	0.362681	0.173526
C_{10}	0	0	0	0.075118	0.241089	0

Then we compute the weighted positive and negative distance as shown in Tables 11-12.

Table 11. The weight positive distances.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0	0.028964	0.001151	0.013867	0.012111	0
C_2	0	0.043366	0	0.020484	0	0
C_3	0	0	0	0.015861	0.05762	0.038124
C_4	0.044855	0.011206	0	0	0.048764	0
C_5	0.120121	0	0	0	0.037576	0.120121
C_6	0	0	0	0.036898	0.028824	0.042992
C_7	0.037294	0	0.060554	0.046981	0	0.017188
C_8	0	0.015047	0.022245	0	0	0
C_9	0	0.011558	0.014133	0	0.0098	0
C_{10}	0	0.002594	0.013098	0.009228	0	0.01388

Table 12. The weight negative distances.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.024736	0	0	0	0	0.031355
C_2	0.028057	0	0.000813	0	0.010025	0.024954
C_3	0.003749	0.064023	0.043832	0	0	0
C_4	0	0	0.069487	0.013455	0	0.021883
C_5	0	0.148257	0.077827	0.051735	0	0
C_6	0.023259	0.069894	0.01556	0	0	0
C_7	0	0.04305	0	0	0.118968	0
C_8	0.019326	0	0	0.011178	0.004857	0.001931
C_9	0.020999	0	0	0.00468	0	0.009812
C_{10}	0.034816	0	0	0	0.003986	0

We compute the weighted normalized positive and negative matrix. Then we computed the appraisal score. Then we rank the alternatives as shown in Table 13.

Table 13. The rank of alternatives.

	Weighted normalized positive distance	*Weighted normalized negative distance*	*Appraisal score*	*Rank*
A_1	0.202271	0.154941	0.178606	5
A_2	0.112734	0.325223	0.218979	6
A_3	0.11118	0.207519	0.159349	2
A_4	0.143318	0.081048	0.112183	1

A_5	0.194694	0.137836	0.166265	4
A_6	0.232305	0.089936	0.16112	3

This methodology shows the alternative 2 is the best and alternative 4 is the worst.

4. Discussion

This part shows the sensitivity analysis to show the different rank of alternatives. This study proposed 11 cases in changing the criteria weights as shown in Table 14. Then we applied the proposed methodology into different weights of criteria. Figure 2 shows the different of alternatives.

Cases 5,9,10,11 show the alternative 1 is the best and alternative 4 is the worst. All other cases show the alternative 2 is the best and alternative 4 is the worst. All cases show the alternative 4 is the worst.

Table 14. The different criteria weights.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0.1	0.2	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889
C_2	0.1	0.088889	0.2	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889
C_3	0.1	0.088889	0.088889	0.2	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889
C_4	0.1	0.088889	0.088889	0.088889	0.2	0.088889	0.088889	0.088889	0.088889	0.088889
C_5	0.1	0.088889	0.088889	0.088889	0.088889	0.2	0.088889	0.088889	0.088889	0.088889
C_6	0.1	0.088889	0.088889	0.088889	0.088889	0.088889	0.2	0.088889	0.088889	0.088889
C_7	0.1	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.2	0.088889	0.088889
C_8	0.1	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.2	0.088889
C_9	0.1	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.2
C_{10}	0.1	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889	0.088889

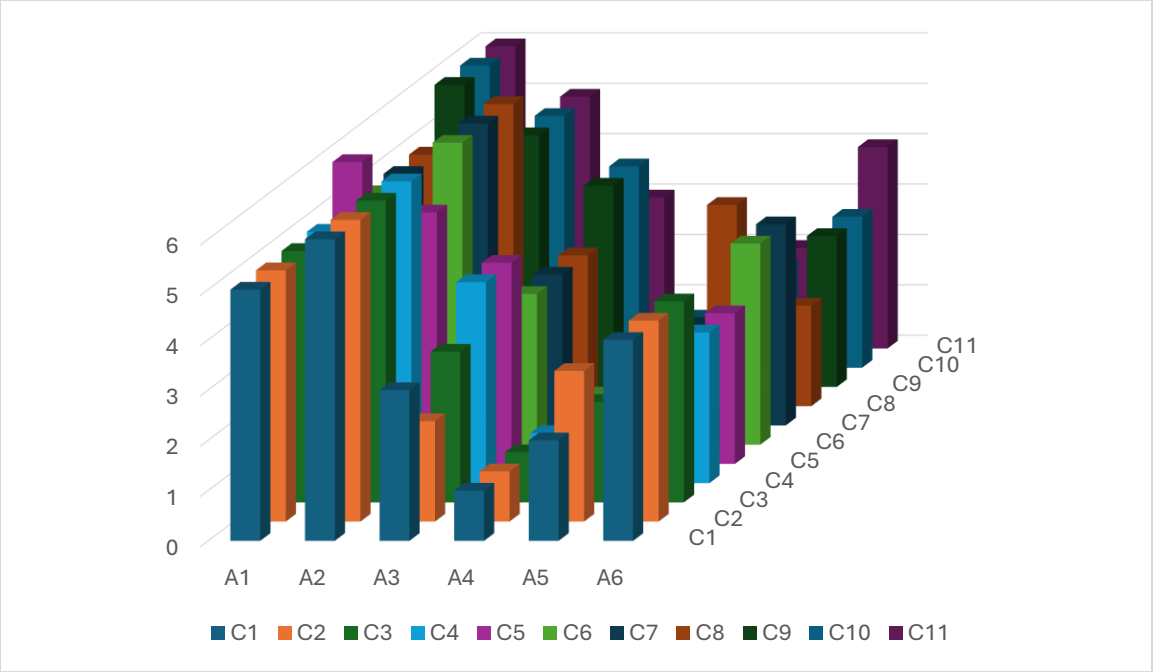

Figure 2. The different rank of alternatives.

5. Conclusions

This study proposed a uncertainty methodology to compute the criteria weights and rank the alternatives for psychological education effectiveness. We used the plithogenic sets to deal with uncertainty and vague data. The EDAS method is used to rank the alternatives. Four experts are invited to evaluate the criteria and alternatives. Ten criteria and six alternatives are used in this study. We used plithogenic numbers to evaluate the criteria and alternatives. Then we combined then by using the operations of plithogenic. Then we obtain crisp values. The criteria show criterion 5 has the highest importance and criterion 9 has the lowest importance. Then we applied the steps of the EDAS method to compute the positive and negative distance from all alternatives. Then we compute the weighted normalized of this distance. The results show the alternative 2 is the best and alternative 4 is the worst. We applied the sensitivity analysis to show the stability of the ranks. The results show the rank of alternatives is stable in different cases.

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Enhanced Approach for Japanese Teaching Quality Evaluation in Higher Education under Plithogenic Sets

Yifan Liang

Zhejiang Yuexiu University, Shaoxing, 312000, Zhejiang, China

*Corresponding author, E-mail: Gogowadgogortd@163.com

Abstract: Japanese language instruction has drawn more attention in the twenty-first century, a time of frequent international interactions and the rapid advancement of knowledge. Colleges and universities are now concentrating on raising the standard of Japanese education going forward as a result. To improve teaching quality, we must strengthen the entire management of teaching quality, particularly the evaluation of instructors' instruction. However, it is challenging to translate the evaluation results into a mathematical analytical formula because there are several elements that affect the quality of instruction, and the weight of each factor changes. We proposed a multi-criteria decision making (MCDM) methodology for evaluation the Japanese teaching quality. We proposed the MCDM method under the Plithogenic sets to deal with vague and uncertainty information. We applied the steps of the MULTIMOORA method under the Plithogenic sets to rank the alternatives. This study collects seven criteria and eight alternatives to be ranked. The criteria weights are computed. The comparative analysis is performed to show the effectiveness of the proposed methodology compared with other MCDM methods. The results show the proposed methodology is effective.

Keywords: Japanese Teaching Quality Evaluation; Higher Education; MCDM; MULTIMOORA Method; Evaluation Method.

1. Introduction

The 21st century is a time of frequent worldwide interactions and the quick creation of information. Foreign languages are now an essential tool for worldwide communication because of the impact of global globalization. Given this context, there has been a growing

interest in teaching Japanese, and the curriculum's overarching objective is to help students become more proficient in the language[1], [2].

While there is few research on the Japanese teaching quality evaluation system, teachers are expected to develop students' cultural literacy, linguistic knowledge, emotional attitude, language abilities, and learning techniques during the teaching session. However, the scientific and logical nature of the educational evaluation system is one of the key elements limiting the advancement of curriculum reform[3], [4].

The primary means by which schools accomplish their educational objectives is through classroom instruction. School administrators and management can better comprehend the extent to which teaching objectives are being met, gain a thorough and accurate understanding of the school's teaching work, and enhance teaching quality by evaluating teachers' classroom instruction. The evaluation of school teaching level is a quite complicated process since the level of teaching job greatly influences the level of people training[5].

In this case, faculty members in higher education might have their yearly performance modeled and assessed using MCDM. This method offers a comparison rating of the faculty members in a fuzzy environment, which is helpful for evaluating individual performance and promoting faculty members[6], [7]. Fuzzy sets, soft sets, and their hybrid models have been widely combined with MCDM techniques to address ambiguity and uncertainty in decision-making processes[8], [9].

Smarandache created plithogenic sets as a generalization of intuitionistic, fuzzy, and crisp sets and advanced the concept of plithogeny. With set P, dominating attribute a, set of attribute values V, degree of appurtenance d, and degree of contradiction c, the plithogenic sets are represented as a quintuple of the form (P, a, V, d, c)[10], [11]. The traits and attribute values that help resolve complex decision-making processes are used to characterize lithogenic sets[12], [13].

Brauers and Zavadskas combined the Reference Point Approach and the Ratio System to introduce MOORA (Multi-Objective Optimization based on a Ratio Analysis) in 2006. To obtain a final integrative ranking based on the outcomes of these triple subordinate methods, Brauers and Zavadskas enhanced MOORA to MULTIMOORA (Multi-Objective Optimization based on Ratio Analysis plus the full Multiplicative form) by adding the Full Multiplicative Form and using Dominance Theory[14], [15]. The first group of MCDM techniques, known as Value Measurement Methods, includes the Ratio System

and Full Multiplicative Form, while the second group, known as Goal or Reference Level Models, includes the Reference Point Approach[16], [17].

The following paper is organized so that the reader can understand the basic ideas behind the suggested concepts of proposed method. A thorough description of each of these ideas is given in Section 2. Section 3 presents comparative analysis. Section 4 of the article introduces results and discussion. Section 5 brings the conclusion article.

2. Japanese Teaching Quality Evaluation in Higher Education (JTQE) problem

In this section, we develop an application to rank the problem of JTQE. We applied the proposed method for ranking the alternatives and computing the criteria weights. We integrated the plithogenic set with the MULTIMOORA method to rank the alternatives.

2.1 Flowchart of JTQE

The flowchart of the JTQE problem is given in Figure 1.

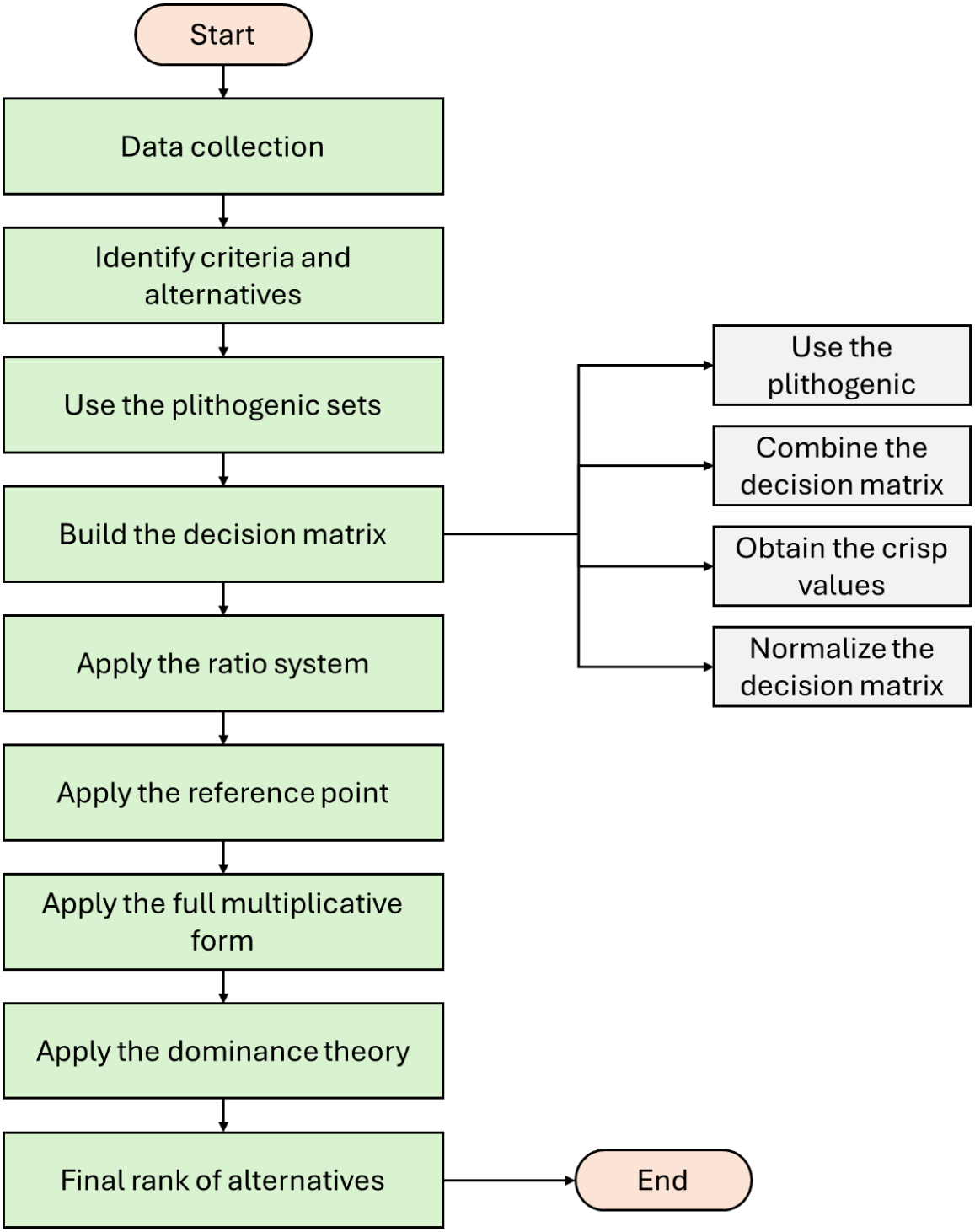

Figure 1. The flowchart of JTQE problem.

2.2 Algorithm of MULTIMOORA Method

MULTIMOORA uses three subordinate ranking techniques—the Ratio System, Reference Point Approach, and Full Multiplicative Form—as well as the vector normalizing technique to produce similar ratings. Since each of the three ranking techniques has advantages and disadvantages, MULTIMOORA employs many strategies. To make the MULTIMOORA approach easier to understand, we describe these three subordinate ranking methods in this section.[14], [15]

Creating a decision matrix and weight vector is the initial step in solving an MCDM problem. As a result, the decision matrix for MULTIMOORA is initially created using the ratings x_{ij} of the problem's m candidate options in relation to n criteria.

Step 1. Build the decision matrix.

The decision matrix is built by a set of experts and decision makers to evaluate the criteria and alternatives.

Step 2. Combine the plithogenic numbers.

We apply the plithogenic operator to combine these numbers.

Step 3. Apply the plithogenic score function to obtain crisp values.

$$S(A) = \frac{(2+T-I-F)}{3} \tag{1}$$

Step 4. Normalize the decision matrix.

Prior to being included in an MCDM model, the ratings of alternatives on the problem's many criteria should be normalized because they may have different dimensions. MCDM techniques have used a variety of normalization strategies. The most reliable option for use in MULTIMOORA is normalization. The following is a representation of vector normalization:

$$y_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m}(x_{ij})^2}} \tag{2}$$

The Ratio System, a completely compensating paradigm, is helpful in situations where the problem involves "independent" criteria. The Full Multiplicative Form, an incompletely compensatory model, is a useful tool in situations where "dependent" requirements are present. As a non-compensatory model, the Reference Point Approach provides a "conservative" way to compare the Full Multiplicative Form and the Ratio System. Although the Reference Point Approach does not permit it, the Ratio System and Full Multiplicative Form both offer the chance to make up for an alternative's subpar

performance on one criterion by its performance on other criteria (the amount of compensation associated with the two approaches is not equal).

Ratio system

The ratio system is a fully compensating model that employs the arithmetic weighted aggregation operator. It implies that the same degree of high values could fully offset small, normalized values of an alternative. Stated differently, an alternative that performs poorly on certain criteria and well on the other criteria can be replaced by one that performs moderately on all criteria. The weighted normalized ratings are added for beneficial criteria and subtracted for non-beneficial criteria in the following manner to calculate the Ratio System's utility:

$$u_i = \sum_{j=1}^{g} w_j y_{ij} - \sum_{j=g+1}^{n} w_j y_{ij} \tag{3}$$

Reference point

$$r_j = \left\{\max y_{ij}, j \le g; \min y_{ij}, j > g\right\} \tag{4}$$

According to the Reference Point Approach, the option with the lowest value across all criteria is the best one. As a non-compensatory model, this method first identifies the alternative ratings that perform the poorest across all criteria, and then it chooses the best overall value (i.e., the lowest value) from these worst ratings. The Min-Max Metric is the foundation of the Reference Point Approach. The general theory of Murkowski Metric, the foundation of various decision analysis techniques found in literature, including Goal Programming, is where the Min-Max Metric got its start. First, the Maximal Objective Reference Point (MORP) Vector is defined as follows to determine the utility:

The distance is computed as follows:

$$d_{ij} = \left|w_j r_j - w_j y_{ij}\right| \tag{5}$$

We compute the utility reference point as:

$$z_i = \max d_{ij} \tag{6}$$

The best alternative is computed based on the lowest value in utility reference point.

Full Multiplicative Form

The utility of full multiplicative form is computed as follows:

$$e_i = \prod_{j=1}^{g} \left(y_{ij}\right)^{w_j} / \prod_{j=g+1}^{n} \left(y_{ij}\right)^{w_j} \tag{7}$$

The alternatives are ranked based on the largest value of utility degree.

2.2 Application of MULTIMOORA Method

We applied the steps of the proposed methodology to rank the alternatives and compute the criteria weights. This study collected seven criteria and eight alternatives as shown in Figure 2.

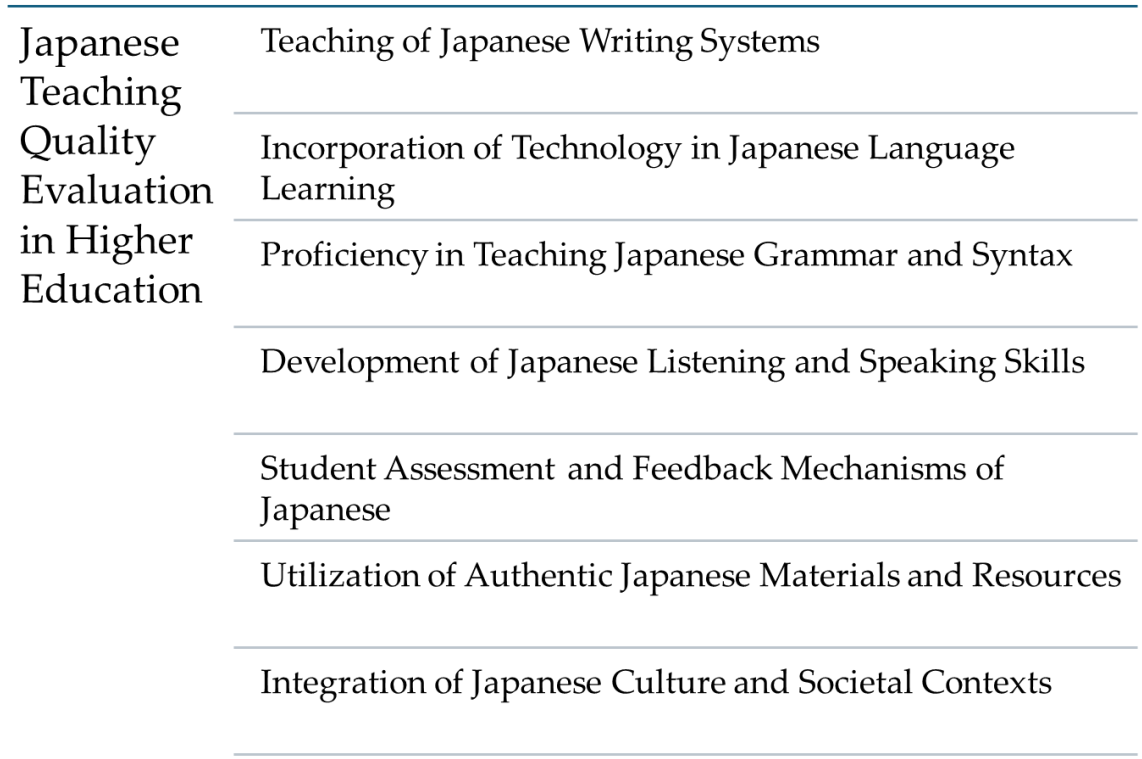
Japanese Teaching Quality Evaluation in Higher Education
Teaching of Japanese Writing Systems
Incorporation of Technology in Japanese Language Learning
Proficiency in Teaching Japanese Grammar and Syntax
Development of Japanese Listening and Speaking Skills
Student Assessment and Feedback Mechanisms of Japanese
Utilization of Authentic Japanese Materials and Resources
Integration of Japanese Culture and Societal Contexts

Figure 2. The seven criteria of this study.

Three experts are evaluated the criteria and alternatives are given in Table 1. Then we combine the plithogenic numbers into a single matrix. Then we obtain crisp values. Then we apply the steps of the MULTIMOORA method under the plithogenic sets.

Table 1. The decision table.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_2	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_3	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)

A4	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)
A5	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A6	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A7	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)
A8	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
	C1	C2	C3	C4	C5	C6	C7
A1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A2	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A3	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A4	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A6	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A7	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A8	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
	C1	C2	C3	C4	C5	C6	C7
A1	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A2	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A3	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A4	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A5	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A6	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A7	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A8	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)

Then we normalize crisp values to obtain the criteria weights. The criteria weights results show as: w1= 0.14568974, w2= 0.138605331, w3= 0.147314866, w4= 0.146706465, w5= 0.13945791, w6= 0.140668997, w7= 0.141556692.

Then we obtain the normalization matrix as shown in Table 2. Then we obtain the results of the ratio system, reference point, and full multiplicative form. Then we apply the dominance theory to obtain the final rank as shown in Figure 3.

Table 2. The normalization matrix.

	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**
A1	0.327246784	0.139104	0.32762	0.26819	0.316917	0.35284	0.383968
A2	0.455320078	0.363696	0.310741	0.417741	0.489426	0.437572	0.490106
A3	0.200552121	0.363469	0.379867	0.338942	0.387416	0.428887	0.385892
A4	0.380440264	0.397234	0.44384	0.374226	0.334131	0.20334	0.199733

A_5	0.319966372	0.372368	0.244477	0.44801	0.204288	0.472705	0.383968
A_6	0.402622286	0.306165	0.398138	0.159914	0.47491	0.164782	0.383968
A_7	0.396875407	0.440415	0.319575	0.416217	0.165551	0.320393	0.164077
A_8	0.280435283	0.363469	0.366608	0.31483	0.321887	0.32479	0.319022

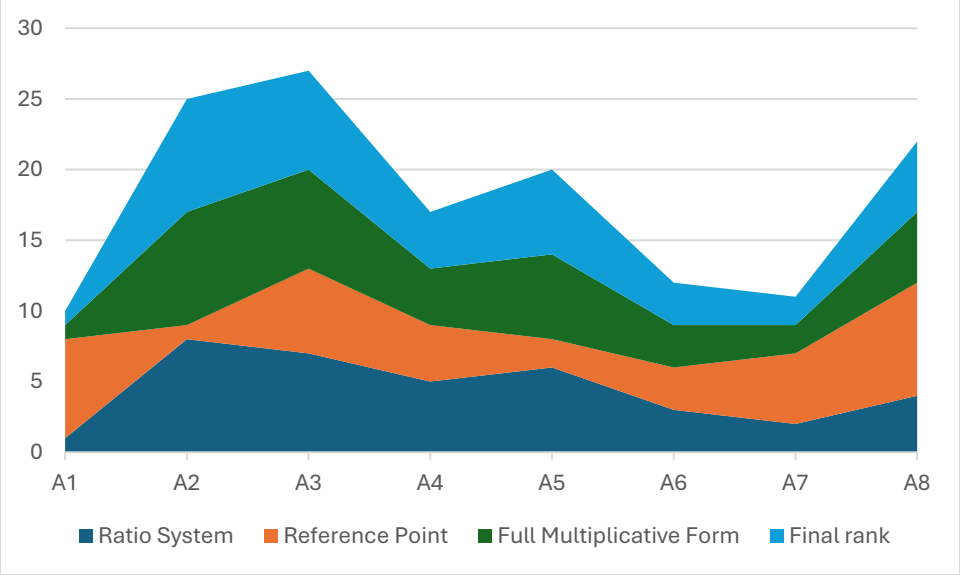

Figure 3. The final rank.

3. Comparative Analysis

This section shows the comparative analysis between the proposed methodology and other MCDM methods to show the effectiveness of the proposed methodology. The proposed methodology and other MCDM methods show the best alternative is A2 and alternative 1 is the worst. The results show the our methodology is effective compared to other MCDM methods. Figure 4 shows the comparative analysis results.

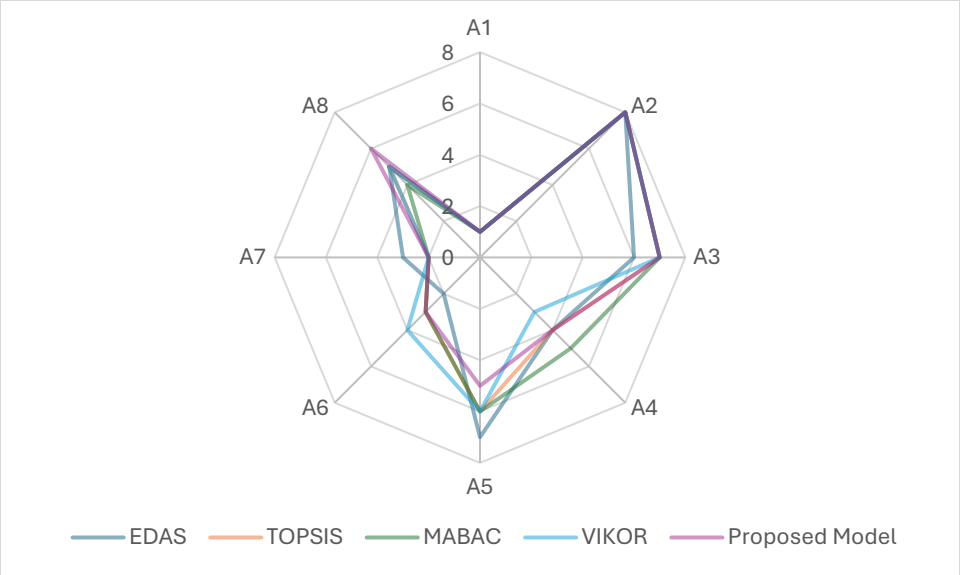

Figure 4. The rank of comparative study.

4. Results and Discussion

In the rank of alternatives, we applied three elements of the MULTIMOORA method such as ratio system, reference points and full multiplicative form. In the ratio system, we show the alternative 2 is the best and followed by the alternative 3 and alternative 5. The alternative 1 is the worst and followed by the alternative 7.

In the reference point, we show the alternative 8 is the best and followed by the alternative 1 and alternative 3. The alternative 2 is the worst and followed by the alternative 6.

In the full multiplicative form, we show the alternative 2 is the best and followed by the alternative 3 and alternative 5. The alternative 1 is the worst and followed by the alternative 7.

In the final rank, we show the alternative 2 is the best and followed by the alternative 3 and alternative 5. The alternative 1 is the worst and followed by the alternative 7.

5. Conclusions

Although many teachers are aware of the significance of implementing cultural literacy considering the ongoing progress of the new curriculum reform, they typically assume that it refers to the Japanese cultural background or certain aspects of Japanese nonverbal behavior. Some teachers do not even incorporate characteristics into cultural literacy lessons, and they are not well understood. Assessing the quality of Japanese instruction

is particularly important because teachers usually grasp certain speech act aspects as grammatical knowledge right away while teaching Japanese courses. One of the most important steps in motivating educators to raise the standard of instruction and education is assessing their effectiveness as teachers. This study used the MULTIMOORA method under the Plithogenic sets to deal with the vague and uncertainty information. Three experts and decision makers are evaluating the criteria and alternatives. They used the Plithogenic numbers to be evaluated the criteria and alternatives. We used the Plithogenic operators to combine the Plithogenic numbers. We normalize the crisp values to obtain the criteria weights. Then we applied the MULTIMOORA method under the Plithogenic sets to rank the alternatives. The results showed that the alternative 2 is the best and alternative 1 is the worst. We compared our model with other MCDM methods. The results show our model is effective.

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A Hybrid MCDM Approach for Evaluating Financial Performance Based on A Set of Business Companies

Lei Sun*, Hong Zheng

Henan University of Engineering, Zhengzhou, 451191, Henan, China
*Corresponding author, E-mail: henangcxysl@163.com; sunlei@haue.edu.cn

Abstract: This paper uses multi-criteria decision making (MCDM) to evaluate the financial performance based on a set of business companies. This study proposed a decision-making methodology for computing the criteria weights and ranking the alternatives. The decision-making methodology is integrated with plithogenic sets to deal with uncertainty information. We collected seven criteria and eleven alternatives to be ranked. We used the WSM method under the plithogenic sets to rank the alternatives. The results show the Profitability criterion has the highest importance and the Market Valuation criterion has the lowest importance. In the ranking of alternatives, we show the alternative 11 has the highest ranking of alternatives and alternative 5 has the lowest ranking of alternatives. We compared our model with other MCDM methods. The results show that our model is effective compared to other MCDM methods.

Keywords: Financial Performance; MCDM Methodology; Decision making; Uncertainty Framework.

1. Introduction

The general state of the company's finances is referred to as financial performance. Every firm uses financial assets, which can take many different forms, to support operations that lead to revenue and, eventually, profits[1], [2].

In its most basic form, financial performance can be defined as the efficiency with which the company makes money, but it encompasses much more. All the factors that affect profitability are reflected in it, both holistically as a group dynamic and independently as line items[3], [4].

Various stakeholders, including lenders, investors, and management, will evaluate a company's financial performance from different angles and with different goals. Therefore, it is impossible to gauge a company's financial performance using a single metric. To do this, several metrics are employed, both separately and in combination.

To evaluate financial performance, businesses generate a variety of financial statements. Financial statements come in a variety of ways since a company's financial activities encompass a wide range of aspects[5], [6].

Given the significance of evaluation of financial management, their performance needs to be assessed using standards that gauge how well these systems work in various scenarios. Like other decision-making issues, evaluating financial management is one of the challenges that arise while making decisions in an unpredictable environment. To account for any ambiguity in the evaluation process, this study attempts to suggest an integrated framework for assessing the financial management performance in an uncertain environment[7], [8].

Performance evaluation according to a set of criteria is regarded as an MCDM challenge. Three of the MCDM techniques will be used in this study to more precisely account for every facet of the assessment[9], [10].

The main contributions of this study are:

- Demonstrates the importance of financial management in enhancing set of business and firms.
- Assess the metrics used to gauge financial management performance.
- To improve evaluation accuracy, use the suggested framework based on a plithogenic set that considers the degree of disagreement between the criteria and alternatives.
- To create a more uniform framework, financial management is evaluated from three distinct perspectives using three MCDM techniques.

2. Methods

2.1 Plithogenic set

Smarandache's neutrosophy was expanded upon by plithogeny. A set of elements defined by a set of attributes A = {α1, α2,…, αm}, m ≥ 1 is called a plithogenic set (P, A, V, d, c). Each attribute is defined by a set of values V = {v1, v2, …, vn}, for n ≥ 1. The two characteristics that define the plithogenic set are the degree of appurtenance and the degree of contradiction. The relationship between each attribute value and the most significant attribute value (dominant) is described by the contradiction degree function c(v,D). D(x,v) is the appurtenance degree function of the element × with regard to the set of specified criteria[11], [12], [13].

$let\ a = (a_1, a_2, a_3)\ and\ b = (b_1, b_2, b_3)$ be two plithogenic numbers. Then the set of operations of plithogenic sets can be computed as:

$$\big((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\big) \wedge p\big((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n\big) = \begin{pmatrix} (a_{i1} \wedge_F b_{i1}), \\ \frac{1}{2}\Big((a_{i2} \wedge_F b_{i2}) + \frac{1}{2}(a_{i2} \vee_F b_{i2})\Big), \\ (a_{i3} \vee_F b_{i3}) \end{pmatrix} \tag{1}$$

$$\big((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\big) \vee p\big((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n\big) = \begin{pmatrix} (a_{i1} \vee_F b_{i1}), \\ \frac{1}{2}\Big((a_{i2} \wedge_F b_{i2}) + \frac{1}{2}(a_{i2} \vee_F b_{i2})\Big), \\ (a_{i3} \wedge_F b_{i3}) \end{pmatrix} \tag{2}$$

2.2 WSM Method

A methodical approach to assessment and decision-making that considers both qualitative and quantitative factors is the weighted sum model. It is frequently employed to make well-informed judgments and to impartially assess the benefits and drawbacks of various options. The weighted sum model is now widely used in a variety of fields, such as product creation, project management, and business appraisal. This approach weighs and evaluates several aspects to provide transparent and understandable decision-making[14], [15], [16].

Build the decision-matrix

The decision matrix between the criteria and alternatives.

$$T = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix}_{m \times n} ; i = 1, \dots, m; j = 1, \dots, n \tag{3}$$

Three experts used the plithogenic terms to evaluate the decision matrix. Then we replace these terms by using the Plithogenic numbers. Then we used the Plithogenic operators to combine these numbers into a single matrix.

Obtain the crisp values.

$$S(t) = \frac{(2+a_1-a_2-a_3)}{3} \tag{4}$$

Normalize crisp values.

We normalize the crisp values in the decision matrix to compute the criteria weights.

Then we compute the criteria weights.

Compute the weighted decision matrix.

$$r_{ij} = w_j * t_{ij} \tag{5}$$

Compute the sum of each row in weighted decision matrix.

$$y_i = \sum_{j=1}^{n} r_{ij} \tag{6}$$

Rank the alternatives

$$e_i = \max y_i \tag{7}$$

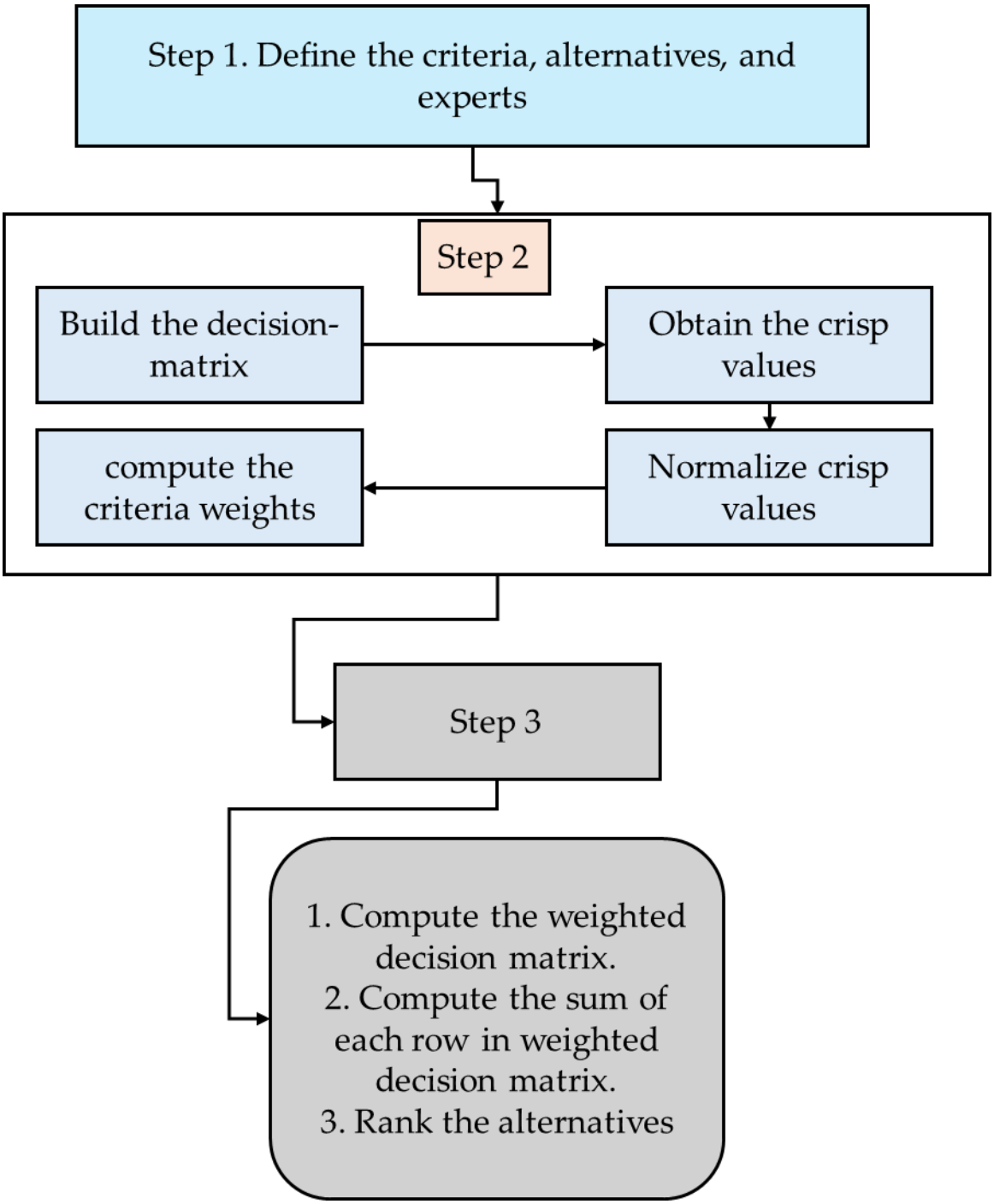
Step 1. Define the criteria, alternatives, and experts
Step 2
Build the decision-matrix
Obtain the crisp values
compute the criteria weights
Normalize crisp values
Step 3
1. Compute the weighted decision matrix.
2. Compute the sum of each row in weighted decision matrix.
3. Rank the alternatives

Figure 1. The Proposed framework steps.

3. Proposed Framework

In this study, we proposed a framework with plithogenic MCDM to evaluate the criteria and alternatives. We used the plithogenic sets to deal with uncertainty information in the evaluation process. Figure 1 shows the steps of the proposed framework.

Stage 1. Construct a group of experts. Define the problem dimensions including a set of criteria and alternatives.

Stage 2. Compute the criteria weights.

Step 1. Build the decision matrix between the criteria and alternatives.

Step 2. Experts use plithogenic terms to evaluate the criteria and alternatives.

Step 3. We replace these terms with plithogenic numbers.

Step 4. We combine these numbers into a single matrix.

Step 5. We obtain crisp values.

Step 6. We normalize crisp values.

Step 7. We compute the criteria weights.

Stage 3. Rank the alternatives.

Step 1. Compute the weighted decision matrix.

Step 2. Sum of each row in weighted decision matrix.

Step 3. Rank the alternatives.

4. Illustrative Example

4.1 Evaluating Financial Performance

The proposed framework based on the plithogenic set is used to evaluate a set of criteria and alternatives. Three experts have evaluated the criteria and alternatives. These experts have experience in the field of this problem.

Stage 1: The problem dimensions are clearly illustrated by criteria and alternatives. Seven criteria and eleven alternatives are defined in this study. The criteria and defined as:

C1: Asset Efficiency

C2: Market Valuation

C3: Cash Flow Management

C4: Profitability

C5: Revenue Growth

C6: Liquidity

C7: Debt Management and Leverage

Stage 2. Compute the criteria weights. Decision matrix is built based on opinions of three experts. Then we replaced their opinions with plithogenic numbers as Tables 1-3. Then we combined these values in one matrix. Then we obtain crisp values. We normalize these values. Then we compute the criteria weights as in Figure 2.

Table 1. The first plithogenic numbers

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)
A_2	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A_3	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_4	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)
A_5	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_6	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_7	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_8	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)
A_9	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)
A_9	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)

Table 2. The second plithogenic numbers

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A_2	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_3	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_4	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_6	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_7	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_8	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)
A_9	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)

A_9	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)

Table 3. The third plithogenic numbers

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_2	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_3	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_4	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_5	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_6	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)
A_7	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_8	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_9	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)
A_9	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)

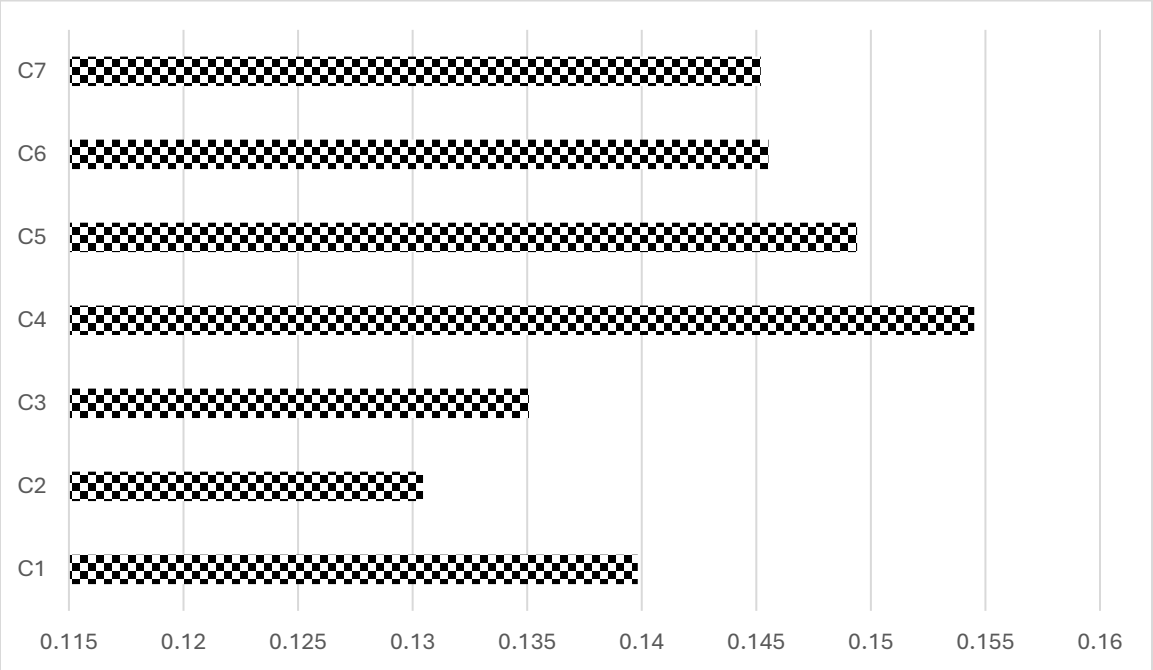

Figure 2. The criteria weights.

Stage 3. Rank the alternatives. We multiply the criteria weights by the combined decision matrix as in Table 4. Then we rank the alternatives as in Table 5. The results show the alternative 11 has the highest rank and alternative 5 has the lowest rank.

Table 4. The weighted decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.075431	0.046736	0.088297	0.113284	0.125029	0.136229	0.168551
A_2	0.141331	0.090074	0.141124	0.176455	0.173434	0.075409	0.148539
A_3	0.09591	0.078273	0.110439	0.161428	0.092267	0.156046	0.075234
A_4	0.159101	0.159776	0.059052	0.191133	0.108066	0.16063	0.115356
A_5	0.159683	0.143991	0.144203	0.083354	0.159808	0.14963	0.173554
A_6	0.159543	0.102864	0.118904	0.16495	0.116003	0.136314	0.184516
A_7	0.1586	0.155938	0.146291	0.092708	0.12699	0.087319	0.123039
A_8	0.105477	0.122117	0.116258	0.155775	0.14193	0.146719	0.127806
A_9	0.118854	0.131858	0.154125	0.184695	0.176466	0.147568	0.066547
A_9	0.130972	0.069424	0.138231	0.170544	0.189862	0.089877	0.128109
A_{11}	0.026095	0.057504	0.025209	0.130937	0.10961	0.156046	0.12385

5. Comparative Analysis

This section compares the proposed model with other MCDM methods such as VIKOR, TOPSIS, EDAS, COPRAS, and MABAC methods. Table 5 shows the comparative analysis. The results show the proposed methodology is effective compared to other MCDM methods due to its outcome the alternative 11 is the best and alternative 5 is the worst. We show all MCDM methods agree the alternative 11 is the best and alternative 5 is the worst.

Table 5. The comparative analysis results.

	EDAS method	**TOPSIS Method**	**VIKOR Method**
A_1	9	10	10
A_2	5	6	4
A_3	9	9	9
A_4	10	4	5
A_5	1	1	1
A_6	2	2	2
A_7	8	8	6
A_8	7	7	7
A_9	3	3	3
A_{10}	6	5	8
A_{11}	11	11	11

	COPRAS method	MABAC Method	Our Model
A_1	10	10	10
A_2	4	4	5
A_3	9	9	9
A_4	5	5	4
A_5	1	1	1
A_6	2	2	2
A_7	7	7	8
A_8	8	8	7
A_9	3	3	3
A_{10}	6	6	6
A_{11}	11	11	11

6. Conclusions

The aim of this paper is to propose a MCDM framework for evaluating the performance of financial based on a set of business. We use the plithogenic set with the MCDM methodology to deal with uncertainty environment. We used the WSM to rank the alternatives. Three decision makers evaluate the criteria and alternatives. Then we used the plithogenic numbers to evaluate the criteria and alternatives. We integrated these numbers into a single matrix. We compute the criteria weights. Then we rank the alternatives. The alternative 11 has the highest rank and alternative 5 has the lowest rank. We compared our model with other MCDM methods. The results show our model is effective compared to other MCDM methods.

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Evaluating Construction Projects of Urban Road and Bridge Based on Environmental Sustainability with MCDM Methodology

Zhenhua Sun*

Henan College of Transportation, Zhengzhou, 451460, Henan, China

*Corresponding author, E-mail: hnszh-cn@163.com

Abstract:

Evaluation construction projects of road and bridge has various criteria so, we used the concept of multi-criteria decision making (MCDM) methodology to deal with these criteria. The MOORA methodology is a MCDM methodology used to rank various projects. MOORA methodology is integrated under the Plithogenic set to deal with uncertainty data. The Plithogenic set is an extension of neutrosophic sets. Three experts are invited to evaluate the criteria and alternatives. This study gathered seven criteria and 13 alternatives. We compute the criteria weights and rank the alternatives. The results show the Environmental Impact Mitigation criterion has the highest weights and the Long-Term Environmental Monitoring criterion has the lowest weights.

Keywords: Decision Making; Environmental Sustainability; Construction Projects of Road and Bridge; MCDM Method; MOORA Method.

1. Introduction

The lengthy and intricate process of every development project, from initial planning to the ultimate approval of the work, is well-established in the customs and practices of road design and construction, but the ex-post review procedure is little understood and frequently overlooked. It is now crucial to assess, upon completion, whether the infrastructure has achieved the goals for which it was first intended[1], [2].

To better guide the selection of measures to be recommended in the future and to address any objections that may be raised in the field by various actors, such as roadside residents, traders, and representatives of various associations and organizations, it is also crucial

that this evaluation draw both positive and negative lessons from completed projects. Under some conditions, it would be possible to prevent undue delays in the development process by using clear communication to offer an impartial foundation for discussion regarding these objections[1], [3], [4].

Since assessing Construction Projects of Road and Bridge is a multi-criteria decision making (MCDM) problem, the evaluation metrics might serve as the criteria, and these sets of metrics could be used to choose the alternatives[5], [6].

One of the most accurate approaches to decision-making is MCDM, also known as Multi-Criteria Decision Analysis (MCDA), which has revolutionized the area. When Benjamin Franklin released his research on moral algebra notion, he created one of the earliest studies on multi-criteria decision-making. Since the 1950s, several theoretical and empirical researchers have studied MCDM techniques to investigate their mathematical modeling potential to offer a framework that can assist in organizing decision-making issues and producing preferences from options. MCDM encompasses a variety of approaches that vary from one another in many ways, which will be covered in the sections that follow[7], [8], [9].

When contradictory or non-contradictory multiple ancient entities are combined to create new entities, this process is known as plithogeny. Smarandache presented it in 2017 as a neutrosophical generalization. A plithogenic set is one whose elements are defined by the attribute values, and it is a generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets. Between vj and the dominant (most significant) attribute value vD, each attribute value has a contradiction degree value c(vj,vD). The degree of attribute disagreement helps the model produce more accurate findings. As generalizations of neutrosophic sets, logic, probability, and statistics, respectively, plithogeny yields the plithogenic set, logic, probability, and statistics that Smarandache also developed in 2017[10], [11], [12].

2. Plithogenic Set Characteristics

Plithogeny is the process by which contradictory (dissimilar) or non-contradictory combinations of several old entities arise, develop, germinate, and evolve into new entities. A plithogenic set (P, A, V, d, c) is a collection of items with values V = {v1, v2, …, vn}, for n ≥ 1 and several attributes A = {α1, α2, …, αm}, m ≥ 1. The value of each attribute, V, has two primary characteristics. The first is the element x's appurtenance degree function d(x,v), according to a set of specified condition. The second degree function that

is achieved between each attribute value and the most significant (dominant) one is the contradiction (dissimilarity) degree function c(v,D)[11].

The primary component of the plithogenic aggregation procedures (union, intersection, complement, inclusion, and equality) that improve aggregation accuracy is the contradiction degree function.

Let A be a non-empty set of uni-dimensional attributes, such as A = {α1, α2, …, αm}, m ≥ 1, and let $\alpha \in A$ be an attribute whose value spectrum is the set S. S can be a finite discrete set, such as S = {s1, s2, …, sl}, $1 \leq l < \infty$, or an infinitely countable set S = {s1, s2, …, s∞}, or an infinitely uncountable (continuum) set S =]a,b[, a < b, where]…[is any open, semi-open, or closed interval from the set of real numbers or from other general sets.

3. Proposed Model

In this study, we proposed a model to evaluate the Construction Projects of Road and Bridge Based on Environmental Sustainability based on combination of MCDM methodology and plithogenic aggregation operators. We aggregate the advantages of the MOORA method to evaluate the criteria and alternatives and the plithogenic sets. The aim of this model derives from the plithogenic sets due to this method ensure more accurate outcomes under uncertainty environments[13], [14], [15]. The steps of the proposed models are shown in Figure 1 and explained in detail in this part.

- Step 1: Decision makers and experts identify a set of criteria and alternatives. The most popular criteria and summarized in Table 1 are based on the opinions of experts and decision makers. These criteria can reflect the sustainability environmental factors.

Table 1. Popular environmental factors for this study.

Criteria	Name	Type
C_1	Debt Management and Leverage	Cost
C_2	Liquidity	Beneficial
C_3	Revenue Growth	Beneficial
C_4	Cash Flow Management	Beneficial
C_5	Profitability	Beneficial
C_6	Asset Efficiency	Beneficial
C_7	Market Valuation	Beneficial

Experts can compute the importance of criteria based on the linguistic terms.

- Step 2: We replace the linguistic terms with the plithogenic numbers. Then we combine these numbers using the plithogenic operators. This step can increase the accuracy of the outcomes.
- Step 3: Obtain crisp values from the aggregated plithogenic numbers.
- Step 4: Normalize crips values to compute the criteria weights
- Step 5: Build the decision matrix.
- Step 6: Normalize the decision matrix.

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}; i = 1, \dots, m; j = 1, \dots n.$$

- Step 7: Compute the reference points for the positive and negative criteria.
- Step 8. Compute assessment values.

$$q_j = \sum_{j=1}^{g} r_{ij} w_j - \sum_{j=g+1}^{n} r_{ij} w_j$$

- Step 9. Final rank of alternatives

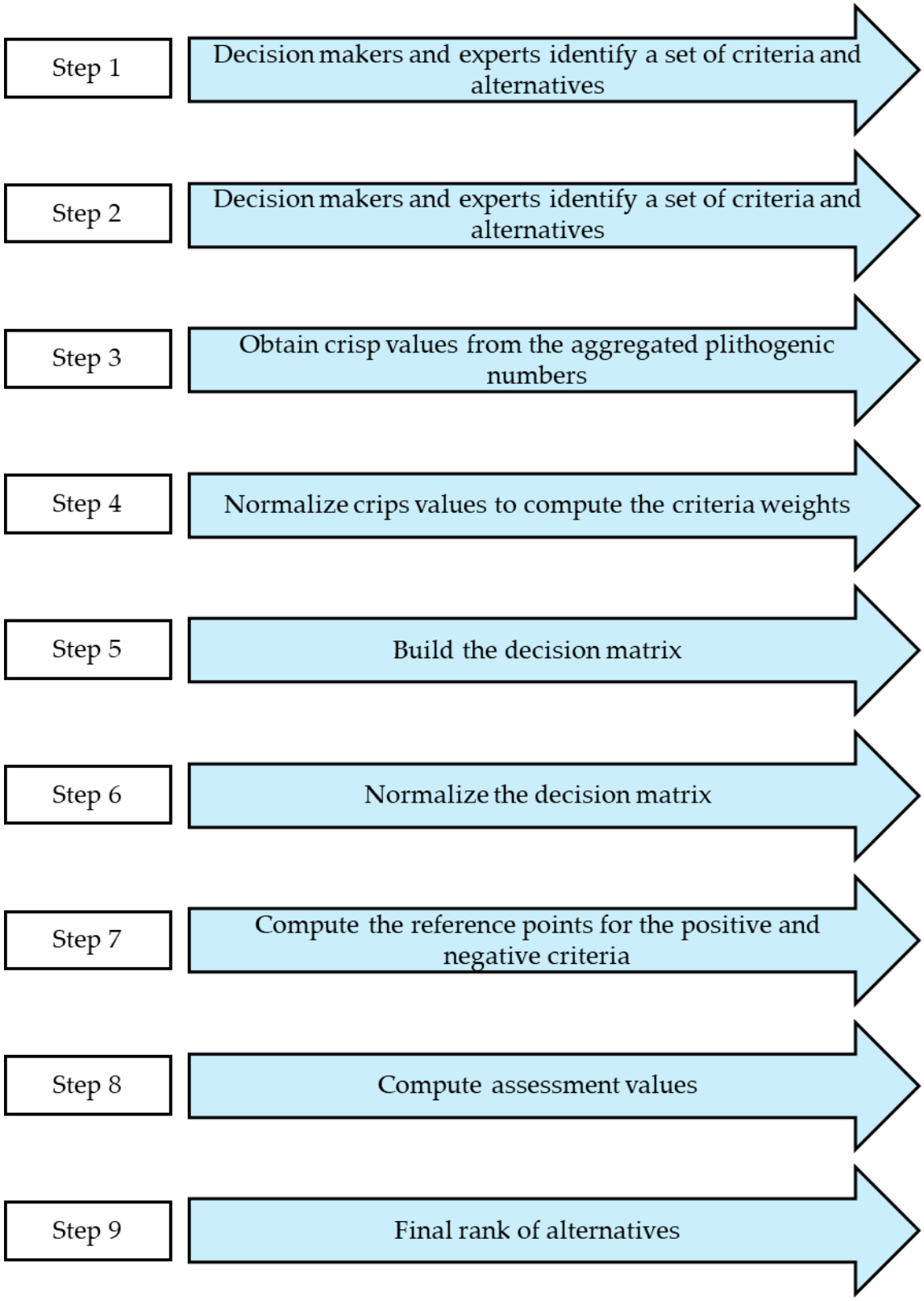
Step 1
Decision makers and experts identify a set of criteria and alternatives
Step 2
Decision makers and experts identify a set of criteria and alternatives
Step 3
Obtain crisp values from the aggregated plithogenic numbers
Step 4
Normalize crips values to compute the criteria weights
Step 5
Build the decision matrix
Step 6
Normalize the decision matrix
Step 7
Compute the reference points for the positive and negative criteria
Step 8
Compute assessment values
Step 9
Final rank of alternatives

Figure 1. The steps of the MOORA method with plithogenic numbers.

4. Real World Case Study

In this study the proposed model has been applied in the application on China to evaluate the Construction Projects of Road and Bridge Based on Environmental Sustainability. In this application, three experts were assisted by their experience in solving such cases to assess the sustainability environment on construction projects. The main goal of this study to evaluate the Construction Projects of Road and Bridge Based on Environmental Sustainability. Firstly, three experts identified a group of seven criteria for environmental sustainability and threaten alternatives. Figure 2 shows the criteria and alternatives.

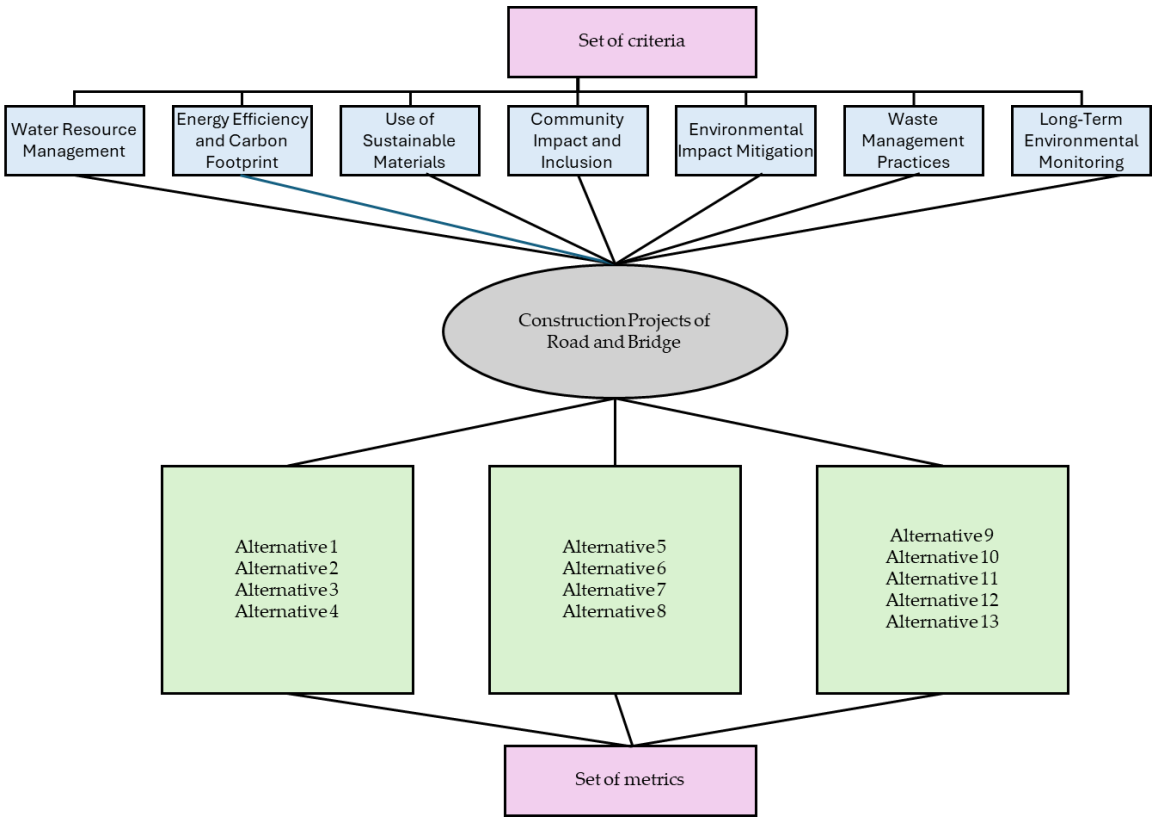

Figure 2. The Construction Projects of Road and Bridge criteria and alternatives.

The criteria by the experts based on linguistic terms are evaluated by using the plithogenic sets. Then we replace these terms by using the plithogenic numbers as shown in Table 2.

Table 2. The evaluation criteria by three experts.

D1	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)
A_2	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)
A_3	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_4	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_5	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)
A_6	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A_7	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_8	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_9	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)
A_{10}	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_{11}	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)
A_{12}	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)
A_{13}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
D2	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)
A_2	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)
A_3	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_4	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_5	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)
A_6	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_7	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_8	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_9	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_{10}	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_{11}	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
A_{12}	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_{13}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
D3	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_2	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_3	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_4	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A_5	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_8	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_9	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_{10}	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)

A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_{12}	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_{13}	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)

We compute the criteria weights as in Table 3.

Table 3. The weights of criteria.

Criteria	Weights	Rank
C_1	0.143529	4
C_2	0.143656	5
C_3	0.149052	6
C_4	0.137201	2
C_5	0.150105	7
C_6	0.139396	3
C_7	0.137061	1

Then we applied the steps of the MOORA method to show the rank of alternatives. We normalize the criteria weights as shown in Table 4. Then we compute the weighted normalized decision matrix as in Table 5. Then we rank the alternatives as in Figure 3.

Table 4. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.297576	0.141862	0.251397	0.278621	0.321986	0.394257	0.276094
A_2	0.277152	0.420262	0.361353	0.323071	0.26289	0.208337	0.383031
A_3	0.241771	0.17127	0.249973	0.302656	0.351958	0.140489	0.268299
A_4	0.11523	0.329344	0.139476	0.089226	0.330461	0.059481	0.154777
A_5	0.272568	0.344256	0.303448	0.117018	0.159696	0.369995	0.174983
A_6	0.255548	0.318321	0.319005	0.409629	0.288675	0.260581	0.276094
A_7	0.273399	0.193101	0.28292	0.329304	0.311507	0.361509	0.345575
A_8	0.27683	0.222213	0.303448	0.334258	0.211395	0.253223	0.253834
A_9	0.396402	0.242771	0.237843	0.195982	0.24486	0.14608	0.231631
A_{10}	0.420016	0.301452	0.272649	0.194321	0.159696	0.16515	0.174983
A_{11}	0.211777	0.273559	0.3006	0.173825	0.159696	0.260581	0.226256
A_{12}	0.222083	0.242771	0.258701	0.217104	0.288495	0.326064	0.361222
A_{13}	0.210249	0.274525	0.264805	0.404049	0.386501	0.394257	0.352807

Table 5. The weighted normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.042711	0.020379	0.037471	0.038227	0.048332	0.054958	0.037842

A_2	0.039779	0.060373	0.053861	0.044326	0.039461	0.029041	0.052499
A_3	0.034701	0.024604	0.037259	0.041525	0.052831	0.019584	0.036773
A_4	0.016539	0.047312	0.020789	0.012242	0.049604	0.008291	0.021214
A_5	0.039121	0.049454	0.04523	0.016055	0.023971	0.051576	0.023983
A_6	0.036678	0.045729	0.047548	0.056202	0.043332	0.036324	0.037842
A_7	0.039241	0.02774	0.04217	0.045181	0.046759	0.050393	0.047365
A_8	0.039733	0.031922	0.04523	0.045861	0.031731	0.035298	0.034791
A_9	0.056895	0.034875	0.035451	0.026889	0.036755	0.020363	0.031748
A_{10}	0.060284	0.043305	0.040639	0.026661	0.023971	0.023021	0.023983
A_{11}	0.030396	0.039298	0.044805	0.023849	0.023971	0.036324	0.031011
A_{12}	0.031875	0.034875	0.03856	0.029787	0.043305	0.045452	0.04951
A_{13}	0.030177	0.039437	0.03947	0.055436	0.058016	0.054958	0.048356

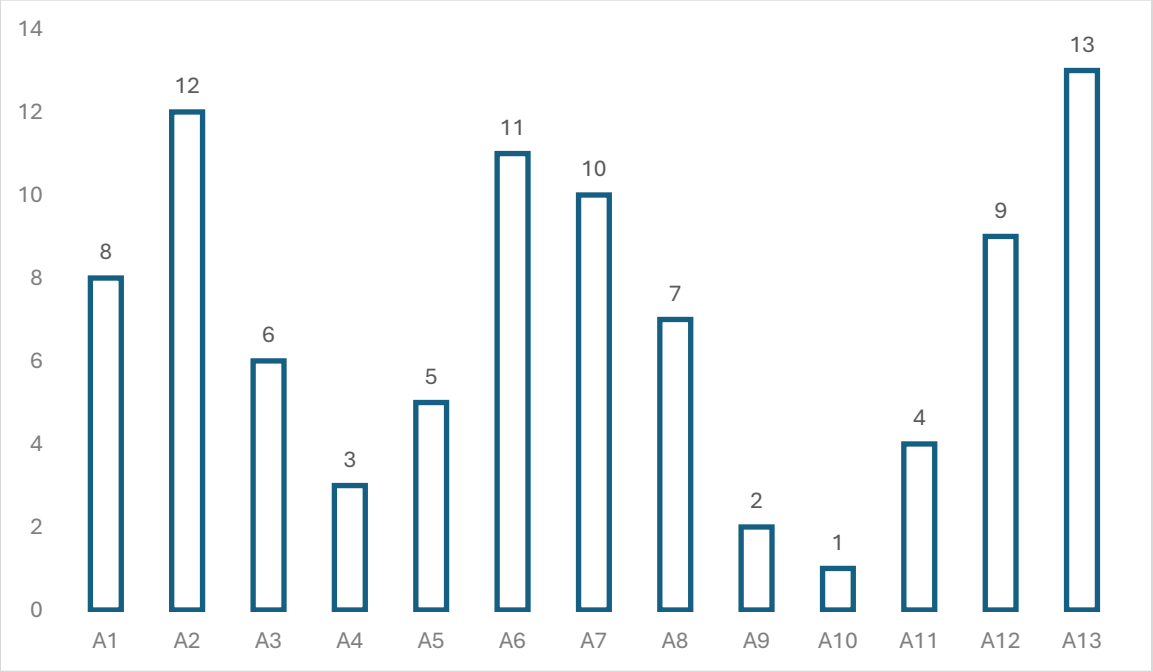

Figure 3. The rank of alternatives.

4. Results and Discussion

Based on the opinion of experts, the ranking of criteria Environmental Impact Mitigation, Use of Sustainable Materials, Energy Efficiency and Carbon Footprint, Water Resource Management, Waste Management Practices, Community Impact and Inclusion, Long-Term Environmental Monitoring. We show that Environmental Impact Mitigation has the highest weights, and Long-Term Environmental Monitoring has the lowest weights.

Three experts are evaluated the seven criteria and 13 alternatives. Based on these criteria weights, we applied the MOORA method to rank the alternatives. We show the alternative 13 has the highest rank and alternative 10 has the lowest rank.

5. Conclusions

This study proposed a MCDM methodology to evaluate the construction of projects or road and bridges. The MCDM methodology is integrated into Plithogenic sets to deal with uncertainty information. Three experts are invited to evaluate the criteria and alternatives. They used linguistic terms in their evaluation. Then we replaced their evaluation using the Plithogenic numbers. Then we combine these numbers into one matrix. Then we apply the MOORA method to rank the alternatives. The decision matrix is built between the criteria and alternatives. The criteria weights are computed by normalizing crisp values. The results show the alternative 13 has the highest rank and alternative 10 has the lowest rank.

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Development of Neutrosophic Gamma Distribution for Modeling Neonatal Mortality Data

Fuad S. Alduais[1] and Zahid Khan[2*]

[1] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj, 11942, Saudi Arabia

[2] Department of Quantitative Methods, Pannon Egyetem, Veszprem, H-8200, Hungary.

[1] zahidkhan@hu.edu.pk
[2] f.alduais@psau.edu.sa

* Correspondence: zahidkhan@hu.edu.pk

Abstract: This study explores and investigates the key characteristics of the neutrosophic gamma model (NGM) to analyze neonatal mortality data. The proposed model has the ability to handle imprecision, vagueness and uncertainty in data which often exist in health statistics. The key characteristics of the proposed model such as the probability density function, cumulative distribution function, statistical moments and some basic shape coefficients are discussed to clarify its difference from the classical model. Air pollution mortality data are commonly encountered imprecise and incomplete information due to factors such as missing values, measurement errors, reporting inconsistencies. The proposed NGM has the inherent ability to model such ambiguous data as robust tool for addressing these challenges. Through a detailed statistical analysis of mortality data linked to air pollution in Saudi Arabia, we demonstrate that the NGM outperforms traditional models in managing uncertainty and providing more accurate mortality analysis. This study not only enhances the theoretical structure of the NGM but also provides practical implications for policy formulation and healthcare management.

Keywords: Neutrosophic probability; gamma function; estimation; simulation

1. Introduction

Probability distributions and random variables are not only a core concept in probability theory and statistics, but they also serve a critical function in helping model uncertainty, as well as in aiding data-driven decisions in different fields [1]. A random variable is a variable that takes numerical values on the outcomes of a random phenomenon [2]. These can be variable either discrete, it will take specific values, or continuous that can take the value in a range. The probability distribution, meanwhile, specifies the probability of each of the possible outcomes of a random variable. Probabilistic models can be used to understand and predict various systems in the real world, including the weather, the stock market, healthcare data, and more. Probability distributions are critical in conducting hypothesis tests, making risk assessments and optimizing models since they provide the framework for uncertainty analysis [3]. Thorough understanding of random variables and probability distributions is of utmost importance in domains in different applications such as

engineering, economics and machine learning and artificial intelligence [4]. One of the most common continuous distributions is gamma distribution [5]. The gamma distribution is one of the most flexible that is used in many statistical applications and plays a unique role among the hundreds of models that we use in many applications. The gamma distribution is a more generalized form of exponential distribution [6]. It has many applications in survival and reliability studies. The gamma distribution is useful for instruments or equipment that needs to be calibrated [7]. Its adaptability to different shapes and scales makes it a vital tool in fields including finance, environmental science, and healthcare. In addition to being prominent in statistical literature, the gamma distribution has been extensively used in statistical methodologies; it is essentially used as conjugate prior for rate parameters in Bayesian analysis, an additional point of its importance [8].

Among various models widely applied, the gamma distribution is an unparalleled one with diverse modelling capabilities in real-world phenomena, making it the most relevant statistical tool [9]. It is vital when dealing with such data, including wait time, survival analysis, and reliability studies [10]. The flexibility of the shape and scale makes the gamma distribution a great model for situations that can be found in finance, environmental science, and health care [11]. It shows how popular it is in the modern-day analytics data applications. Random uncertainty, representable with random variable, is common in the way of a stochastic environment; however fuzzy uncertainty appears when the information is not random but vague or unclear. Fuzzy uncertainty refers to situations in which precise values cannot be ascertained, and where the boundaries between possible outcomes are not clearly defined [12]. Fuzzy set theory is employed to model this kind of uncertainty; its underlying principle is that it is more appropriate to describe elements by degrees of membership rather than a definitive classification [13]. For example: The temperature is high. In classical probability we might define a random variable with respect to temperature values whereas in fuzzy logic, we might simply refer to the concept of "high temperature", which we would now express as a fuzzy set in which temperature is high to different extents depending on context. Other examples would be customer satisfaction ratings (where satisfaction could be "low," "medium," or "high," with overlapping meanings) or the age of a person (where an age could be "young" or "old," but the switch between the categories is gradual and fuzzy). Fuzzy uncertainty is applied when exact measurement is infeasible, such as in expert/knowledge-based systems, controllers, decision-making processes, etc., where human perception and intangibles are translated into quantitative measures.

Neutrosophic theory is an extension of fuzzy set theory which helps in dealing with indeterminate and inconsistent information where it fails [14]. Fuzzy set theory is used when elements are assigned membership degrees in range of 0 to 1 point; neutrosophic theory, on the other hand, considers truth, indeterminacy, and falsity, presenting a further granulated means of dealing with uncertainty [15]. While in fuzzy sets, the elements can only have a certain degree of truth or membership, neutrosophic sets can contain information that is indeterminate, as truth, indeterminacy, and falsity which are independent components and take values between 0 and 1. This was a major shortcoming of fuzzy set theory as it failed to model situations where the information is conflicting or uncertain and cannot be represented by a single membership function [16]-[18]. In fuzzy set theory, for instance, an answer to the question “Is the weather cold?” would be a fuzzily defined concept, and could be denoted with a membership value, for example, 0.7 for "cold." In neutrosophic theory, however, the answer might have a truth degree of 0.7 (e.g., "cold"), falsity degree of 0.2 (e.g., "not cold"), and indeterminacy degree of 0.5 (e.g., "uncertain") at the same time, and this represents a more realistic and multi-dimensional representation of uncertainty. The neutrosophic theory is, therefore, a very convenient opportunity in decision-making, artificial intelligence, finance, business, engineering, healthcare and many real field applications where vagueness and contradictory information commonly exist [19]-[21]. Due to neutrosophic logic being able to manage truth, falsity, and indeterminacy separately, its use is widespread in all fields of science and

engineering, particularly where uncertainty, vagueness, and incomplete information are concerned. In the domain of neutrosophic statistics, a wide range of indecisive data cannot be modeled using exact statistics and can rather induce a neutrosophic logic in decision making process [22]. Neutrosophic statistics generalizes classical and traditional statistical techniques by introducing uncertainty and considering the uncertain and contradictory nature of information and data, which is potentially most valuable in many complex scenarios, decision-making, risk analysis, and pattern recognition [23]. For example, a medical diagnostic in which signs and test results may be conflicting or uncertain; neutrosophic statistics can evaluate more accurately the probability of diagnosis because it correlates indeterminate and conflicting evidence. Neutrosophic logic can also be utilized in machine learning and artificial intelligence to process incomplete or ambiguous data, enhancing model accuracy and enabling improved decision-making processes [24]. Neutrosophic statistics can further find application in finance for risk assessment and also in engineering for fault detection and in environmental studies for modeling certain uncertain and fluctuating data due to the capability and effectiveness to signifying degrees of truth, falsity and uncertainty separately. Under the neutrosophic logic many extensions of the classical probability distributions have been developed in literature [25]-[29]. These extensions are important because such distributions exhibit properties that are well-suited to modeling types of information that involve vagueness (incomplete, inconsistent, or unclear information). Neutrosophic probability distributions utilize neutrosophic logic, providing a flexible framework for statistical modeling in difficult real-life situations. Their versatility renders them best proven for applications in domains where classical probabilistic methods are challenged in areas including social sciences, engineering studies, survival analysis, health care studies, and environmental studies and much more. Although many neutrosophic structures have been developed so far, there is still a need to provide real-life applications for many developed neutrosophic probability distributions, particularly for commonly used models.

In this paper, we explore the properties of the NGM and its application to air pollution mortality rates. Incorporating neutrosophic parameters in the standard gamma distribution allows the neutrosophic gamma distribution to effectively handle uncertainties within the data. This allows for a more accurate and nuanced understanding of the data behind the mortality rates associated with air pollution in relation to other forms of pollution, capturing both the unknowns in terms of environmental and health conditions.

This work is organized as follows: Section 2 provides some useful properties of classical gamma model. In Section 3 the proposed model with essential properties is discussed. In Section 4, estimation methodology of the neutrosophic version of the gamma model is demonstrated. Section 5 relates the applications of the suggested model to air pollution mortality rate data. Eventually, the findings of the work are concluded in Section 6.

2. Gamma Distribution

In this section we will discuss the classical structure of gamma distribution. The gamma distribution has its unique role in statistical analysis. A random variable $T > 0$ is said to follow gamma distribution with scale parameter θ and shape parameter ρ if it has the probability density function (pdf) given below:

$$f_T(t;\rho,\theta) = \frac{t^{\rho-1}e^{-\frac{t}{\theta}}}{\Gamma(\rho)\,\theta^{\rho}}, \quad t > 0 \tag{1}$$

where $\Gamma(\rho)$ represent the gamma function with the defined value:

$$\Gamma(\rho) = \int_0^{\infty} z^{\rho-1}e^{-z}\,dz$$

A pdf is central to statistics and refers to the probability of a continuous random variable taking on the value or range of values. The distribution of the data refers to the definition of the relative frequency of occurrence which specifies a complete characterization of (the) sampled data. In probability, the area under the curve of the pdf over any interval corresponds to the probability that the random variable falls within that interval, making it fundamental for probability calculation. The pdf additionally elucidates important aspects of the data, including central tendencies, spread, and whether skewness or multimodality is present. In reliability analysis, for example, the pdf of a lifetime distribution illustrates how failure rates are age-dependent, which in turn, helps with decision-making. It connects theory and practice and facilitates modeling and prediction in engineering, economics, and the natural sciences. The shape of the gamma pdf is shown in Figure 1.

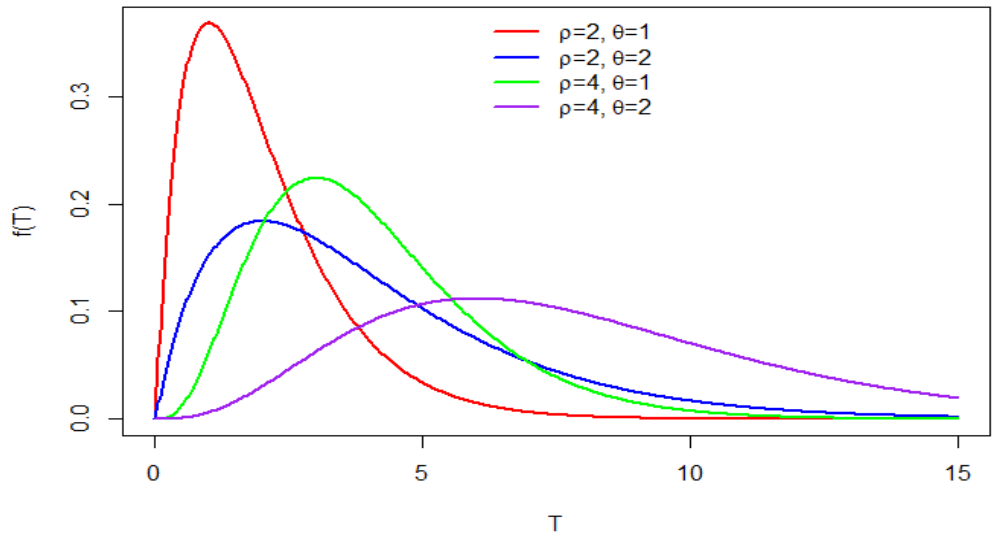

Figure 1. The pdf of the gamma distribution with different parameter settings

Figure 1 shows the curves of the pdf given in Eq (1) for different values of shape and scale parameters. It can be noticed that shape turns to symmetry for lager value of the shape parameter. Similarly the cumulative function (cdf) is another key concept related to pdf of any distribution. The cdf of the gamma distribution can be written as:

$$F_T(t;\rho,\theta) = \frac{1}{\Gamma(\rho)}\int_0^{\frac{t}{\theta}} x^{\rho-1}e^{-x}\,dx, \quad t>0 \tag{2}$$

The cdf in Eq (2) shows that it involves incomplete gamma fuction and not in closed form. However, due to existing software we can easily sketch plots of the cdf as shown in Figure 2.

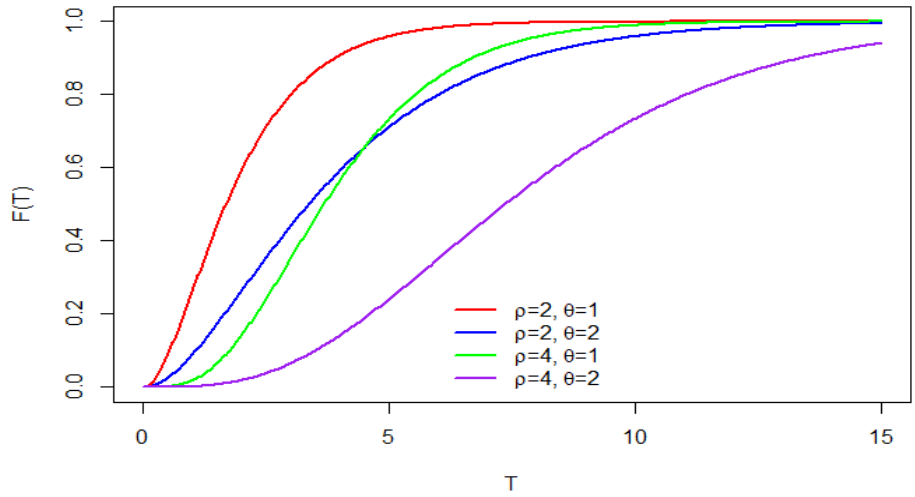

Figure 2. The cdf plot of the gamma distribution with different parameter settings.

The cdf is a key concept in probability theory, providing the cumulative probability that a random variable T is less than or equal to a specific value. While the pdf demonstrates the probability of certain values. The cdf is a full portrayal of the distribution and displays probabilities over intervals. This is crucial for reading probability and quantiles and percentiles. The cdf can be used in a reliability analysis to find the probability of failure over a period, for instance. Its meaning is intuitive: for any value t it provides us the proportion of the population or data less than tt which gives way to real-world applications like risk assessment, decision-making, and hypothesis testing.

The moment generating function of the gamma distribution can be written as:

$$M_T(s) = (1 - \theta s)^{-\rho}, \quad s < \frac{1}{\theta} \tag{3}$$

The moments about mean can easily be established using Eq (3) that may help to write its other properties like mean, variance, skewness and kurtosis coefficients.

$$\text{Mean: } E[T] = \rho\theta \tag{4}$$

$$\text{Variance: } \text{Var}(T) = \rho\theta^2 \tag{5}$$

$$\text{Skewness} = \frac{2}{\sqrt{\rho}} \tag{6}$$

$$\text{Excess Kurtosis} = \frac{6}{\rho} \tag{7}$$

Two important coefficients in statistics that show us something about the distribution of a probability distribution, are skewness and kurtosis. Skewness refers to the imbalance of the distribution, where the data could be skewed to the left side or right of the mean. A positive skew means the tail on the right side is longer, whereas a negative skew means the left tail is longer. Kurtosis measures the sharpness of the distribution. High kurtosis indicates a distribution that is more outlier-prone than the normal distribution, while low kurtosis reveals less outliers than the normal distribution. Those two coefficients together give us an insight to general shape of the distribution considering higher moment values, that is why they are useful for task overview for model selection, data analysis and risk assessment, particularly in finance, engineering and natural sciences. Now in the next Section we illustrate the characteristics of the proposed NGM with essential functions.

3. Neutrosophic Gamma Model

In this section, neutrosophic structure of the gamma model has been discussed in connection with the classical model. By contrast, a NGM uses neutrosophic logic to model uncertainty, indeterminacy, and impreciseness in the data. The NGM is an extension of the classical distance principle with

neutrosophic parameters as opposed to classical numbers; Its numeric parameters can be expressed in ranges, rather than fixed numbers. This augmented framework is capable of accounting for richer uncertainty layers than the classical gamma distribution, which is ideal considering the ambiguity of the data.

A gamma random variable T follows the NGM with the following pdf:

$$f_n(t) = \frac{1}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}}, \quad t > 0 \tag{8}$$

where incomplete neutrosophic function is given by: $\Gamma\rho_n = \int_0^\infty z^{\rho_n - 1} e^{-z} dz$.

The distributional parameters θ_n and p_n and θ_n are the scale and shape parameter respectively of the NGM. By assuming different values to shape and scale parameters , pdf of the suggested model can be depicted in Figure 3

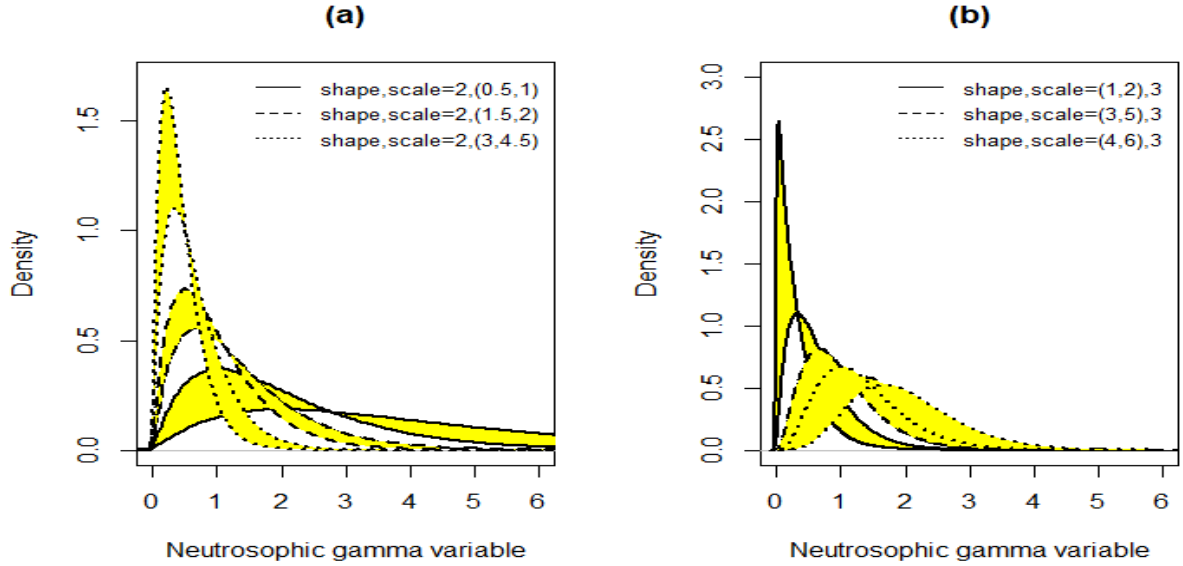

Figure 3. Shapes of the pdf of NGM with varying imprecise parameter setting

Figure 3 shows the sturdy curves of the NGM for varying imprecise parameter settings. In Figure 3 (a), scale parameter θ_n varies to three different values but shape parameter is taken as a crisp value, i.e., $\rho_n = [8, 8]$. In Figure 3(b) we assume shape parameter as an imprecise value and varies but scale parameter is set to fixed [6,6]. The pdf of the NGM in either case has the same properties as it exists in case of classical gamma distribution curve. Although we have limited our curves to some fixed values of shape and scale parameters, however; one can sketch these curves easily by using the "neutrostat" R library [22].

Some basic properties of the proposed model can be determined under the neutrosophic algebra. For example the kth moment about origin of the proposed model can be established as follows:

The kth moment of NGM can be defined as:

$$\vartheta'_{kn} = \int_o^\infty \frac{1}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}} t^k \; dt \tag{9}$$

Eq (9) further can be extended as:

$$= \left[\int_0^\infty \frac{1}{\Gamma\rho_l \theta_l^{\rho_l}} t^{\rho_l - 1} e^{-\frac{t}{\theta_l}} t^k \; dt, \; \int_0^\infty \frac{1}{\Gamma\rho_u \theta_u^{\rho_u}} t^{\rho_u - 1} e^{-\frac{t}{\theta_u}} t^k \; dt \right] \tag{10}$$

Considering the transformation $y = \frac{t}{\theta_n}$ in Eq (10) and simplification yielded:

$$\vartheta'_{kn} = \left[\frac{\theta_l^k}{\Gamma\rho_l} (\Gamma k + \rho_l), \; \frac{\theta_u^k}{\Gamma\rho_u} (\Gamma k + \rho_u) \right] = \frac{\theta_n^k}{\Gamma\rho_n} \; ; \quad k = 1, 2, \ldots \tag{11}$$

Assigning different values to k in Eq (11) generates moments of origin which can be extended as:

$\vartheta_{1n} = \theta_n$ (First moment of origin)
$\vartheta_{2n} = \theta_n$ (Second moment of origin
$\vartheta_{3n} = 2\theta_n$ (Third moment of origin)
$\vartheta_{4n} = 6\theta_n - 3\theta_n^2$ (Fourth moment of origin)
The other important function of the proposed model is cumulative distribution function(cdf) which can be defined asL

$$F_n(x) = \int_0^x f_n(t)\, dt. \tag{12}$$

$$\mathrm{F_n(t)} = \int_0^x \frac{1}{\Gamma(\rho_n)\,\theta_n^{\rho_n}}\, \mathrm{t}^{\rho_n - 1}\, e^{-\frac{\mathrm{t}}{\theta_n}}\, dx. \tag{13}$$

Simplification of Eq (13) provided:

$$F_n(x) = \frac{\gamma\left(\rho_n, \frac{x}{\theta_n}\right)}{\Gamma(\rho_n)} \tag{14}$$

where $\gamma\left(\rho_n, \frac{x}{\theta_n}\right)$ is incomplete gamma function which can easily be found by using function in R library. The construction of the Eq (14) can be done by using the function of neutrostat package. The cdf of the proposed model for some specific values of shape and scale parameter is shown in Figure 4.

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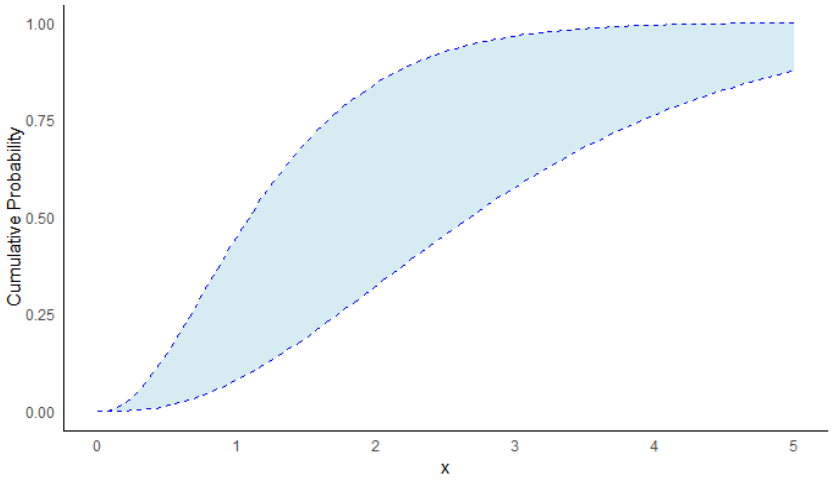

Figure 4. The cdf curve of the NGM at $\theta_n = [2, 3.5]$, and $\rho_n = [0.5, 1]$

Figure 4 shows the cdf curve of the proposed model when both parameters of the distribution are not exactly defined. Different cdf curves can be depicted with different parameter settings using the neutrostat package in R.
By using the definition of expected value, mean of the proposed NGM can be derived as:

$$E(T) = \int_0^\infty \frac{t}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}} dt \tag{15}$$

Extend form Eq (15) can be written as;

$$E(T) = \int_0^\infty \left[\frac{t}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}},\ \frac{t}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}}\right] dz \tag{16}$$

Considering the transformation $y = \frac{t}{\theta_n}$ provided:

$$E(T) = [\rho_l \theta_l,\ \rho_u \theta_u] = \rho_n \theta_n \tag{17}$$

Now if we define the variance, we can write

$$var(T) = E(T^2) - [E(T)]^2 \quad (18)$$

where $E(T^2)$ can be obtained as:

$$E(Z^2) = \int_0^{\infty} \frac{t^2}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}}\, dt$$

$$= \int_0^{\infty} \left[\frac{t^2}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}},\ \frac{t^2}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}} \right] dt \quad (19)$$

Simplification provided:

$$E(T^2) = [\theta_l^2 \rho_l(\rho_l + 1),\ \theta_u^2 \rho_u(\rho_u + 1)] \quad (20)$$

Thus Eq (18) becomes

$$var\ (T) = [\rho_l \theta_l^2,\ \rho_u \theta_u^2] = \rho_n \theta_n^2 \quad (21)$$

The other important measures of any distribution are the shape coefficients. The neutrosophic coefficient of skewness and the neutrosophic coefficient of kurtosis can be successfully used to study asymmetry and tail behavior, offering a solid framework for uncertainty and imprecision in data analysis. While skewness quantifies the skew of a traditional probability distribution, neutrosophic skewness reflects the interval (when the data components are not exactly defined) or amount of skew in a neutrosophic probability distribution. Likewise, kurtosis measures the pointiness or flatness of the peak of a distribution, while neutrosophic kurtosis makes this measure applicable to uncertain environments, quantifying the degree of sharpness as an interval or identifying indeterminacy. This flexibility of interpretation enables these measures to be especially beneficial when the data is affected by incompleteness and vagueness.

The skewness coefficient of the proposed model can be defined as

$$\beta_{1n} = \frac{\vartheta_{3n}}{(\vartheta_{2n})^{3/2}} \quad (22)$$

where the values of ϑ_{2n} and ϑ_{3n} can be determined from Eq (11).

Thus Eq (22) becomes

$$\beta_{1n} = \frac{2}{\sqrt{\theta_n}}$$

where $\beta_{1n} = [\beta_{1l}, \beta_{1u}]$.

Similarly kutorsis or access kurtosis coefficient of the NGM is defined as:

$$\beta_{2n} = \frac{\vartheta_{4n}}{(\vartheta_{2n})^2} \quad (23)$$

Eq (11) can be utilized to simplify Eq (23)

$\beta_{2n} = \frac{6}{\theta_n}$, which is the required result.

where $\beta_{2n} = [\beta_{2l}, \beta_{2u}]$ is neutrosophic measure of the proposed model.

4 Random Sample Generation and Estimation

In this section, we discuss the random generation and estimation procedure of the proposed model. The inverse cdf or the inverse transform method is a widely used technique to generate random samples from a specified probability distribution. It uses that fact that for any continuous random variable X with some F(x) there exists a way to sample from it, by drawing a uniform random variable and then applying the inverse cdf. This approach will work because the cdf "transforms" the probability range [0, 1] to the values of the random variable. First, uniform random values are generated and then, the inverse cdf is applied to transform these random values into the target distribution.

The cdf of the proposed function is defined in Eq (14) so the quantile function is given by:

$$Q(p; \rho_n, \theta_n) = F^{-1}(p; \rho_n, \theta_n) = U \quad (24)$$

where $\boldsymbol{p}$ follows the uniform distribution.

Now assigning different values to scale and shape parameters, we can draw random samples from the NGM. Assume that $\theta_n = [2, 3.5]$, and $\rho_n = [0\ .5, 1]$ we draw the random sample of 40 values the NGM, neutrosophic characteristics of simulated data are shown results Table 1.

Table 1 Random samples from the proposed NGM

[0.268, 1.425]	[0.7, 2.78]	[2.875, 8.214]	[1.575, 5.097]	[0.48, 2.125]
[2.097, 6.376]	[1.721, 5.459]	[1.121, 3.934]	[0.645, 2.622]	[0.958, 3.499]
[1.13, 3.958]	[0.564, 2.38]	[1.08, 3.826]	[0.727, 2.858]	[1.003, 3.62]
[0.549, 2.335]	[0.962, 3.508]	[1.218, 4.188]	[0.159, 1.009]	[2.939, 8.362]
[1.279, 4.346]	[1.241, 4.247]	[0.093, 0.714]	[0.726, 2.855]	[1.1, 3.878]
[0.939, 3.446]	[1.124, 3.94]	[0.684, 2.735]	[1.301, 4.401]	[0.32, 1.606]
[1.513, 4.94]	[0.264, 1.41]	[1.473, 4.842]	[1.998, 6.139]	[1.183, 4.096]
[1.778, 5.6]	[0.832, 3.154]	[0.709, 2.807]	[2.441, 7.197]	[0.924, 3.406]

Table 1 presents the random samples generated at specific shape and scale parameter values. Note that these values are randomly drawn using a function from the nutrostat package. Different seed settings in the program can yield different random samples, even for the same parameter settings. To study the different statistical characteristics of the proposed model , 10000 random samples are drawn at the same parameters setting and results are shown in Table 2.

Table 2 Characteristics of the simulated data generated from the NGM

Characteristics	Estimated value
Mean	[1.001, 3.501]
Variance	[0.713, 1.888]
Median	[0.836, 3.164]
Skewness	[1.058, 1.396]
Kurtosis	[4.572, 5.758]

Table 2 shows the neutrosophic characteristics of the NGM. It can be seen from the results that all characteristics are in interval values due to existence of indeterminacy in the shape and scale parameters of the proposed model. If this indeterminacy becomes zero, results of the proposed model match with the classical model.

Now we discuss the maximum likelihood (ML) approach for estimating the parameters of the NGM under the uncertain environment. ML is a common statistical approach used to estimate the parameters of a probabilistic model. This means finding the parameter values that maximize the likelihood function, a measure of how well the model accounts for the observed data. This approach works on the principle of maximizing the probability of the observed data, conditional on the model being used to describe the data. It is important because of its versatility and theoretical properties. The estimators obtained from ML approach are consistent and usually efficient (i.e. they converge to the true parameters with the lowest variance among biased estimators under regular conditions. Let $t_1\ , t_2, \ldots . t_n$ are sample of randomly values taken from the NGM, the log likelihood function is defined as

$$\vartheta(\rho_n, \theta_n\ |T) = -n\rho_n log(\theta_n) + (\rho_n - 1)\sum_{i=1}^{n} ln(t_i) - nln(\Gamma\rho_n) - \frac{\sum_{i=1}^{n} t_i}{\theta_n} \quad (24)$$

Maximizing (24) with unknown θ_n and ρ_n provided:

$$\frac{d\vartheta(\rho_n, \theta_n\ |T)}{d\theta_n} = n\left[-ln(\theta_n) - \frac{dln(\rho_n)}{d\rho_n} + \frac{\sum_{i=1}^{n} ln(t_i)}{n}\right] \quad (25)$$

$$\frac{d\vartheta(\rho_n, \theta_n\ |T)}{dd_n} = -\frac{n\rho_n}{\theta_n} + \frac{\sum_{i=1}^{n} t_i}{{\theta_n}^2} \quad (26)$$

Equating the expressions (25) and (26) to zero yielded:

$$\theta_n = \frac{\bar{t}}{\rho_n} \tag{27}$$

$$\overline{ln(t)} - ln(\bar{t}) + ln(\rho_n) - \frac{dln(\rho_n)}{d\rho_n} = 0 \tag{28}$$

The structures of Eq (27) and Eq (28) show that closed form does not exist so some iterative algorithm is required to achieved these estimates. Numerical set up to solve these unclosed forms can be achieved easily in R. In addition, the reliability study of these estimators can be assessed using the following metrics [26]:

$$Bias = \frac{\sum_{i=1}^{N}(\hat{\vartheta}_{ni} - \vartheta_n)}{N} \tag{29}$$

$$RSME = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\vartheta}_{ni} - \vartheta_n)^2}{N}} \tag{30}$$

where $RSME$ is formula used to find the root mean squared error of the estimator calculated from the sample. Bias and RMSE (Root Mean Square Error) are two important metrics in (non-)theoretically measuring the performance of your estimators. Bias (as a measure of systematic error) quantifies the expected difference between the value of the estimator and the true value of the parameter. A low bias in an estimator means that it is accurate on average, which is one reason why it is an important aspect of making predictions with confidence. In contrast, RMSE is a single value that incorporates both bias and variability; it is the square root of the average of the squared difference between the estimated and true values. This metric offers a comprehensive assessment of the accuracy of an estimator, as it accounts for both systematic and random errors. To assess the accuracy of the ML estimator, a simulation study has been conducted with $\theta_n = [0.5, 2]$ and $\rho_n = [3, 3]$ and results are shown in Table 3.

Table 3. Reliability assessment of ML estimator of the NGM

Sample	Bias	RMSE
10	[0.0011, 0.0046]	[6.8920, 10.2065]
25	[0.0005, 0.0019]	[6.8893, 10.1882]
50	[0.0003, 0.0011]	[6.8891, 10.1852]
75	[0.0000, 0.0000]	[6.8893, 10.1849]

Table 3 provides the accuracy of the ML estimators crossing different sample sizes. The Bias and RMSE assign a numerical value to the accuracy of an estimation, but these numbers are hard to interpret without fixed values. That is why estimation accuracy is assessed using the baseline values $\theta_n = [0.5, 2]$ and $\rho_n = [3, 3]$. Numerical results show that Bias and RMSE decrease with increasing sample size. This shows that as sample sizes increase, neutrosophic estimators provide better estimates. Consequently, the differences between the baseline values and the estimated values diminish as the sample size grows.

5 Real Data Example

In this section, we provide the real-data application of the proposed model. The proposed model is applied to neonatal mortality rate specific to Saudi Arabia. Neonatal mortality is the death of a newborn within the neonatal period which is the first 28 days of life [31]. This period is decisive for infant survival and establishes the basis for lifelong health. Neonatal mortality is a key indicator of the quality of maternal and newborn health care and, more broadly, of the society. Since early neonatal deaths can be prevented by timely and effective intervention including attendance of skilled/ trained personnel at birth, effective neonatal care and access to basic medical facilities. Intervening in this hugely impactful situation is therefore of urgent concern: scaling up interventions to reduce neonatal mortality is critical to saving lives for healthy societies and to achieving global health goals. Neonatal mortality data have significant attributes of health statistics under which they served to ascertain quality of a country in terms of health facilities, maternal and child health services and socio-economic status. Understanding this allows policymakers and healthcare providers to recognize gaps, prioritize

resources, and design targeted interventions to improve newborn survival rates. Neonatal mortality has dramatically decreased in Saudi Arabia over the years as a result of improved healthcare infrastructure, maternal care, and access to neonatal intensive care units. But the access to health care is not equal in all parts of the region. Consequently, projects within Vision 2030 have been enacted by the Saudi government as part of a strategy to tackle neonatal mortality in newborns through the expansion of health services, training of health professionals, and improved awareness around prenatal and postnatal care. This way, they can help strengthen safeguards for newborns, leading hopefully to a healthier next generation. The upper and lower values of the neonatal mortality from the source are given [32] in Table 4.

Table 4. Neonatal mortality rate for Saudi Arabia for the year 2001-2020

[5.97, 9.60]	[5.25, 8.56]	[4.63, 7.52]	[3.43, 5.73]	[2.47, 5.09]
[8.71, 13.26]	[7.68, 11.91]	[6.75, 10.82]	[4.90, 8.05]	[4.34, 7.01]
[2.77, 5.22]	[7.26, 11.35]	[6.37, 10.20]	[4.05, 6.53]	[8.17, 12.58]
[5.60, 9.08]	[3.76, 6.07]	[3.11, 5.44]	[2.26, 4.99]	[2.11, 4.89]

It can be seen from Table 4 that classical gamma model is not suitable for data given in interval forms. To analysis that we use the proposed model. Results of the proposed model are shown in Table 5.
Table 5 Statistical characteristics of the neonatal mortality rate using NGM

Table 5. Statistical characteristics of the neonatal mortality data

Characteristics	Estimated values
Shape	[5.91, 9.17]
Scale	[0.65, 0.72
Mean	[3.82, 6.61]
Median	[3.61, 6.37]
Skewness	[0.66, 0.82]
Kurtosis	[3.65, 4.01]

Table 5 indicates that all sample statistics are in interval form. The proposed method effectively analyzes data in interval forms and is versatile enough to handle cases where zero indeterminacy exists. This adaptability ensures its applicability to a wide range of datasets, providing robust analysis under varying conditions of indeterminacy.

6 Conclusions

This work explores the utility of the proposed neutrosophic gamma Model (NGM) in demonstrating the effectiveness of neutrosophic probabilistic models for healthcare datasets. The statistical properties of the NGM are discussed to differentiate it from the classical gamma distribution. The NGM is specifically designed to analyze data involving imprecise and vague information. The quantile function of the proposed model is examined to facilitate the generation of random samples. The maximum likelihood (ML) approach under the neutrosophic framework has been developed, and the reliability of these estimates is evaluated through bias and mean squared error, using a simulation approach. Simulated results show that imprecise data with a larger sample size efficiently estimate the unknown neutrosophic parameters. A practical example using a real dataset illustrates the utility of the proposed model. The results show that the NGM offers a significant improvement over the conventional gamma model. Specifically, its application to neonatal

mortality rates demonstrates that the classical gamma model fails to account for the interval nature of the data, whereas the NGM provides valid and consistent interval-based statistical estimates.

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